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NONSTANDARD ANALYSIS OF DYNAMICAL SYSTEMS. I: LIMIT MOTIONS, STABILITY(1)

BY
A. E. HURD

Abstract. The methods of nonstandard analysis are applied to the study of the qualitative theory of dynamical systems. The nonstandard notions connected with limiting behavior of motions (limit sets, etc.) are developed, and then applied to the study of stability theory, including stability of sets, attracting properties, first prolongations and stability of motions.

0. **Introduction.** Nonstandard analysis originated in 1960 when A. Robinson developed a logical procedure whereby the usual limiting operations of the calculus could be dealt with using a language of infinitesimals and indefinitely large quantities similar to that employed by Leibniz and other early developers of the calculus [7]. In order to do this Robinson imbedded the real numbers R in a larger structure $^*R$ which shared all of the formal properties of R that could be expressed in the lower predicate calculus, and which contained infinitesimals and infinitely large quantities (and hence was non-archimedean). The enlargment $^*R$ was shown to exist using the compactness principle of the lower predicate calculus (but an explicit construction for $^*R$ can be given using ultrafilters [4]).

Since Robinson's first paper nonstandard analysis has developed in many directions. Three books ([4], [5], [8]) and a recent symposium [2] have been devoted to the subject. W. A. J. Luxemburg has developed the ultrafilter techniques which in particular show the power of saturation in the construction of enlargements [3].

It is now clear that many branches of analysis can be profitably developed using nonstandard analysis as a tool. The general idea is to imbed the given mathematical structure in a larger one, an enlargement, which is analogous to a compactification or completion in that certain "ideal" elements have been added. Analysis in the enlargement is usually considerably simpler than in the standard structure. Information can then be carried back to the standard structure by means of a transfer principle, which says that the enlargement and the original structure are formally the same in a specified logical sense (involving higher order languages and type theory in Robinson's framework [8]).

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Although nonstandard analysis almost always yields a simplification of the methods of proof of standard results, its full power seems to appear with the introduction of new concepts in the enlargement, which have no immediate or obvious counterpart in the standard structures. This is certainly the case in the nonstandard theory of dynamical systems. In this paper we will lay the foundations for the theory, which will be developed further in later papers.

After a short introduction to nonstandard analysis in §1, we show how to construct the nonstandard dynamical system in §2.

§3 is devoted to the basic nonstandard notion of limit motion. A standard motion can be considered nonstandardly as still proceeding even when the time is infinite. Many standard notions, for example, limit sets and Lagrange stability, can be characterized in terms of limit motions.

In §4 we present the nonstandard analysis of the basic notions of stability. This allows us to unify the treatment, and extend results from the compact to the noncompact case. We also indicate how the theory of prolongations might be treated nonstandardly, but a full treatment is reserved for a later paper, as are other applications and extensions of the theory.

In this paper we have restricted ourselves to the analysis of dynamical systems over metric spaces, but much of the analysis could be carried through to systems over more general topological spaces, and at several places we indicate how this could be done.

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1. **Nonstandard analysis.**

1.1. *Higher order structures and languages.* We present here an introduction to nonstandard analysis, which is obviously not complete, but which will hopefully provide an intuitive guide to the sequel. Our basic reference is A. Robinson's book [8].

Mathematical structures consist of a basic set (the set of *individuals*), together with certain subsets of the basic set, relations between its elements, relations between relations, etc. (bearing in mind that functions are special types of relations). A systematic description of relations of higher order (i.e., relations between relations, between elements and relations, etc.) may be given in terms of the class of types T which are defined inductively [8, §2.6]. A mathematical structure $M=\{B_\tau\}_{\tau\in T}$, stratified by types, where $B_0=A$ and the elements of B_τ are relations of type τ, is called a *higher order structure*. For technical reasons Robinson allows the possibility that relations may be repeated, and calls the structure *normal* if this does not happen. For our purposes it suffices to assume that all structures are normal. If M contains all the relations of each type, then it is called *full*. If M is full and normal, it will be called the *complete structure* of A, and will be denoted by $M=\{A_\tau\}_{\tau\in T}$.

We will use a *formal language* Λ to make statements about higher order structures. The *atomic symbols* of Λ are (i) constants, (ii) variables (countably many), (iii) connectives ($\neg$, negation; $\vee$, conjunction; $\wedge$, disjunction; $\rightarrow$, implication; $\equiv$, equivalence), (iv) quantifiers (($\forall$), universal; ($\exists$), existential), (v) brackets ([,]), (vi) relation symbols $\Phi_\tau(\ ,\ldots,\ )$, one for each type $\tau\neq 0$, where if $\tau=(\tau_1,\ldots,\tau_n)$ then Φ_τ has $n+1$ places. *Well-formed formulae* (wff) and sentences are defined inductively in a natural way [8, §2.1]. A subclass of the well-formed formulae and sentences will be consistent with respect to type in the following sense. With each place of a relation symbol $\Phi_\tau(\ ,\ldots,\ )$ we assign a type; the type of the first place is τ, and if $\tau=(\tau_1,\ldots,\tau_n)$ then the type of the kth place is τ_{k-1} ($k=2,\ldots,n+1$). A wff or sentence is then said to be *stratified* if any given constant or variable appears in its atomic formulae only at places of the same type. A set K of sentences is stratified if each sentence in K is stratified, and the places in the sentences of K at which a given constant occurs are all of the same type.

In order to use Λ to make statements about M (either true or false) it is necessary to have a consistent way of naming entities in M by constants in Λ. We suppose that there is a *naming map* C from a set of constants of Λ onto all of the individuals and relations of M. If $C(\beta)=b$ for a given entity b in M then β is a *name* for (or *denotes*) b (note that a given entity may have many names). A stratified sentence X is *admissible in* M if every constant β, occurring in X at places of type τ, denotes some element of B_τ. Starting from the atomic sentences and proceeding inductively, it is now easy to define what is meant by saying that an admissible sentence X is *true* (or *holds*) in M (with respect to the mapping C). In applications this notion will be clear and so we will not elaborate [8, §2.6].

Let K be a stratified set of sentences in Λ, and M a higher order structure. If there is a naming map C such that the sentences in K are all admissible and true in M with respect to C, then we say that M is a *model* of K. A basic tool in Robinson's development of nonstandard analysis is the following *compactness* theorem, which for higher order model theory is due to L. Henkin, and is proved in §2.8 of [8].

THEOREM 1.1. *If K is a stratified set of sentences such that every finite subset K' of K has a model, then K has a model.*

In general the model M of K will not be full. For example, if A is the set of individuals of M, one would normally interpret a statement containing the phrase "for all subsets of A" to refer to *all* subsets. However, in the model M such phrases must be read "for all admissible subsets of A", where the admissible subsets do not necessarily include all subsets of A. Similar remarks apply to entities of any type $\tau\neq 0$. The admissible entities will be called *internal.*

1.2. *Enlargements.* Given a higher order structure $M=\{B_\tau\}_{\tau\in T}$, and a naming map C with respect to the language Λ, the basic idea of nonstandard analysis, as developed by Robinson, is to construct an appropriate higher order structure ${}^*M=\{{}^*B_\tau\}_{\tau\in T}$ and a naming map ${}^*C$ (again with respect to Λ) which

(i) is such that any sentence which is admissible and true in M is also true in $^*M$, and any sentence which is true in $^*M$, and admissible in M, is true in M (*transfer principle*),

(ii) can be regarded as a proper extension of M in the sense that the set of individuals $^*B_0$ in $^*M$ properly contains the set of individuals B_0 in M.

The structure $^*M$ will be called an enlargement of M. The mapping $^*C$ is a mapping from some of the constants of Λ onto all of the entities of $^*M$, so that Λ must be supposed initially to have sufficiently many constants. It is natural from (ii) that we require an agreement between C and $^*C$ on the individuals of M: if α denotes an individual a in M (i.e. $C(\alpha)=a$) then $^*C(\alpha)=a$. Thus when transferring true statements from M to $^*M$, the individuals carry over unchanged. However, if β is the name of an entity b in M of higher type, then $^*C(\beta)$ is not in general the same as b and will be denoted by $^*b$. For example, if b is a set of individuals in M, then $^*b$ will be a set containing b, i.e., an extension of b. Entities of this kind in $^*M$ will be called *standard*.

Before indicating how the enlargements $^*M$ are shown to exist it may be helpful to note the manner in which they are used in practice. The construction is in fact reminiscent of many similar procedures in classical analysis in which one enlarges a mathematical structure by the introduction of new "ideal" elements which complete the given structure in a natural way (i.e., which are closely related to its structure), and which make the analysis much easier. As examples from different areas, we may take the theory of compactifications of Riemann surfaces and the theory of distributions. The problem is then to see how results obtained in the enlargement are related to results in the original structure. In the standard theory this is usually dependent on the particular manner in which the ideal elements are introduced, whereas in the nonstandard theory the connection is achieved by use of the transfer principle.

The enlargements $^*M$ are shown to exist using the compactness theorem. Enlargements, in the technical sense of the word as used by Robinson, are richer structures than is implied by (ii). To construct them we need the notion of a concurrent relation. Let b be a binary relation of type $(\tau_1, \tau_2)=\tau$ in $M=\{B_\tau\}_{\tau\in T}$. We define the *domain* and *range* of b by dom $b=\{x\in B_{\tau_1}$: there exists a $y\in B_{\tau_2}$ such that $b(x, y)$ holds in $M\}$, ran $b=\{y\in B_{\tau_2}$: there exists an $x\in B_{\tau_1}$ such that $b(x, y)$ holds in $M\}$. Then b is said to be *concurrent* or *finitely satisfiable* if, for every finite set $x_1, \dots, x_n \in$ dom b, there is a $y\in$ ran b such that $b(x_i, y)$ holds in M for all $i=1, \dots, n$. A similar definition can be given for n-ary relations, $n>2$. The relation of inequality $\neq$ on the real line is an example of a concurrent relation. For every concurrent relation b in M, Robinson requires that there exists an entity y of type τ_2 in $^*M$ such that $^*b(x, y)$ for all $x\in$ dom $b\subseteq M$.

