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ON KNOTS WITH NONTRIVIAL INTERPOLATING MANIFOLDS

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Abstract. If a polygonal knot K in the 3-sphere S^3 does not separate an interpolating manifold S for K, then $S-K$ does not carry the first homology of either closed component of S^3-S. It follows that most knots K with nontrivial interpolating manifolds have the property that a simply connected manifold cannot be obtained by removing a regular neighborhood of K from S^3 and sewing it back differently.

0. **Introduction.** A polygonal knot K in the 3-sphere S^3 is said to have *Property* P [1] if it is impossible to obtain a simply connected manifold by removing a regular neighborhood of K from S^3 and sewing it back differently. It has been conjectured that all nontrivial knots have Property P, and large classes of knots with this property have been described by Hempel [5], Bing and Martin [1], Noga [10], Connor [2], Gonzales [4], and the author [12]. If K has Property P, then the knot type of K is determined by the topological type of S^3-K. Furthermore, it would be interesting to know that no fake 3-sphere could be constructed by surgery along a knot, since [7] any closed, orientable 3-manifold can be realized by surgery along some finite link in S^3.

In [12], a *Property* Q is defined for knots and it is shown there (Theorem 5) that Property Q, along with an additional technical requirement, implies Property P. Property Q and Neuwirth's notion of an *interpolating manifold* [9] for a knot are similar in that both require the knot to be contained in a closed 2-manifold in a "sufficiently complicated" manner. It is conjectured in [12] that a knot K has Property Q iff K has a nontrivial interpolating manifold. This conjecture is established by the following theorem, which is the main result of this paper:

THEOREM. *If S is a polyhedral, closed 2-manifold in S^3, K a nonseparating simple closed curve in S, and A is either closed complementary domain of S, then K generates a free factor of $H_1(A)$ iff $H_1(A, S-K)=0$. It follows that most knots with nontrivial interpolating manifolds have Property* P.

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§1 contains the proof of the above theorem and its application to Property P. §2 considers generalizations of the main result: Theorem 2 extends Theorem 1 to knots in manifolds other than S^3; in Lemma 3.1 necessary criteria are found for an endomorphism of the first homology group of a closed, orientable 2-manifold to be induced by a map. These are used in Theorem 3 to obtain results analogous to Theorem 1 for loops, rather than simple closed curves.

Conventions. All topological spaces, subspaces, and maps considered here are polyhedral, and all manifolds are orientable. A *knot* is a simple closed curve in S^3 that does not bound a disk. A manifold is *closed* if it is compact, connected, and has no boundary. Homology groups are taken with integer coefficients unless otherwise specified. An *interpolating manifold* for a knot K is a closed 2-manifold $S\subset S^3$ such that $K\subset S$ and K does not generate a free factor of the first homology group of either closed complementary domain of S; since, as noted in [9], every knot K has an interpolating manifold S such that K separates S, we call S *nontrivial* if $S-K$ is connected. If K is contained in a closed 2-manifold S such that $S-K$ is connected and $S-K$ does not carry the first homology of either closed complementary domain of S, then K is said to have *Property* Q. If x, y are elements of a group G, the *commutator of* x *and* y, denoted $[x, y]$, is $x^{-1}y^{-1}xy$; the *commutator subgroup*, denoted G', of G is the subgroup generated by $\{[x, y]\}_{x,y\in G}$. If $R_1, R_2, \ldots$ are elements of the free group F generated by $a_1, a_2, \ldots$, the symbol $P=(a_1, a_2, \ldots \mid R_1, R_2, \ldots)$ will denote the quotient group G of F by its smallest normal subgroup containing $R_1, R_2, \ldots$; if H is a group isomorphic to G, then P is called a *presentation of* H with *generators* $\{a_i\}$ and (*defining*) *relators* $\{R_i\}$.