Given the higher order structure $M=\{B_\tau\}_{\tau\in T}$, we choose a language Λ and a naming map C such that the constants not in the domain of C form a set of cardinality greater than $\sum_{\tau\in T} |B_\tau|$, where $|B_\tau|$ is the cardinality of B_τ. Let K be the set of sentences in Λ which are true in M under C. By the cardinality assumption, a constant a_b not in the domain of C may be assigned to each concurrent relation b. Let K_0 denote the collection of all sentences $\Phi_\tau(\beta, g, a_b)$ where β is a name for b, b ranges over all concurrent relations in M, and g ranges over the preimage under C of the domain of b. Finally let $H=K \cup K_0$. Then it is not too hard to see, using the concurrentness of the relations, that M itself is a model of any finite subset of H. Hence by the compactness theorem H has a model M'. An individual $a' \in M'$ which is denoted by a name α already denoting an individual $a \in M$ can be replaced by a. By making corresponding changes in relations of higher type we obtain a new model ${}^*M=\{{}^*B_\tau\}$ of H in which B_0 is imbedded in ${}^*B_0$. The model ${}^*M$ is then called an *enlargement* of M.

The fact that ${}^*M$ is a model of H, and in particular of K, immediately ensures that (i) holds. More far reaching implications follow from the fact that ${}^*M$ is a model of K_0. For example, let b be a fixed concurrent relation of type (0, 0) whose domain is all of A. The fact that ${}^*M$ is a model of K_0 now ensures the existence of an individual a_b in ${}^*B_0$ such that ${}^*b(x, a_b)$ holds for *all* $x \in B_0$, where ${}^*b$ is the binary relation corresponding to β in ${}^*M$. Thus, for example, if B is infinite then the relation of "not-identical" of type (0, 0) is concurrent, and so in this case ${}^*B_0$ contains individuals which are different from all individuals in B_0, so that (ii) holds.

Any element of ${}^*B_\tau$ will be called an *internal* relation of type τ. In general, as remarked before, ${}^*M$ will not be a complete structure, and we will call elements of ${}^*A_\tau - {}^*B_\tau$ *external.* Each internal element R of ${}^*M$ has a name ρ in L. If ρ also denotes an element S of M then R is called *standard*, and will be denoted by ${}^*S$. A function of n variables in M can be represented by an $(n+1)$-ary relation R, and it is easy to see that ${}^*R$ defines a function ${}^*f$ in ${}^*M$ which will be called a *standard function.* Also ${}^*f$ is an extension of f in the obvious sense. Strictly speaking, all extensions of standard entities should be starred. However, when it is clear that we are in the enlargement, and no confusion will result, we will often omit the star (see §1.6).

1.3. *Enlargements of the reals.* We will use the symbol R to denote the standard real numbers, both as a set and as a mathematical structure, namely an archimedean totally-ordered field; the context will resolve any possible confusion. The relations of equality and inequality are binary relations of type (0, 0), and the operations of addition and multiplication can be defined by ternary relations of type (0, 0, 0).

Enlargements of the reals are discussed in [8, §3.2]. An enlargement ${}^*R$ of R is a totally-ordered field, but it is not archimedean, since it is a proper extension of R. The numbers in $R \subset {}^*R$ are called *standard.* The relation of inequality is concurrent, so there are numbers in ${}^*R$ which are larger in absolute value than any standard real number. Such numbers are called *infinite*, and the set of infinite numbers will be denoted by R_∞. The positive and negative infinite numbers will be denoted by R_∞^+ and R_∞^- respectively, or by R_∞^ω, where ω may be either $+$ or $-$. They are both external sets. A number $t \in {}^*R$ is *finite* if it is not infinite. The set of finite numbers will be denoted by M_0, and we have

$$M_0 = \{r \in {}^*R : |r| \leqq b \text{ for some } b \in R\}.$$

Similarly there are numbers in $^*R$ which are smaller in absolute value than any standard positive real; such numbers are called *infinitesimal,* and will be denoted collectively by M_1 so that

$$M_1 = \{r \in {}^*R : |r| \leqq \varepsilon \text{ for all } \varepsilon > 0 \text{ in } R\}.$$

M_0 is a ring, M_1 is a prime maximal ideal in M_0, and M_0/M_1 is isomorphic to R.

If two numbers r and s in $^*R$ are infinitesimally close, i.e., $r-s$ is infinitesimal, then we write $r \simeq s$. The relation $\simeq$ is an (external) equivalence relation on $^*R$. For every finite number $r \in {}^*R$ there is a unique standard number ^0r, called the *standard part of* r, such that $r \simeq {}^0r$. We will sometimes denote ^0r by st r.

Robinson has shown that topological notions in R can be defined in terms of monads in $^*R$. The monad $\zeta(r)$ of *any* $r \in {}^*R$ is defined by

$$\zeta(r) = \{t \in {}^*R : t \simeq r\}.$$

Monads will be fundamental to our discussion.

$^*R$ may be divided into galaxies. Two numbers r and s are in the same galaxy if $|r-s|$ is finite. This equivalence relation divides $^*R$ into disjoint subsets, each of which is called a *galaxy*. The finite points of $^*R$ constitute the *principal galaxy.*

If N denotes the higher order structure of the natural numbers as a substructure of R, and $^*N$ is the corresponding structure in $^*R$, then $^*N$ is an enlargement of N. There are individuals in $^*N$ which are greater than all individuals in N. The numbers in N are called standard, while the numbers in $^*N-N$ are called infinite. The sets N and $^*N-N$ are external in $^*N$.

1.4. *Enlargements of metric spaces.* Let X denote a metric space (as a point set) with metric ρ. The definition of a metric space involves the real numbers R as well as X, and the full structure based on $X \cup R$, in which the metric function has been singled out as of special importance, will be denoted by (X, R, ρ). The nonstandard theory of metric spaces is developed in an enlargement $^*(X, R, \rho)$ of (X, R, ρ) (see [8, §4.3]). Enlargements $^*X$ and $^*R$ of X and R respectively will be contained in $^*(X, R, \rho)$. The individuals in $X \subset {}^*X$ will be called *standard* points. The metric function in $^*(X, R, \rho)$ will again be denoted by ρ.

A point $y \in {}^*X$ is *finite* if there is a standard point x such that $\rho(x, y)$ is finite. $^*X$ is divided into galaxies, each *galaxy* consisting of points which are finitely far apart. The finite points constitute the principal galaxy.

We say that $x \simeq y$ if $\rho(x, y)$ is infinitesimal. The relation $\simeq$ is an equivalence relation. For any point $x \in {}^*X$, the *monad* of x is defined by

$$\mu(x) = \{y \in {}^*X : x \simeq y\}.$$

Note that the monads are defined for nonstandard as well as standard points. A point $x \in {}^*X$ is *near-standard* if there is a standard point y such that $x \simeq y$. The point y is unique and is called the *standard part* of x; we will denote it by ${}^0x$ or st (x). If E is any subset of ${}^*X$ (internal or not) then st (E), the standard part of E is the set $\{x : x \in X$ and there exists a $y \in E$ with $x \simeq y\}$. According to a result of Robinson [8, Corollary 4.1.15] a standard set S is compact if and only if every point $y \in {}^*S$ is near-standard to some point in S.

In §4 we will need two types of "monads" of sets of points $E \subset {}^*X$. The first is a direct generalization of the monads of points defined above. We put

$$\mu(E) = \{y \in {}^*X : y \simeq x \text{ for some } x \in E\}.$$

To introduce the second type of monad, which is new, we denote the open sphere of radius $\varepsilon > 0$ about an arbitrary set $E \subset {}^*X$ by

$$S(E, \varepsilon) = \{y \in {}^*X : \text{there is an } x \in E \text{ such that } \rho(x, y) < \varepsilon\}$$

and define

$$\nu(E) = \bigcap \{S(E, \varepsilon) : \varepsilon > 0 \text{ and standard}\}.$$

It might be supposed at first glance that the two types of monad are identical, but this is not the case. For example, if E is the set $\{1/n : n = 1, 2, \ldots\}$ on the real line then $\mu(E)$ does not contain 0 but $\nu(E)$ does. However, we do have

THEOREM 1.2. (a) *For any set* $E \subset {}^*X$, $\mu(E) \subseteq \nu(E)$.
(b) *If E is standard then* $\bar{E} = \text{st}(\mu({}^*E)) = \text{st}(\nu({}^*E))$.
(c) *If E is standard then* $\mu(\bar{E}) \subseteq \mu({}^*E)$.

Proof. (a) Suppose that $x \in \mu(E)$. Then there exists a $y \in E$ such that $x \simeq y$. In particular $\rho(x, y) < \varepsilon$ for all standard $\varepsilon > 0$ and so $x \in \nu(E)$.

(b) We have ${}^*E \subseteq \mu({}^*E) \subseteq \nu({}^*E)$ and so st $({}^*E) = \bar{E} \subseteq$ st $(\mu({}^*E)) \subseteq$ st $(\nu({}^*E))$, where the first equality results from [8, Theorem 4.3.4]. Suppose that y is a standard point which is not in $\bar{E}$. Then $\mu(y) \cap {}^*E = \varnothing$ by [8, Theorem 4.1.5], so that no point of $\mu({}^*E)$ is infinitesimally close to y, i.e. $y \notin$ st $(\mu({}^*E))$, proving the first equality in (b). To prove the second equality, we note that since $y \notin \bar{E}$, there is a standard $\varepsilon > 0$ so that $\rho(y, E) \geqq \varepsilon$ is standardly true. Thus in ${}^*X$ there is no point $z \in {}^*E$ such that $\rho(y, z) < \varepsilon$. On the other hand, if $y \in$ st $(\nu({}^*E))$ then there is a point z' in $\nu({}^*E)$ such that $y \simeq z'$, and $z' \in S({}^*E, \varepsilon)$ by definition of $\nu({}^*E)$, so that there is a $z \in {}^*E$ with $\rho(z, z') < \varepsilon$ and hence $\rho(y, z) < \varepsilon$ (contradiction).

(c) If $x \in \mu(\bar{E})$ then x is infinitesimally close to some point in $\bar{E}$, which itself is infinitesimally close to some point in ${}^*E$ by Theorem 4.1.6 in [8], showing that $x \in \mu({}^*E)$. ∎

Part (c) of Theorem 1.2 can be strengthened if E has a compact cover, i.e., if there is a compact set $K \supset E$.

THEOREM 1.3. *If E is a standard set with a compact cover, then* $\mu({}^*E) = \mu(\bar{E})$. *In particular, if E is compact then* $\mu({}^*E) = \mu(E)$.

Proof. By virtue of (c) of Theorem 1.2 we need only prove that $\mu(^*E)\subseteq\mu(\bar{E})$. If $x\in\mu(^*E)$ then x is infinitesimally close to a point in $^*E$, which, by virtue of the fact that E has a compact cover, is infinitesimally close to a standard point y. Now y must lie in $\bar{E}$ since st $(^*E)=\bar{E}$ by Theorem 4.3.4 in [8], completing the proof. ▌

1.5. *Enlargements of topological spaces.* Most of this paper will be concerned with dynamical systems in metric spaces, but we will in several places indicate how the results can be generalized to hold for systems in topological spaces. The generalizations will invariably involve the substitution of topological monads for metric monads.

Let (X, τ) be a topological space, where X is the basic point set, and τ is the collection of open subsets of X, which can be defined by a relation of type ((0)). We identify (X, τ) with the full structure which is based on X, and which we will also, for convenience, sometimes denote by X, again letting the content clear up any ambiguities.

Let $^*(X, \tau)$ be an enlargement of (X, τ) with $^*X$ being its set of individuals and $^*\tau$ being its open sets. Some sets in $^*\tau$ are standard, being the extensions $^*U$ of open sets $U\in\tau$, while others are internal.

If x is any *standard point* and Ω_x is the set of neighborhoods of x, i.e. the sets E which contain an open set $U\in\tau$ continuing x, then we define the monad

$$\mu_\tau(x) = \bigcap\,(^*E : E\in\Omega_x).$$

Robinson has shown how the standard topological properties of (X, τ) can be defined in terms of these monads of standard points. However, in working with dynamical systems we almost immediately encounter the need to define monads for nonstandard points. The appropriate definition for metric spaces is in terms of infinitesimals as in §1.4. Now Luxemburg [3, Chapter III, §1] has given a definition of monads for an arbitrary $x\in{}^*X$ as follows. If $x\in{}^*X$, let Ω_x denote the collection of all subsets E of X for which there is an open set U such that $x\in{}^*U\subset{}^*E$, and define $\mu_\tau(x)=\bigcap\,(^*E : E\in\Omega_x)$. A similar definition is given for the monad $\mu_\tau(S)$ of an arbitrary nonempty subset $S\subset{}^*X$.

We use such definitions in the nonstandard characterization of limit sets in §3, and remark how they may be used in extending our results on stability to the topological context in §4. However, the status of such possible generalizations is at present unclear, for the following reason. For standard points the two types of monads coincide, and hence for metric spaces they both coincide with the metric monads. However, for nonstandard points in a metric space, Luxemburg's definition of monads is in general different from the metric monads.

1.6. *Conventions.* In writing down statements about our structures we will use an informal language rather than the full formal language of §1.1; it will be clear that these statements could be written formally.

The star notation is used to refer to an entity or fact about the enlargement. Thus we shall use expressions like *open, *true, etc., to mean "open in the enlargement", "true in the enlargement", etc. Also, standard entities in the enlargement, that is, entities which are extensions to the enlargement of standard entities, are usually starred. Thus if E is a (standard) subset of B, then it becomes $^*E$ in the enlargement. However, we will not be entirely consistent in this notation. Thus, when it is clear from the context whether a given standard entity is being considered in the standard structure or in the enlargement, we will sometimes omit the star.

The ω notation which was briefly used in §1.3 will be used extensively throughout this paper. The symbol ω is to be interpreted as either $+$ or $-$, but the interpretation is supposed to be consistently the same in a given context (e.g., theorem or formula).

2. **Dynamical systems.** In this section we will present the basic definitions concerning dynamical systems, and show how they are nonstandardized. Excellent basic references for this section and the sequel are the books by Nemyckiĭ and Stepanov [6] and Bhatia and Szegö [1]. These books deal with dynamical systems in metric spaces, and most of our results will be established in that context. However, several of our more basic results on limit orbits will be proved in the more general topological context, and we will remark on possible generalizations to the topological case which the nonstandard methods allow.

Let X be a Hausdorff topological space with topology τ. A *dynamical system* $\mathscr{D}$ *on* X is a pair (X, π), where π is a mapping from $R\times X$ into X satisfying the following conditions:

(a) $\pi(0, x)=x$ for all $x \in X$;

(b) $\pi(t_2, \pi(t_1, x))=\pi(t_1+t_2, x)$ for all $x \in X$ and $t_1, t_2 \in R$ (this is called the *group property*);

(c) π is continuous.

For each $x \in X$ the mapping π induces a continuous mapping $\pi_x: R \to X$, defined by $\pi_x(t)=\pi(t, x)$, which is called the *motion through* x. For conciseness we will usually write $\pi_x(t)$ as xt. If π_x is restricted to the set $S\subset R$ we obtain the *motion over* S. In the special case that $S=R^\omega$ ($\omega=+$ or $-$) we obtain the *positive and negative motions through* x.

For any motion π_x we may consider the *orbit of the motion over a set* $S\subset R$, defined by

$$\gamma_x(S) = [xt : t \in S].$$

If S is either R, R^+, or R^- respectively we obtain the *orbit through* x, and the *positive and negative orbits through* x, which we denote by γ_x, γ_x^+, and γ_x^-, respectively.

For each $t \in R$ the mapping π induces a continuous map $\pi^t: X \to X$ defined by $\pi^t(x)=\pi(t, x)$ and called an *action*. It is easy to see that π^t is a one-to-one bicontinuous map from X onto X with inverse π^{-t}, and that the set of actions form a group which is called the *flow* defined by $\mathscr{D}$.

We now construct a nonstandard enlargement of the dynamical system $\mathscr{D}$. If M is the complete higher order structure based on the set of individuals $A_0=R \cup X$ then all of the notions in $\mathscr{D}$ can obviously be defined in M. The nonstandard theory of $\mathscr{D}$ can then be developed in any enlargement $^*M$ of M, and the "nonstandard dynamical system" so obtained will be denoted by $^*\mathscr{D}$. By Theorem 2.11.4 in [8], $^*M$ contains substructures which are enlargements $^*R$ and $^*X$ of the structures R and X respectively.

$^*\mathscr{D}$ consists of a mapping $^*\pi\colon {}^*R\times {}^*X\to {}^*X$ which agrees with π when restricted to $R\times X$ (i.e., $^*\pi$ is an extension of π) and which satisfies the obvious extensions of conditions (a), (b) and (c) above. We will usually omit the * and write $^*\pi$ as π, letting the context clear up any possible ambiguity.

In $^*\mathscr{D}$ there are motions through every $x\in {}^*X$, with the associated positive and negative motions, orbits, etc. If x is standard the motions, orbits, etc., in $^*\mathscr{D}$ will be termed *standard*, where it is to be remembered, of course, that a standard motion xt in $^*\mathscr{D}$ is defined for all $t\in {}^*R$, and not just for standard t. Similarly, if S is any set in $^*R$ (not necessarily internal) then $\gamma_x(S)$, the orbit over S, is well defined in $^*\mathscr{D}$. If S is the extension $^*A$ of a standard set $A\subset R$ and x is standard then $\gamma_x(S)$ is standard, and if S is internal then $\gamma_x(S)$ is internal for any $x\in {}^*X$.

The continuity of the mapping π translates to a simple nonstandard statement using Theorem 4.2.7 in [8]. Let σ denote the product topology on $R\times X$. Then we have

THEOREM 2.1. *If x and t are standard then $\pi(\mu_\sigma(t, x))\subset\mu_\tau(\pi(t, x))$.*

Noting that $\mu_\sigma(t, x)=\zeta(t)\times\mu_\tau(x)$ for standard x and t, we obtain the following obvious corollaries:

COROLLARY 2.1.1 (CONTINUITY IN TIME). *Let $x\in X$ and $t_0\in R$ be fixed. Then $xt\in\mu_\tau(xt_0)$ for all $t\in\zeta(t_0)$.*

COROLLARY 2.1.2 (CONTINUITY OF ACTIONS). *Let $x\in X$ be fixed. If $y\in\mu_\tau(x)$ then $yt\in\mu_\tau(xt)$ for all standard t.*

In metric spaces the second corollary may be strengthened. Using the compactness of finite time intervals, Corollary 2.1.2 can be shown to hold for all finite (not necessarily standard) t. Also if X is a metric space, we can prove, using Theorem 4.5.3 in [8], the following corollary.

COROLLARY 2.1.3. *If X is a metric space and B is a compact subset of X, then for all $x\in {}^*B$, if $y\in\mu(x)$ then $y\tau\in\mu(x\tau)$ for all finite τ.*

3. Limit motions and orbits.

3.1. *Introduction.* The notion of the limit set of a motion plays a central role in the standard theory of dynamical systems. The nonstandard model $^*\mathscr{D}$ provides a larger and more general setting in which to analyse the behavior of the motion as t approaches infinity.

DEFINITION 3.1. The positive (negative) limit motion (orbit) of $x\in {}^*X$ is the motion (orbit) of x over R_∞^+ (R_∞^-). The limit orbits will be denoted by Ω_x^ω ($\omega=+$ or $-$).

In the standard theory motions may not be followed for an infinite length of time. The structure of the limit sets is then used to convey information about the limiting behavior of the motion. However, the limit set is often not an orbit, even locally. Indeed, the limit set may coincide with X itself. On the other hand, in the nonstandard dynamical system the limit orbits and motions have the nonstandard equivalent of the structure of orbits and motions, and this fact can be used very effectively to study the limiting behavior of the motion. The notion of limit motion has no counterpart in the standard theory and will be used, in particular, to study standard motions in the limit set.

In general the limit orbits are external, but they are closely related to the standard notion of limit set.

DEFINITION 3.2. The limit sets (positive and negative respectively) of $x \in {}^*X$ are the standard sets $\Lambda_x^\omega = \mathrm{st}\,(\Omega_x^\omega)$ ($\omega = +$ or $-$ respectively).