It will also be useful to define a "standard basis" for a closed 2-manifold S of genus $n\geqq 1$. Let $a_1, \ldots, a_{2n}$ be a system of simple closed curves in S such that (1) if $|i-j|=n$ then a_i and a_j are transverse, and (2) $a_i \cap a_j=\varnothing$ otherwise. Choose a base point $s\in S$, and, for $i=1, \ldots, n$, let t_i be an arc in S from s to $a_i \cap a_{n+i}$ such that for $i\neq j$, $t_i \cap t_j=\{s\}$. For $i=n+1, \ldots, 2n$, let $t_i=t_{i-n}$. Orient the curves a_i, $i=1, \ldots, 2n$, and let α_i be the loop obtained from a_i by tracing out t_i, a_i, and then t_i^{-1}. Then $\{\alpha_i\}_{i=1,\ldots,2n}$ generates $\Pi_1(S, s)$, and, with possible changes of orientations and renumbering of curves *within the pairs* a_i, a_{n+i}, the function $\lambda\colon \alpha_i \to x_i$, $i=1, \ldots, 2n$, defines an isomorphism of $\Pi_1(S, s)$ onto

$$\left(x_1, \ldots, x_{2n} \;\middle|\; \prod_{i=1}^{n} [x_i, x_{n+i}]\right).$$

With such orientations and numbering, the curves $\{a_i\}_{i=1,\ldots,2n}$ will be called a *standard basis* for S. It may be the case, however, as in §2, that necessary properties of $\{a_i\}$ would be lost by renumbering. It is still possible to orient the curves so that $[\lambda(a_i), \lambda(a_{n+i})]$ is conjugate to $[x_i, x_{n+i}]$. With such orientations, $\{a_i\}$ will be called a *prestandard basis* for S. This choice of orientations is independent of the base point s and the arcs t_i. With any orientations, $\{a_i\}_{i=1,\ldots,2n}$ is a basis for $H_1(S)$.

1. **The main result.** Let S be a closed 2-manifold of genus $n \geqq 1$ in S^3 containing a nonseparating simple closed curve K, and let A be the closure of a complementary domain of S.

THEOREM 1. *$K \in pH_1(A)$ for some prime $p \in Z$ iff there exists a homomorphism of $H_1(A, S-K)$ onto Z_p. In particular, K generates a free factor of $H_1(A)$ iff $H_1(A, S-K)=0$.*

COROLLARY 1. *A knot K has a nontrivial interpolating manifold iff K has Property* Q.

COROLLARY 2. *If a knot K has a nontrivial interpolating manifold S such that a boundary component J of a regular neighborhood of K in S is not* 0, *a generator, or twice a generator in $H_1(S^3-K)$, then K has Property* P.

Proof of Corollary 2. By Theorem 1, S and J satisfy the requirements of Theorem 5 of [12], and so K has Property P.

Proof of Theorem 1. If $n=1$, the result is obvious, so it will be assumed throughout that $n \geqq 2$. Since $A \subset S^3$ and bdy (A) is connected, by a theorem of Fox [3], A can be re-embedded in S^3 so that $(S^3-A)^-$ is a regular neighborhood of a finite graph. From a theorem of Papakyriakopoulos (Theorem (4.1) of [11]), it then follows that there is a prestandard basis $\{a_i\}$ for S such that

$$(1)\quad \begin{array}{ll} \text{(i)} & a_1, \ldots, a_n \text{ is a basis for } H_1(A), \text{ a free abelian group of rank } n, \\ \text{(ii)} & a_i \sim 0 \text{ in } A \text{ for } n+1 \leqq i \leqq 2n. \end{array}$$

For notational convenience, it will be assumed that no renumbering is necessary to change $\{a_i\}$ to a standard basis for S; the arguments below will accommodate any such complication by appropriate alteration of subscripts.