THEOREM 3.1. *If x is standard then Λ_x^ω ($\omega = +$ or $-$ respectively) coincides with the standard limit set (positive and negative respectively).*

Proof. According to the standard definition (see [1, §2.2.15]), the limit sets are given by $\tilde{\Lambda}_x^\omega = \bigcap\, [\bar{\gamma}_{xt}^\omega : t \in R^\omega]$. Suppose that $y \in \mathrm{st}\,(\Omega_x^\omega)$; then $xt_0 \in \mu_\tau(y)$ for some $t_0 \in R_\infty^\omega$. If $y \notin \tilde{\Lambda}_x^\omega$ then $y \notin \bar{\gamma}_{xT}^\omega$ for some standard $T \in R^\omega$, and so there is a standard neighborhood U of y such that $U \cap \gamma_{xT}^\omega = \varnothing$. Thus the statement

$$(\forall t)[[t \in R \wedge \omega t \geqq \omega T] \rightarrow xt \notin U]$$

is true in $\mathscr{D}$ and hence, by transfer, in ${}^*\mathscr{D}$, where it reads

$$(\forall t)[[t \in {}^*R \wedge \omega t \geqq \omega T] \rightarrow xt \notin {}^*U].$$

But $t_0 > T$ and $xt_0 \in \mu_\tau(y) \subset {}^*U$ [8, Theorem 4.1.4], which contradicts the last sentence.

Conversely, suppose that $y \in \tilde{\Lambda}_x^\omega$. Then the statement

$$(\forall V)(\forall T)[[V \text{ an open set containing } y \wedge T \in R^\omega] \rightarrow (\exists t)[\omega t \geqq \omega T \wedge xt \in V]]$$

holds in $\mathscr{D}$. This transfers in ${}^*\mathscr{D}$ to the statement

$$(\forall V)(\forall T)[[V \text{ a }{}^*\text{open set containing } y \wedge t \in {}^*R^\omega] \rightarrow (\exists t)[\omega t \geqq \omega T \wedge xt \in V]].$$

Now by [8, Theorem 4.1.2], there is a *open set V containing y and contained in $\mu_\tau(y)$. Using this V and a $T \in R_\infty^\omega$ in the last statement, we see that $xt_0 \in \mu(y)$ for some $t_0 \in R_\infty^\omega$, i.e., $y \in \mathrm{st}\,(\Omega_x^\omega)$. ▌

3.2. *Applications.* One disadvantage of working with limit sets is that they may be empty. Such will be the case when, loosely speaking, the motion xt goes to infinity (i.e., leaves any preassigned compact set) as t approaches infinity. Thus we may expect to study such phenomena more effectively using the limit orbits rather than the limit sets, even though the former are not, in general, internal sets.

To indicate how limit orbits are used we will give nonstandard proofs of some familiar properties of limit sets. In order to do so we need to know the following fact about standard limit sets.

LEMMA 3.2. *If x is standard and $z \in {}^*\Lambda_x^\omega$ and V is any *open set in ${}^*X$ containing z, then V intersects Ω_x^ω.*

Proof. By looking at the standard definition of Λ_x^ω we see that the statement

$$(\forall z)(\forall V)(\forall T)[[z \in \Lambda_x^\omega \wedge V \text{ is an open set in } X \text{ containing } z \wedge \omega T > 0] \\ \rightarrow (\exists t)[\omega t \geqq \omega T \wedge xt \in V]]$$

is true in $\mathscr{D}$ and hence in ${}^*\mathscr{D}$. Letting T be an infinite number we obtain

$$(\forall z)(\forall V)[[z \in {}^*\Lambda_x^\omega \wedge V \text{ is a *open set in } {}^*X \text{ containing } z] \\ \rightarrow (\exists t)[t \in R_\infty^\omega \text{ and } xt \in V]]$$

as desired. ∎

Notice that the lemma is true, by definition, for all points $z \in \Lambda_x^\omega$. When X is a metric space the same method of proof yields the following corollary.

COROLLARY 3.2.1. *If X is a metric space and x is standard then every point in ${}^*\Lambda_x^\omega$ is infinitesimally close to some point in Ω_x^ω.*

Simple examples show that the converse of the corollary is not necessarily true, even if "infinitesimally" is replaced by "finitely". In this connection we have the following result.

THEOREM 3.3. *If X is a metric space, x is standard, and Λ_x^ω is not empty, then every point in Ω_x^ω is infinitesimally close to some point in ${}^*\Lambda_x^\omega$ if and only if $\lim_{t \to \omega\infty} \rho(xt, \Lambda_x^\omega) = 0$.*

Proof. Suppose that every point in Ω_x^ω is infinitesimally close to some point in ${}^*\Lambda_x^\omega$. Let $\varepsilon > 0$ be a fixed standard number. Then the statement

$$(\exists T)[\omega T > 0 \wedge (\forall t)[[\omega t \geqq \omega T] \rightarrow (\exists y)[y \in {}^*\Lambda_x^\omega \wedge \rho(xt, y) < \varepsilon]]]$$

is true in ${}^*\mathscr{D}$, as we see by taking T to be any number in R_∞^ω. Transferring this statement to $\mathscr{D}$ yields the desired standard statement.

Conversely suppose that $\lim_{t \to \omega\infty} \rho(xt, \Lambda_x^\omega) = 0$. Then the statement

$$(\forall \varepsilon)[\varepsilon > 0 \rightarrow (\exists T)[\omega T > 0 \wedge (\forall t)[\omega t > \omega T \rightarrow (\exists y)[y \in \Lambda_x^\omega \wedge \rho(xt, y) < \varepsilon]]]]$$

is true in $\mathscr{D}$. With ε and corresponding T fixed and standard we thus obtain

$$(\forall t)[t \in R_\infty^\omega \rightarrow (\exists y)[y \in {}^*\Lambda_x^\omega \wedge \rho(xt, y) < \varepsilon]]$$

in ${}^*\mathscr{D}$, i.e., any ball of standard radius ε about xt intersects ${}^*\Lambda_x^\omega$. Thus the internal set $\{n \in {}^*N : S(xt, 1/n) \cap {}^*\Lambda_x^\omega \neq \varnothing\}$ then contains all standard integers, and so must contain some infinite n, since otherwise the set of standard integers would be internal, contradicting [8, Theorem 3.1.7]. Thus every point in Ω_x^ω is infinitesimally close to some point in ${}^*\Lambda_x^\omega$ as desired. ∎

Nemyckiĭ and Stepanov [6, p. 401] say that in this case the ω-orbit through x approaches the limit set uniformly (not to be confused with the notion of uniform approximation of the limit set—[6, Definition 9.0.4]). This suggests the following definition.

DEFINITION 3.3. We say that the motion xt approaches its limit set Λ_x^ω uniformly if for all $z \in \Omega_x^\omega$, $\mu_\tau(z)$ intersects $^*\Lambda_x^\omega$.

This definition is applicable in the topological case.

As an illustration of nonstandard techniques, we now use our nonstandard results to prove the following standard and well-known result.

THEOREM 3.4. *If x is standard then Λ_x^ω is closed and invariant.*

Proof. To prove closure we need to show that if $y \in {}^*X$ is a standard point which is not in Λ_x^ω, then $\mu_\tau(y)$ does not intersect $^*\Lambda_x^\omega$ [8, Theorem 4.1.5]. If this is not true then there is a $z \in {}^*\Lambda_x^\omega$ such that $z \in \mu_\tau(y)$. Now by [3, Corollary 2.1.6], there is a *open set $V \subset \mu_\tau(y)$ containing z. By Lemma 3.2, V intersects Ω_x^ω. But then $y \in \operatorname{st}(\Omega_x^\omega) = \Lambda_x^\omega$ (contradiction).

To prove invariance suppose that $y \in \Lambda_x^\omega$. We want to show that $yt_0 \in \Lambda_x^\omega$ for any standard t_0. But $xt \in \mu_\tau(y)$ for some $t \in R_\infty^\omega$, and hence $x(t+t_0) \in \mu_\tau(yt_0)$ by Corollary 2.1.2 and the group property. Clearly $t+t_0 \in R_\infty^\omega$ and so $x(t+t_0) \in \Omega_x^\omega$ and hence $yt_0 \in \Lambda_x^\omega$. ▮

The rest of the results in this section will be devoted to dynamical systems on metric spaces. Some extensions to the topological case are possible but will not be pursued here.

Among the more tractable motions in the standard theory are the Lagrange stable ones, those whose orbits have compact closures [1, Definition 2.5.1]. These can be characterized nonstandardly as follows.

DEFINITION 3.4. The motion xt is ω-Lagrange stable if every point of Ω_x^ω is near-standard (to some point in Λ_x^ω).

THEOREM 3.5. *If x is standard then the motion xt is ω-Lagrange stable in the standard sense if and only if it is ω-Lagrange stable in the sense of Definition* 3.4.

Proof. Suppose that the motion xt is ω-Lagrange stable in the standard sense, i.e., $\bar{\gamma}_x^\omega$ is compact. Now the sentence $(\forall t)[t \in R^\omega \to xt \in \bar{\gamma}_x^\omega]$ is true, and hence by transfer the sentence $(\forall t)[t \in {}^*R^\omega \to xt \in {}^*\bar{\gamma}_x^\omega]$ is *true. In particular, $xt \in {}^*\bar{\gamma}_x^\omega$ for $t \in R_\infty^\omega$. But since $\bar{\gamma}_x^\omega$ is compact, every point in $^*\bar{\gamma}_x^\omega$ is near-standard [8, Corollary 4.1.15] so that every point in Ω_x^ω is near-standard.

Conversely suppose that every point in Ω_x^ω is near-standard. If $\bar{\gamma}_x^\omega$ is not compact then $^*\bar{\gamma}_x^\omega$ has a point y which is not in the monad of a point in $\bar{\gamma}_x^\omega$. Now $\bar{\gamma}_x^\omega = \gamma_x^\omega \cup \Lambda_x^\omega$, and so $^*\bar{\gamma}_x^\omega = {}^*\gamma_x^\omega \cup {}^*\Lambda_x^\omega$. Suppose that $y \in {}^*\Lambda_x^\omega$. Then y is infinitesimally close to some point in Ω_x^ω by Corollary 3.2.1. But every such point is near-standard to some point in Λ_x^ω and hence to some point in $\bar{\gamma}_x^\omega$ (contradiction). Finally, suppose that $y \in {}^*\gamma_x^\omega$, i.e., $y = xt$ for some $t \in {}^*R^\omega$. If t is finite then $y \simeq x\,{}^0t \in \gamma_x^\omega$ (contradiction), while if t is infinite then $y \in \Omega_x^\omega$, and we reach a contradiction as above. We conclude that $\bar{\gamma}_x^\omega$ is compact. ▮

COROLLARY 3.5.1. *If the motion through the standard point x is ω-Lagrange stable then* $\lim_{t\to\infty} \rho(xt, \Lambda_x^\omega)=0$.

Proof. Every point in Ω_x^ω is near-standard by Definition 3.4, hence infinitesimally close to a point in Λ_x^ω (Definition 3.2). Apply Theorem 3.3. ▮

If X is locally compact we obtain a stronger result.

THEOREM 3.6. *If X is locally compact and Λ_x^ω is compact for a standard x, then every point of Ω_x^ω is near-standard.*

Proof. Since X is locally compact and Λ_x^ω is compact, there is a neighborhood of Λ_x^ω whose closure K is compact. Now for some T we must have $\gamma_{xT}^\omega \subset K$; otherwise, for each t, $\bar{\gamma}_{xt}^\omega \cap \partial K \neq 0$ and so $\bigcap \bar{\gamma}_{xt}^\omega \cap \partial K \neq \varnothing$ since ∂K is compact. Then in $^*\mathcal{D}$ we have $^*\gamma_{xT}^\omega \subset {}^*K$ and in particular $\Omega_x^\omega \subset {}^*K$ and every point of $^*K$ is near-standard since K is compact [8, Corollary 4.1.15]. ▮

COROLLARY 3.6.1. *If X is locally compact then Λ_x^ω is compact iff the motion through x is ω-Lagrange stable.*

3.3. *Finite motions in the limit motion.* Given a motion xt ($t \in {}^*R$) through a point $x \in {}^*X$, we may define a collection of what might be called finite motions associated with it. Recall that if $t_0 \in {}^*R$ then the galaxy containing t_0 is the subset of $^*R$ consisting of those t such that $|t-t_0|$ is finite. The principal galaxy is defined by a finite t_0. If S is any galaxy in $^*R$ we call the motion (orbit) over S a *finite motion* (*orbit*). In particular, if S is the principal galaxy in $^*R$ we call the motion (orbit) over S the *principal motion* (*orbit*).

If some point $xt_0 \in {}^*\gamma_x$ is near-standard to a point y in X, then it follows from Corollary 2.1.2 that each point in the finite orbit over the galaxy defined by t_0 is also near-standard, and in fact $x(t+t_0) \in \mu(yt)$ for all t such that $|t-t_0|$ is finite. Such finite motions (orbits) will be called *near-standard finite motions* (*orbits*). Each near-standard finite motion (orbit) in the motion xt is clearly associated with a standard motion (orbit), i.e., the motion through the standard point y defined for all standard t, but a given standard motion may be associated with many near-standard finite motions in the motion xt. All standard motions in the limit set Λ_x^ω are associated in this way with (perhaps many) near-standard finite motions in Ω_x^ω. The near-standard finite motions are obviously ordered in time, giving us another tool for the analysis of the motions in the limit sets.

4. Stability.

4.1. *Introduction.* The classical work on stability initiated by Liapunov has recently been extended to a stability theory for arbitrary closed sets in X (see [1] for references). However, the development for noncompact sets is more difficult, and still somewhat incomplete, due mainly to the standard difficulty with limit sets, namely that they may be empty. In this section we present a unified treatment of the theory using nonstandard analysis. The definitions are the same for noncompact (and also, incidentally, for not necessarily closed sets) as for compact sets. The insight which is provided by the method allows us to generalize many of the results of the standard theory, with proofs which are much more geometrical and intuitively clear.

4.2. *Stability of sets.* We begin with the nonstandard definitions of stability of arbitrary sets $M \subset X$ (see [1] for the standard definitions).

DEFINITION 4.1. Let M be a standard set in the metric space X. M is said to be (ω-)

(a) *stable* if $\gamma_y^\omega \subset \nu(^*M)$ for all $y \in \mu(M)$,

(b) *uniformly stable* if $\gamma_y^\omega \subset \nu(^*M)$ for all $y \in \mu(^*M)$.

THEOREM 4.1. *The nonstandard Definitions* 4.1 *are equivalent to the standard definitions* [1, *Definition* 2.12.1].

Proof. According to the definitions in [1], M is standardly called

(1) stable if

$$(\forall \varepsilon)(\forall x)[[\varepsilon > 0 \wedge x \in M] \rightarrow (\exists \delta)[\delta > 0 \wedge (\forall y)[\rho(x, y) < \delta \rightarrow \gamma_y^\omega \subset S(M, \varepsilon)]]]$$

(2) uniformly stable if

$$(\forall \varepsilon)[\varepsilon > 0 \rightarrow (\exists \delta)[\delta > 0 \wedge (\forall x)(\forall y)[[x \in M \wedge \rho(x, y) < \delta] \rightarrow \gamma_y^\omega \subset S(M, \varepsilon)]]].$$

We will only prove equivalence in case (2) since the other case is similar. Suppose that M is standardly uniformly stable. Let $\varepsilon > 0$ and the corresponding $\delta > 0$ be fixed standard numbers. Then in $^*\mathscr{D}$ we obtain by transfer the statement

$$(\forall x)(\forall y)[[x \in {}^*M \wedge \rho(x, y) < \delta] \rightarrow \gamma_y^\omega \subset S(^*M, \varepsilon)].$$

In particular,

$$(\forall x)(\forall y)[[x \in {}^*M \wedge x \simeq y] \rightarrow \gamma_y^\omega \subset S(^*M, \varepsilon)].$$

Thus

$$(\forall y)[y \in \mu(^*M) \rightarrow \gamma_y^\omega \subset S(^*M, \varepsilon)].$$

This is true for all standard $\varepsilon > 0$, and hence for all $y \in \mu(^*M)$ we have $\gamma_y^\omega \subset \nu(^*M)$, i.e., M is uniformly stable in our sense.

Conversely, suppose that M is uniformly stable in the sense of Definition 4.1. Let $\varepsilon > 0$ be fixed and standard. Then the sentence

$$(\exists \delta)[\delta > 0 \wedge (\forall x)(\forall y)[x \in {}^*M \wedge \rho(x, y) < \delta \rightarrow \gamma_y^\omega \subset S(^*M, \varepsilon)]]$$

is *true, as we see by taking $\delta \simeq 0$. Transferring back to $\mathscr{D}$ yields the desired statement showing that M is standardly uniformly stable. ∎

An important simplification in the definition of stability results from the assumption that M is (standardly) (ω-) *invariant*, i.e., if $x \in M$ then for all standard $t \in R^\omega$ we have $xt \in M$. In that case we can prove the following theorem.

THEOREM 4.2. *If M is (ω-) invariant then it is (ω-) stable if and only if $\Omega_y^\omega \subset \nu(^*M)$ for all $y \in \mu(M)$.*

Proof. It is clear from Definition 4.1 that stability implies the given condition. Conversely, let $y \in \mu(M)$; then $\Omega_y^\omega \subset \nu(^*M)$ and to show stability we need only show that $yt \in \nu(^*M)$ for all finite $t \in {}^*R^\omega$. Since $y \in \mu(M)$ there is an $x \in M$ such that $y \simeq x$. By invariance $xt \in M$ for all finite t and $yt \simeq xt$ by Corollaries 2.1.1 and 2.1.2, so that $yt \in \mu(M) \subseteq \nu(^*M)$ for all finite t. ▮

4.3. *Attractors.* We next present the nonstandard definitions of the various types of attractors.

DEFINITION 4.2. Let M be a standard set in the metric space X. Then

(a) M is an (ω-) *semi-attractor* if there is a standard open set $V \supset M$ such that $\Omega_y^\omega \subset \nu(^*M)$ for all $y \in V$,

(b) if there is a standard $\delta > 0$ such that we may take $V = S(M, \delta)$ (in the standard sense) in (a), then we say that M is an (ω-) *attractor*,

(c) if the conclusions of (a) and (b) are true for all $y \in {}^*V$, then we say that M is a (ω-) *uniform attractor* of the corresponding type (e.g., uniform semi-attractor).

(d) if the conclusion in the above definitions is weakened to the condition that $\Omega_y^\omega \cap \nu(^*M) \neq \varnothing$, then we say that M is a *weak attractor* of the corresponding type (e.g., a weak uniform semi-attractor is one for which $\Omega_y^\omega \cap \nu(^*M) \neq \varnothing$ for all $y \in {}^*V$).

THEOREM 4.3. *The nonstandard Definitions* 4.2 *are equivalent to the standard definitions* [1, *Definition* 2.12.12].

Proof. M is standardly called an (ω-)

(a) semi-attractor if

$$(\exists V)[V \text{ open} \wedge V \supset M$$
$$\wedge\ (\forall y)[y \in V \to (\forall \varepsilon)(\exists T)(\forall t)[[\omega T > 0 \wedge \omega t \geqq \omega T] \to \rho(yt, M) < \varepsilon]]],$$

(b) attractor if $V = S(M, \delta)$ for some $\delta > 0$,

(c) uniform attractor if

$$(\exists \delta)(\forall \varepsilon)(\exists T)[\delta > 0 \wedge \varepsilon > 0 \wedge \omega T > 0$$
$$\wedge\ (\forall y)[y \in S(M, \delta) \to (\forall \varepsilon)[\omega t \geqq \omega T \to \rho(yt, M) < \varepsilon]]].$$

The standard definitions for the corresponding weak notions of attraction are somewhat more complicated in that they involve sequences $t_n \to \infty$, and so we will not explicitly present them here. We will establish equivalence in case (c); the proof in the other cases is similar. Incidentally, the standard notions as presented in [1] include no definition of uniform semi-attractor, although the standard definition corresponding to the nonstandard one given above would be easy to formulate.