Since K is a nonseparating simple closed curve in S, there exists a homeomorphism $f: S \to S$ taking a_1 to K. Since f is a homeomorphism, f induces a conjugacy class of automorphisms of $\Pi_1(S)$, which in turn induces an automorphism f_* of $H_1(S)$. Let $E=(e_{ij})$ be the $2n \times 2n$ integer matrix of f_*, where $f_*(a_i)=\sum_{j=1}^{2n} e_{ij}a_j$. By a theorem of Magnus (Corollary 5.15 of [8]), since f_* is induced by a homeomorphism, and $\{a_i\}$ is a standard basis, the matrix E must be symplectic. That is, if I is the $n \times n$ identity matrix, J is the $2n \times 2n$ matrix with block diagram $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, and E' is the transpose of E, then $EJE'=\pm J$ (although $EJE' \neq E'JE$ in general, E is symplectic iff E' is symplectic). Thus, in particular, for all s, t such that $1 \leqq s < t \leqq 2n$ and $|s-t| \neq n$, it must be the case that

$$(*)\qquad \sum_{i=1}^{n} \begin{vmatrix} e_{i,s} & e_{i,t} \\ e_{i+n,s} & e_{i+n,t} \end{vmatrix} = 0.$$

Since it has been assumed that no renumbering among a_i, a_{i+n} was necessary, the remainder of the proof will only make use of equations $(*)$ for $1 \leqq s < t \leqq n$.

Let $f_A\colon H_1(S)\to H_1(A)$ be the homomorphism induced by f followed by inclusion of S into A. Then in $H_1(A)$, $K=f_A(a_1)$ and

$$\begin{aligned}H_1(A, S-K) &= H_1(A)/f_AH_1(S-a_1)\\ &= H_1(A)/\{f_A(a_i) : i = 1,\dots, 2n, i \neq n+1\}.\end{aligned}$$

Case 1. Assume that for some prime $p\in Z$, $K\in pH_1(A)$. To map $H_1(A, S-K)$ onto Z_p, it suffices to find a homomorphism of $H_1(A)$ onto Z_p annihilating $\{f_A(a_i) : i=2,\dots, 2n, i\neq n+1\}$, since any homomorphism of $H_1(A)\to Z_p$ will annihilate $f_A(a_1)$.

First define $\sigma\colon H_1(A)\to H_1(A;Z_p)$ by $\sigma(a_i)=a_i$. Then $\sigma f_A(a_i)=\sum_{j=1}^n e_{ij}a_j$, where now e_{ij} is a residue class mod (p). The equations $(*)$ remain valid over Z_p; in particular, since $\sigma f_A(a_1)=0$, we have for $1\leqq s<t\leqq n$,

$$(**)\qquad \sum_{i=2}^{n}\begin{vmatrix} e_{i,s} & e_{i,t}\\ e_{i+n,s} & e_{i+n,t}\end{vmatrix} = 0\in Z_p.$$

By Lemma 1.1 below, the $(2n-2)\times n$ matrix $(e_{ij})_{i=2,\dots,2n,i\neq n+1;j=1,\dots,n}$ has rank at most $n-1$ over Z_p. Thus there exists an epimorphism $\theta\colon H_1(A;Z_p)\to Z_p$ annihilating $\{\sigma f_A(a_i) : i=2,\dots, 2n, i\neq n+1\}$, and so $\theta\circ\sigma$ induces a map of $H_1(A, S-K)$ onto Z_p.

Case 2. Assume now that for some prime $p\in Z$, there exists an epimorphism $\rho\colon H_1(A)/f_AH_1(S-a_1)\to Z_p$. We wish to show that $f_A(a_1)\in pH_1(A)$. Let Π be the natural projection of $H_1(A)$ onto $H_1(A)/f_AH_1(S-a_1)$, and let σ be as in Case 1. Since Z_p has characteristic p, $\theta=\rho\circ\Pi\circ\sigma^{-1}$ is a well-defined homomorphism of $H_1(A;Z_p)$ onto Z_p. Since S contains generators for $H_1(A)$, specifically $a_1,\dots, a_n$, and f is a homeomorphism, f_A must be surjective, and so $\{\sigma f_A(a_i) : i=1,\dots, 2n\}$ generates $H_1(A;Z_p)$. Thus it must be the case that $\theta\sigma f_A(a_{n+1})$ generates Z_p. On the other hand, if $\sigma f_A(a_1)\neq 0$, then by Lemma 1.2 below, equations $(*)$ imply that $\sigma f_A(a_{n+1})$ is a linear combination of $\{\sigma f_A(a_i) : i=1,\dots, 2n, i\neq n+1\}$, and so $\theta\sigma f_A(a_{n+1})=0$. We conclude that $\sigma f_A(a_1)=0$, i.e., $K\in pH_1(A)$.