Suppose that M is a standard (ω-) uniform attractor. Let $\delta>0$, $\varepsilon>0$, and the corresponding T in sentence (c) be standard and fixed. Then by transfer the sentence

$$(\forall y)[y \in {}^*S(M, \delta) \to (\forall t)[t \in R_\infty^\omega \to \rho(yt, {}^*M) < \varepsilon]]$$

holds in ${}^*\mathscr{D}$. For any $t \in R_\infty^\omega$ the last sentence is *true for all standard $\varepsilon>0$, and hence $\Omega_y^+ \subset \nu({}^*M)$ for all $y \in {}^*S(M, \delta)$ as desired.

Conversely, suppose that M is a uniform attractor in the sense of Definition 4.2. Then with $V=S(M, \delta)$ for the appropriate standard $\delta>0$, and with $\varepsilon>0$ any fixed standard number, the statement

$$(\exists T)(\forall y)[y \in {}^*V \to (\forall t)[\omega t \geqq \omega T \to \rho(yt, {}^*M) < \varepsilon]]$$

is true in ${}^*\mathscr{D}$, as we see by taking $T \in R_\infty^\omega$. By transfer back to $\mathscr{D}$ we obtain the desired result. ▮

Associated with attractors and weak attractors of the various types are regions of attraction defined as follows.

Definition 4.3. Let M be a standard set. The subsets of ${}^*X$ defined by

$$A^\omega(M) = \{y \in {}^*X : \Omega_y^\omega \subset \nu({}^*M)\}$$

and

$$A_w^\omega(M) = \{y \in {}^*X : \Omega_y^\omega \cap \nu({}^*M) \neq \varnothing\}$$

are called the (ω-) *region of attraction* and the (ω-) *region of weak attraction* of M, respectively.

Theorem 4.4. *The sets of standard points in $A^\omega(M)$ and $A_w^\omega(M)$, coincide with the standard regions of attraction and weak attraction, respectively* [1, *Definition* 1.6.12].

Proof. Suppose that $y \in A^\omega(M)$ is standard. Then, with $\varepsilon>0$ a fixed standard number, the sentence $(\exists T)(\forall t)[\omega t \geqq \omega T \to yt \in S({}^*M, \varepsilon)]$ is *true, as we see by taking $T \in R_\infty^\omega$. Transferring back to $\mathscr{D}$ yields the desired characterization of a point in the standard region of positive attraction.

Conversely, suppose that y is in the standard region of ω-attraction. Then the sentence $(\forall \varepsilon)(\exists T)(\forall t)[\omega t \geqq \omega T \to yt \in S(M, \varepsilon)]$ is true and so in ${}^*\mathscr{D}$ we have, in particular, that $[t \in R_\infty^+ \to yt \in S({}^*M, \varepsilon)]$. This is true for all standard $\varepsilon>0$, and hence $y \in A^\omega(M)$.

The equivalence for regions of weak attraction is similarly established. ▮

Theorem 4.5. (i) *$A^\omega(M)$ is finitely invariant, i.e., if $y \in A^\omega(M)$ and $\tau \in {}^*R$ is finite, then $y\tau \in A^\omega(M)$.*

(ii) *$A^\omega(M)$ is ω-invariant, i.e., if $y \in A^\omega(M)$ and $\tau \in {}^*R^\omega$, then $y\tau \in A^\omega(M)$.*

Proof. (i) Let $y \in A^\omega(M)$ and let τ be finite. Then for any $t \in R_\infty^\omega$ we see that $t+\tau \in R_\infty^\omega$ so that $y(t+\tau) \in \Omega_y^\omega \subset \nu({}^*M)$. But any point in $\Omega_{y\tau}^\omega$ is of the form $y(t+\tau)$ for some $t \in R_\infty^\omega$, and we are through. (ii) is similarly proved. ▮

COROLLARY 4.5.1. *The standard regions of attraction are finitely invariant and positively invariant.*

In [1] Bhatia and Szegö present two different definitions of uniformity of attractors, one for compact sets in Definitions 1.5.6 and 2.6.1, and one for general closed sets in Definition 2.12.12. It is the second version that we have used in our nonstandard characterization. A nonstandard characterization of the first type of uniformity would be given by

DEFINITION 4.4. The standard set M is a (ω-) uniform attractor if for all compact sets K in the standard (ω-) region of attraction we have $\Omega_y^{\omega} \subset \nu(^*M)$ for all $y \in {}^*K$.

This definition is in some senses weaker and in other senses stronger than the previous definition.

More interesting results are obtained when various combinations of stability and attraction are assumed to hold jointly. Such combinations lead to various notions of *asymptotic stability*. For example, a standard set M is said to be uniformly asymptotically stable if it is both uniformly stable and a uniform attractor. Combinations of the two kinds of stability and eight kinds of attraction lead to sixteen possible types of asymptotic stability (they are not necessarily all different) which will not be separately studied here.

4.4. *Comparison with the standard definitions.* Some comments are now in order concerning the relation between the above definitions and the standard ones. The first significant difference is that the standard definitions are formally very different when M is compact and when it is not compact. This is most clearly evident for the definitions of attractors. When M is compact the standard definitions of attractors are suggestively similar to ours. For example, M is a positive attractor if there is a $\delta > 0$ such that for $y \in S(M, \delta)$, $\Lambda_y^+ \neq \varnothing$ and $\Lambda_x^+ \subset M$. Thus we see Λ_x^+ playing somewhat the same role as Ω_y^+ in our definition (but even here notice the extra assumption that $\Lambda_y^+ \neq \varnothing$). For noncompact M the possibility that the limit sets may be empty is crucial, and so attraction is characterized, not in terms of these, but by certain limiting conditions, which are more difficult to deal with. For example, the closed set M is a positive attractor if there is a $\delta > 0$ such that $\rho(yt, M) \to 0$ as $t \to +\infty$ for each $y \in S(M, \delta)$. One then has to show that the two definitions coincide when M is compact or has a compact cover. Our unification of the definitions employs limit orbits in place of limit sets; the geometric concepts and intuition associated with the notion of limit orbit can then be used effectively in proving theorems.

Secondly we should note that the above definitions lead to other natural definitions of stability and attraction which seem to have no counterpart in the present standard theory.

(i) In the conclusion of the definitions of stability and attraction we could replace $\nu(^*M)$ by $\mu(^*M)$. This would lead to a stronger (i.e., more restrictive) form of stability and attraction.

(ii) An even more restrictive definition would be obtained by replacing $\nu({}^*M)$ by $\nu(M)$ or $\mu(M)$ (i.e., we could remove the stars in the definitions).

This in fact leads to natural definitions of stability and attraction for *arbitrary* sets $M \subset {}^*X$. Thus we could say that $M \subset {}^*X$ is stable if $\gamma_y^\omega \subset \nu(M)$ for all $y \in \mu(M)$ (or $y \in \nu(M)$). This possibility is interesting because it would allow us to discuss, for example, the stability and attracting properties of limit orbits.

(iii) We could replace the two types of metric monads, μ and ν, by topological monads μ_τ.

The last alteration yields definitions which would be significant for dynamical systems on topological spaces. The alternative definitions suggested above will, however, not be studied in this paper, since our main concern is to provide connections with the standard theory. We hope to devote a later paper to topological dynamics.

4.5. *Simplifications in special cases.* Before proceeding further we will note the simplifications in the definitions and corresponding theorems which result when M is subject to certain commonly occurring assumptions.

(i) *M is invariant.* Recall that the standard set M is ω-invariant if for any $x \in M$ and any $t \in R^\omega$ we have $xt \in M$. In this case we see that for any $x \in {}^*M$ and any $t \in {}^*R^\omega$ we have $xt \in {}^*M$. Thus to check stability it is only necessary to check that $\Omega_y^\omega \subset \nu({}^*M)$ for all $y \in \mu(M) - M$. For uniform stability we require that $\gamma_y^\omega \subset \nu({}^*M)$ for all $y \in \mu({}^*M) - {}^*M$. Similarly, to establish that M is an attractor (of the various types) it is only necessary to check that $\Omega_y^\omega \subset \nu({}^*M)$ for all $y \in V - M$ (or $y \in {}^*V - {}^*M$ for uniform attractors).

(ii) *M is compact.* In this case, as noted in Theorem 1.3, we have $\mu({}^*M) = \mu(M)$. Thus there is no distinction between stability and uniform stability under the assumption of compactness.

(iii) *M has a compact cover.* This means that there is a compact set $K \supset M$. We see easily from standard compactness arguments that the distinction between semi-attractors and attractors disappears in this case. A second consequence, if X is locally compact, is that the assumption $\Omega_y^\omega \subset \nu({}^*M)$ for $y \in V$ can be replaced by the standard condition $\Lambda_y^\omega \subset \overline{M}$, showing that our definitions are consistent with Definitions 2.6.1 in [1] (in fact are slightly more general since they apply to sets which are not necessarily closed). For if $\Lambda_y^\omega \subset \overline{M} \subset \mu(\overline{M}) = \mu({}^*M)$ then Λ_y^ω is compact. Also, by Theorem 3.6, every point of Ω_y^ω is near-standard to a point in Λ_y^ω which, by the above inclusions, is near-standard to a point in $\mu({}^*M) \subset \nu({}^*M)$, i.e., $\Omega_y^\omega \subset \nu({}^*M)$.

(iv) *Motions approaching the limit set uniformly.* There is a second case, not involving compactness, in which the condition $\Omega_y^\omega \subset \nu({}^*M)$ can be replaced by $\Lambda_y^\omega \subset M$ for $y \in V$. Assume that for each (standard) $y \in V$ the limit set Λ_y^ω is nonempty, and suppose that the motion yt approaches Λ_y^ω uniformly for each $y \in V$ (Definition 3.3). If $\Lambda_y^\omega \subset M$ then ${}^*\Lambda_y^\omega \subset {}^*M$. But for each $z \in \Omega_y^\omega$ there is a point $x \in {}^*\Lambda_y^\omega$ such that $z \simeq x$, and hence $z \in \mu({}^*M) \subset \nu({}^*M)$, i.e., $\Omega_y^\omega \subset \nu({}^*M)$.

4.6. *Applications.* It is not our intention to reprove all known standard results in dynamical systems using nonstandard analysis, although this could be achieved, usually with considerable economy as compared with the standard methods of proof. We shall, however, give here the nonstandard proofs of several standard results for the sake of illustration.