LEMMA 1.1. *Let*

$$E = (e_{ij})_{i=2,\dots,2n,i\neq n+1;j=1,\dots,n}$$

be a $(2n-2)\times n$ *matrix over a field subject to equations* $(**)$. *Then* rank $(E)\leqq n-1$.

Proof. If $n=2$, the result is obvious. We proceed by induction on n. The rank of E and equations $(**)$ are preserved under the following transformations: (i) permute columns, (ii) divide a column by a nonzero scalar, (iii) add to one column a multiple of another, and (iv) permute rows *in pairs*: row $(i)\rightleftarrows$ row (i') *and* row $(i+n)\rightleftarrows$ row $(i'+n)$. If rank $(E)=n$, then with finitely many transformations of types (i)–(iv), we can obtain a new matrix E, satisfying equations $(**)$, such that $e_{2,1}=e_{2+n,2}=1$, all the other terms in rows $(i=2)$ and $(i=2+n)$ are 0, and the submatrix

$$\bar{E} = (e_{ij})_{i=3,\dots,2n;i\neq 1+n,2+n;j=2,\dots,n}$$

has $(n-2)$ linearly independent rows, the first component of each being 0. But if rows $(i=2)$ and $(i=2+n)$ are as specified, then we have, for $2 \leqq s < t < n$,

$$\sum_{i=3}^{n} \begin{vmatrix} e_{i,s} & e_{i,t} \\ e_{i+n,s} & e_{i+n,t} \end{vmatrix} = 0,$$

so inductively, rank $(\bar{E}) \leqq n-2$. Thus the first component of each row of $\bar{E}$ must be 0, and so $e_{2+n,2}=1$ and all the other terms in column $(j=2)$ of E are 0. Since $e_{21}=1$, this contradicts the equation

$$0 = \sum_{i=2}^{n} \begin{vmatrix} e_{i,1} & e_{i,2} \\ e_{i+n,1} & e_{i+n,2} \end{vmatrix}.$$

Lemma 1.2. *Let $E=(e_{ij})_{i=1,\ldots,2n;j=1,\ldots,n}$ be a $2n\times n$ matrix over a field subject to equations* $(*)$ (*for* $1 \leqq s < t \leqq n$). *If row* (1) *is not identically* 0, *then row* $(n+1)$ *is a linear combination of the other rows of E.*

Proof. In addition to preserving equations $(*)$, the transformations (i)–(iii) described in the preceding proof do not alter the fact of whether or not row $(n+1)$ of E is a linear combination of the other rows. Since row (1) has some nonzero component, we can therefore assume that row $(1)=(1, 0, \ldots, 0)$. Equations $(*)$, for $s=1$, then become

$$\left\{ e_{1+n,t} = \sum_{i=2}^{n} \begin{vmatrix} e_{i+n,1} & e_{i+n,t} \\ e_{i,1} & e_{i,t} \end{vmatrix} \right\}_{t=2,\ldots,n}.$$

It follows easily that if $\alpha_j = e_{j+n,1},\ -e_{j-n,1}$ according as $1 \leqq j \leqq n$ or $n+2 \leqq j \leqq 2n$, then

$$\text{row}\,(n+1) = \sum_{j=1,\ldots,2n;j\neq n+1} \alpha_j \cdot \text{row}\,(j).$$

2. **Generalizations.** Several of the hypotheses of Theorem 1 can be weakened, while maintaining the same or appropriately modified conclusions. First, p need not be prime; Theorem 1 easily extends to the case that p is a product of distinct primes. It is not clear, however, how one might establish a duality theorem of the following sort:

Conjecture. If K, A, S are as in Theorem 1, then the torsion subgroups of $H_1(A, K)$ and $H_1(A, S-K)$ are isomorphic.