THEOREM 4.6. *If M is closed and ω-stable then M is ω-invariant.*

Proof. Let $y \in M \subset \mu(M)$. Then for all $t \in {}^*R^{\omega}$ we have $yt \in \nu({}^*M)$. But if $t \in R^+$ is standard then yt is standard, so that, by Theorem 1.2, $yt \in \text{st}(\nu({}^*M)) = \overline{M} = M$. ▮

From the definitions we obtain the fact that if M is uniformly ω-stable and $y \in {}^*M$ then $yt \in \nu({}^*M)$ for all $t \in {}^*R^+$, which shows that ${}^*M$ is "almost invariant" in an obvious sense.

THEOREM 4.7. *If M is positively invariant and uniformly semi-attracting, then M is stable.*

Proof. We must show, using Theorem 4.2, that $\Omega_y^+ \subset \nu({}^*M)$ for all $y \in \mu(M)$. Now if V is the open set in Definition 4.2, then $y \in {}^*V$ and the result is immediate from Definition 4.2. ▮

4.7. *Characterizations in terms of negative orbits.* It is possible to characterize positive stability and attraction in terms of the negative orbits. In this section we will distinguish standard and nonstandard orbits by stars.

THEOREM 4.8. *The standard set M is positively stable if and only if ${}^*\gamma_x^- \cap \mu(M) = \varnothing$ for all $x \notin \nu({}^*M)$, and positively uniformly stable if and only if ${}^*\gamma_x^- \cap \mu({}^*M) = \varnothing$ for all $x \notin \nu({}^*M)$.*

Proof. Suppose that M is positively stable, and let $x \notin \nu({}^*M)$. If there is a $y = xt$ ($t \in {}^*R^-$) such that $y \in {}^*\gamma_x^- \cap \mu(M)$, then $y(-t) = x$ and so ${}^*\gamma_x^+$ is not contained in $\nu({}^*M)$, contradicting stability. The converse and the uniform case are proved similarly. ▮

THEOREM 4.9. *If the standard set M is invariant, then it is positively stable if and only if $\Omega_x^- \cap \mu(M) = \varnothing$ for all $x \notin \nu({}^*M)$.*

Proof. Similar to the proof of the previous theorem. ▮

As an easy corollary of Theorem 4.8 we obtain the following result.

THEOREM 4.10. *If the standard closed set M is positively uniformly stable then $\overline{\gamma_x^-} \cap M = \varnothing$ for all $x \notin M$.*

Proof. Since M is closed there is a standard $\eta > 0$ such that $x \notin S(M, \eta)$.Then the statement $(\forall y)[y \in M \to \rho(x, y) \geqq \eta]$ is true, and hence in ${}^*\mathscr{D}$, $(\forall y)[y \in {}^*M \to \rho(x, y) \geqq \eta]$; in particular $x \notin \nu({}^*M)$. If $\overline{\gamma_x^-} \cap M \neq \varnothing$ then either $\gamma_x^- \cap M \neq \varnothing$ or $\Lambda_x^- \cap M \neq \varnothing$. In the first case ${}^*\gamma_x^- \cap {}^*M \neq \varnothing$ so that ${}^*\gamma_x^- \cap \mu({}^*M) \neq \varnothing$, contradicting Theorem 4.8. In the second case ${}^*\Lambda_x^- \cap {}^*M \neq \varnothing$, and so $\Omega_x^- \cap \mu({}^*M) \neq \varnothing$ by Corollary 3.2.1, i.e., ${}^*\gamma_x^- \cap \mu({}^*M) \neq \varnothing$, again contradicting Theorem 4.8. ▮

From Theorem 4.8 we also obtain the following standard characterizations of stability.

THEOREM 4.11. (i) *The set M is positively stable if and only if, given $\eta>0$ and $y \in M$, there is a corresponding $\delta>0$ such that $\rho(\gamma_x^-, y) \geqq \delta$ for all $x \notin S(M, \eta)$.*

(ii) *The set M is uniformly positively stable if and only if, given $\eta>0$, there exists a $\delta>0$ such that $\rho(\gamma_x^-, M) \geqq \delta$ for all $x \notin S(M, \eta)$.*

Proof. (i) Suppose that M is positively stable, and let the standard $x \notin S(M, \eta)$ for some standard $\eta>0$. Then as in Theorem 4.10 we see that $x \notin \nu(^*M)$. By Theorem 4.8 $^*\gamma_x^- \cap \mu(M)=\varnothing$. Let y be a fixed point in M. Then the statement $(\exists\delta)[^*\gamma_x^- \cap S(y, \delta)=\varnothing]$ is *true, as we see by taking $\delta>0$ infinitesimal. Transferring this sentence to $\mathscr{D}$ yields the desired condition.

Conversely, let $\eta>0$ and $y \in M$ and the corresponding $\delta>0$ be given standard entities. Then by transfer to $^*\mathscr{D}$ the statement $(\forall x)[x \notin S(^*M, \eta) \rightarrow {}^*\gamma_x^- \cap S(y, \delta) = \varnothing]$ is *true, and in particular, $^*\gamma_x^- \cap \mu(y)=\varnothing$. The statement is true for each $y \in M$ so that $(\forall x)[x \notin S(^*M, \eta) \rightarrow {}^*\gamma_x^- \cap \mu(M)=\varnothing]$ is also *true. Since the last statement is true for all $\eta>0$ we see that $(\forall x)[x \notin \nu(^*M) \rightarrow {}^*\gamma_x^- \cap \mu(M)=\varnothing]$ is also *true, and we conclude by Theorem 4.8 that M is positively stable.

(ii) Suppose that M is positively uniformly stable and let the standard $x \notin S(M, \eta)$ for some standard $\eta>0$. Then as in (i) we see that $x \notin \nu(^*M)$, and so by Theorem 4.8 we have $^*\gamma_x^- \cap \mu(^*M)=\varnothing$. Thus the statement $(\exists\delta)[^*\gamma_x^- \cap S(^*M, \delta)=\varnothing]$ is *true, as we see by taking δ infinitesimal. For if $\delta>0$ is infinitesimal and $y \in {}^*\gamma_x^- \cap S(^*M, \delta)$, then there is a $z \in {}^*M$ such that $\rho(y, z)<\delta$ and hence $y \in \mu(^*M)$. Transferring the last statement to $\mathscr{D}$ yields the desired conclusion.

The proof of the converse is similar to the proof of the converse of (i). ∎

It is clear that in the converse directions the assumption that $x \notin S(M, \eta)$ can be replaced by the weaker assumption that $x \in \partial S(M, \eta)$, the boundary of $S(M, \eta)$. Similar remarks apply in later results.

Part (ii) of Theorem 4.11 is Theorem 7 in Zubov [11], with whom the idea of characterizing positive stability by negative orbits seems to have originated. In the Russian (1957) edition of his book the theorem was stated as "a necessary and sufficient condition for uniform positive stability of a closed invariant set M is that, for all $x \notin M$, $\Lambda_x^- \cap M=\varnothing$." This condition is necessary but not sufficient, as was pointed out by Lefschetz. The necessity follows trivially from our results, and the insufficiency is seen from examples in Ura [10]. However, our original nonstandard Theorem 4.8, in its form, closely resembles Zubov's original (but false) theorem, which in its simplicity was very appealing.

The characterization of positive stability in terms of negative orbits given by Theorem 4.8 suggests a similar characterization of attractors.

THEOREM 4.12. *The standard set M is a positive semi-attractor if and only if there exists a standard open set $V \supset M$ such that $\Omega_x^- \cap V=\varnothing$ for all $x \notin \nu(^*M)$. With*

$V=S(M, \delta)$ for some standard $\delta>0$ we obtain a characterization of attractor, and if $\Omega_x^- \cap V=\varnothing$ is replaced by $\Omega_x^- \cap {}^*V=\varnothing$ we obtain a characterization of uniform attractor of the corresponding type.

Proof. Suppose M is a positive semi-attractor. If for all standard $V\supset M$ there is an $x \notin \nu({}^*M)$ such that $\Omega_x^- \cap V\neq \varnothing$, i.e., there is a $y \in \Omega_x^- \cap V$, then Ω_y^+ contains x contradicting the fact that M is a semi-attractor. The converse and the other cases are similarly proved. ∎

As corollaries we obtain the following standard results.

THEOREM 4.13. *If the standard closed set M is a uniform positive semi-attractor then there exists a standard open set $V\supset M$ such that $\Lambda_x^- \cap V=\varnothing$ for all standard $x\notin M$.*

Proof. Let V be the open set given by Theorem 4.12. If $x\notin M$ is standard, we see as in the proof of Theorem 4.10 that $x\notin \nu({}^*M)$, and so $\Omega_x^- \cap {}^*V=\varnothing$. We now claim that $\Lambda_x^- \cap V=\varnothing$. For if $y\in \Lambda_x^- \cap V$ then since $y \in \Lambda_x^-$ there is a $y' \in \Omega_x^-$ with $y\simeq y'$, and since V is open and $y\in V$, we see that $y'\in {}^*V$ [8, Theorem 4.1.4], i.e., $y'\in \Omega_x^- \cap {}^*V$ (contradiction). ∎

COROLLARY 4.13.1. *If M is a closed uniform positive semi-attractor, then $\Lambda_x^- \cap M=\varnothing$ for all $x\notin M$.*

THEOREM 4.14. *If there is a standard open set $V\supset M$ such that $\Lambda_x^- \cap V=\varnothing$ for all $x\notin M$ then M is a positive semi-attractor.*

Proof. By transferring the given standard condition to ${}^*\mathscr{D}$ we see that ${}^*\Lambda_x^- \cap {}^*V =\varnothing$ for all $x\notin {}^*M$, and in particular for all $x\in \nu({}^*M)$. We now claim that $\Omega_x^- \cap V=\varnothing$ for such x. For suppose that $y \in \Omega_x^- \cap V$. Then y is standard and hence in $\Lambda_x^- \cap V\subset {}^*\Lambda_x^- \cap {}^*V$ (contradiction). ∎

It should be noted that the conclusion of Theorem 4.14 is not valid if we assume the weaker condition that $\Lambda_x^- \cap M=\varnothing$ for all $x\notin M$, even if M is compact. This is shown by Bhatia and Szegö's example [1, 1.5.22].

COROLLARY 4.14.1. *The compact set M is a positive attractor if and only if there exists an open set $V\supset M$ such that $\Lambda_x^- \cap V=\varnothing$ for all $x\notin M$.*

COROLLARY 4.14.2. *If M is compact and positively invariant and there exists an open set $V\supset M$ such that $\Lambda_x^- \cap V=\varnothing$ for all $x\notin M$, then M is positively stable and a positive attractor.*

Proof. Use the last corollary and Theorem 4.7.

4.8. *Prolongations.* The notion of prolongation was first introduced by T. Ura [9], and has since been considerably developed by Ura, J. Auslander, P. Seibert, N. P. Bhatia and others (again [1] is an excellent reference for these developments). We will show how these ideas can be generalized using nonstandard analysis, and how they fit naturally into the analysis developed thus far. It is not our intention to push the analysis as far as it will go; in particular we will not study higher order prolongations. A later paper will be devoted to a more complete exposition.

DEFINITION 4.5. For any $x \in {}^*X$ we define the *first* (ω-) *prolongation* $D^\omega(x)$ by

$$D^\omega(x) = \bigcup \{\gamma_y^\omega : y \simeq x\}.$$

If M is any subset of ${}^*X$, then we define

$$D^\omega(M) = \bigcup \{D^\omega(x) : x \in M\}.$$

THEOREM 4.15. *If x is standard then the standard first ω-prolongation* (*Definition* 2.3.1 *in* [1]) *is given by* $\operatorname{st}(D^\omega(x))$.

Proof. We will only prove the result for the positive prolongation. Suppose that the standard point y is in the standard first prolongation. Then there exists a sequence $\{x_n\} \subset X$ and $\{t_n\} \subset R$ such that $x_n \to x$ and $x_n t_n \to y$. Thus given any standard $\varepsilon > 0$ there is a standard N such that $\rho(x, x_n) < \varepsilon$ and $\rho(y, x_n t_n) < \varepsilon$ for $n \geqq N$. In particular, we see that for n infinite, $\rho(x, x_n) < \varepsilon$ and $\rho(y, x_n t_n) < \varepsilon$ for all standard $\varepsilon > 0$, i.e., $x_n \simeq x$ and $x_n t_n \simeq y$. Thus $x_n t_n \in \gamma_{x_n}^+$ and $y \in \operatorname{st}(\gamma_{x_n}^+)$.

Conversely if $y \in \operatorname{st}(D^+(x))$, then for some $z \simeq x$, and some $t \in {}^*R^+$, we have $zt \simeq x$. Thus given $\varepsilon > 0$ and $\delta > 0$ standard but otherwise arbitrary, the sentence

$$(\exists z)(\exists t)[z \in {}^*X \wedge \rho(x, z) < \varepsilon \wedge t \in {}^*R^+ \wedge \rho(zt, y) < \delta]$$

is *true. Transferring this sentence to $\mathscr{D}$ shows that y is in the standard first positive prolongation. ▮

It is clear that Definition 4.5 could be adapted to arbitrary topological dynamical systems in the spirit of Ura.

Notice that the relations $\mu(M) \subseteq D^\omega(M)$ and $\mu({}^*M) \subseteq D^\omega({}^*M)$ follow immediately from the definition.

THEOREM 4.16. *The standard set M is* (ω-)

(i) *stable if and only if* $D^\omega(M) \subset \nu({}^*M)$,

(ii) *uniformly stable if and only if* $D^\omega({}^*M) \subset \nu({}^*M)$.

Proof. Obvious from the definitions. ▮

This theorem generalizes Theorem 2.6.6 in [1]. As a corollary we have a generalization of Theorem 2.6.5 in that book.

COROLLARY 4.16.1. *If the standard set M is* (ω-) *stable then* $\operatorname{st}(D^\omega(M)) = \overline{M}$.

Proof. We have $\mu(M) \subset D^\omega(M) \subset \nu({}^*M)$ and hence the result follows by taking standard parts and using Theorem 1.2. ▮

DEFINITION 4.6. For any $x \in {}^*X$ we define the *first* (ω-) *prolongational limit set* $J^+(x)$ by

$$J^+(x) = \bigcup \{\Omega_y^\omega : y \simeq x\}.$$

If M is any subset of ${}^*X$, then we define

$$J^+(M) = \bigcup \{J^+(x) : x \in M\}.$$

THEOREM 4.17. *If x is standard then the standard first prolongational limit set* (*Definition* 2.3.6 *in* [1]) *is given by* st $(J^+(x))$.

Proof. Similar to the proof of Theorem 4.15. ▮

The following results follow immediately.

THEOREM 4.18. *The standard set M is a uniform* (ω-) *semi-attractor if and only if there is a standard open set $V \supset M$ such that $J^{\omega}(*V) \subset \nu(*M)$. If there is a standard $\delta > 0$ such that $V = S(M, \delta)$ then M is a uniform attractor.*

THEOREM 4.19. *The standard invariant set M is* (ω-) *stable if and only if $J^{\omega}(M) \subset \nu(*M)$.*

THEOREM 4.20. *The standard set M is* (ω-) *stable* (*uniformly stable*) *if and only if*

$$J^{-\omega}(*M) \cap \mu(M) = \varnothing \qquad (J^{-\omega}(*M) \cap \mu(*M) = \varnothing).$$

This theorem generalizes results in Exercises 2.6.76 in [1].

4.9. *Stability of motions.* The notion of a stable motion (as opposed to a stable set) is very important in the study of almost periodicity. This notion can be characterized nonstandardly as follows:

DEFINITION 4.7. The motion through $x \in *X$ is said to be *positively* (*negatively*) *stable* if for all y such that $y \simeq x$ we have $xt \simeq yt$ for all $t \in *R^+$ ($t \in *R^-$).

THEOREM 4.21. *If x is standard then the motion through x is stable in the sense of Definition* 4.7 *if and only if it is stable in the standard sense* [1, *Definition* 1.11.1].

Proof. The motion through a standard point x is positively stable in the standard sense if given $\varepsilon > 0$ there is a $\delta > 0$ such that for all $y \in X$ such that $\rho(x, y) < \delta$ we have $\rho(xt, yt) < \varepsilon$ for all $t \in R^+$. Suppose that this condition is satisfied. With $\varepsilon > 0$ and the corresponding $\delta > 0$ standard and fixed, the sentence

$$(\forall x)(\forall y)[[y \in X \wedge \rho(x, y) < \delta] \to (\forall t)[t \in R^+ \to \rho(xt, yt) < \varepsilon]]$$

is true in $\mathscr{D}$ and hence, by transfer, in $*\mathscr{D}$. In particular we have

$$(\forall x)(\forall y)(\forall t)[[y \in *X \wedge x \simeq y \wedge t \in *R^+] \to \rho(xt, yt) < \varepsilon].$$

This is true for any standard $\varepsilon > 0$ and hence the desired result.

Conversely, suppose that the nonstandard condition of Definition 4.7 is satisfied. Then given $\varepsilon > 0$ and standard, the statement

$$(\forall y)(\exists \delta)(\forall t)[\delta > 0 \wedge \rho(x, y) < \delta \wedge t \in *R^+ \to \rho(xt, yt) < \varepsilon]$$

is *true, as we see by choosing δ infinitesimal. Transferring to $\mathscr{D}$ yields the desired standard condition. ▮

Before proceeding we make several remarks concerning Definition 4.7:

(i) The definition can be adapted to motions in topological spaces by using topological monads rather than metric monads.

(ii) If x is standard then we need only have $xt \simeq yt$ for all $t \in R^+_\infty$ since the condition is automatically satisfied for all finite $t \geqq 0$ by continuity.

DEFINITION 4.8. The motion through $x \in {}^*X$ is *uniformly positively* (*negatively*) *stable* if for all $\tau \in {}^*R$ and all y such that $y \simeq x\tau$, we have $x(t+\tau) \simeq yt$ for all $t \in {}^*R^+$ ($t \in {}^*R^-$).

THEOREM 4.22. *If x is standard then the motion through x is uniformly stable in the sense of Definition* 4.8 *if and only if it is uniformly stable in the standard sense* [1, *Definition* 1.11.3].

Proof. Similar to the proof of Theorem 4.21. ∎

We get a more general version of the definition by restricting τ to ${}^*R^+$ (or $\tau \in R^+_\infty$ in the case of standard x).

Corresponding to the above definitions we have the following more general notion of stability with respect to a set S.

DEFINITION 4.9. The motion through $x \in {}^*X$ is *positively* (*negatively*) *stable in* $S \subset {}^*X$ if for all $y \in S$ such that $x \simeq y$ we have $xt \simeq yt$ for all $t \in {}^*R^+$ ($t \in {}^*R^-$), and uniformly positively (negatively) stable in S if for all $\tau \in {}^*R$ and all $y \in S$ such that $y \simeq x\tau$ we have $x(t+\tau) \simeq yt$ for all $t \in {}^*R^+$ ($t \in {}^*R^-$).

If x is standard and B is a standard set in X, then Definition 4.9, with $S = {}^*B$, can be shown to be equivalent to Definition 1.11.9 in [1].

DEFINITION 4.10. If R is a set in ${}^*X$ then we say that *the motions in R are positively* (*negatively*) *stable* (*uniformly stable*) *in S* if the corresponding conditions of the definition are satisfied for each $x \in R$.

This definition should be compared with Definition 2.10.3 in [1], in which Bhatia and Szegö introduce the notion of the motions through a standard set A contained in a standard set B being *uniformly* stable in the set B. It is not too hard to see that this notion is equivalent to a special case of Definition 4.10, namely that the motions in ${}^*A$ are stable in ${}^*B$. A stronger sort of uniformity would be obtained if we insisted that the motions in ${}^*A$ were uniformly stable in ${}^*B$. If, however, A is invariant then this latter sort of uniformity coincides with the former since $x\tau \in {}^*A$ for each $x \in {}^*A$ and each $\tau \in {}^*R$.

We could obviously go on to define attracting motions and asymptotically stable motions, and study their properties, but we will not do so since no special difficulties are involved.

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University of California,
Los Angeles, California 90024