It is also unnecessary to require the ambient space to be S^3.

Definition. A compact, connected, 3-manifold A with boundary a closed 2-manifold S of genus n is called a *homology cube-with-holes* (HCWH) if $H_1(A)$ is a free abelian group of rank n and S has a prestandard basis $\{a_i\}_{i=1,\ldots,2n}$ satisfying conditions (1). From the proof of Theorem 1 we have immediately

Theorem 2. *If A is a HCWH and, otherwise, K, S, p, A are as in Theorem* 1, *then* $K \in pH_1(A)$ *iff* $\exists \rho$: $H_1(A, S-K) \twoheadrightarrow Z_p$.

If A can be embedded in a homology 3-sphere as the closure of the complement of a regular neighborhood of a finite graph, then, from Theorem (4.1) of [11] and the Mayer-Vietoris sequence, it follows that A is a HCWH. If A can be embedded in a homotopy 3-sphere, then, modifying Fox's proof in [3], A can be embedded in a (possibly different) homotopy 3-sphere as the closed complement of a cube-with-handles, so, again, A is a HCWH.

Finally, results analogous to Theorem 1 can be obtained in the case that K is the image of a nonseparating simple closed curve under a map of S to S.

THEOREM 3. *Let A be a HCWH with boundary S, K a nonseparating simple closed curve in S, $f\colon S\to S$ a map, and $f_A\colon H_1(S)\to H_1(S)\to H_1(A)$ the induced homomorphism. If, for some prime $p\in Z$, $f_A(K)\in pH_1(A)$, then there is a homomorphism of $H_1(A)/f_AH_1(S-K)$ onto Z_p. The converse holds providing f_A is assumed to be surjective.*

Proof. Let $E=(e_{ij})$ be as in the proof of Theorem 1. The full strength of the fact that E was symplectic was not required for that proof, but only that equations $(*)$ be valid for $1\leqq s<t\leqq 2n$, $|s-t|\neq n$. It thus suffices to verify these equations in the case that f is a map.

Let $\{a_i\}_{i=1,\dots,2n}$ be a prestandard basis for S, $f_*(a_i)=\sum_{j=1}^{2n} e_{ij}a_j$ the endomorphism of $H_1(S)$ induced by f,

$$x_{s,t}=\sum_{i=1}^{n}\begin{vmatrix} e_{i,s} & e_{i,t}\\ e_{i+n,s} & e_{i+n,t}\end{vmatrix},$$

and

$$G_{a,n}=\left(a_1,\dots,a_{2n}\;\middle|\;\prod_{i=1}^{n}[a_i,a_{n+1}],\ \{[u,[v,w]]\}_{u,v,w\in\{a_i\}}\right)$$

(isomorphic to the quotient group of $\Pi_1(S)$ by the third term in its lower central series). Since f is a map and $\{a_i\}$ is a prestandard basis, f induces an endomorphism $\#$ of $G_{a,n}$ which induces f_*. Thus $\prod_{i=1}^{n}[f_\#(a_i),f_\#(a_{n+i})]=1\in G_{a,n}$. But, using the identity $u^pv^q=v^qu^p[u,v]^{pq}$ in $G_{a,n}$, it is easy to show that

$$(\#)\qquad \prod_{i=1}^{n}[f_\#(a_i),f_\#(a_{n+i})] = \Bigg(\prod_{1\leqq s<t\leqq 2n;|s-t|\neq n}[a_t,a_s]^{x_{s,t}}\Bigg)\cdot\Bigg(\prod_{s=1,\dots,n-1;t=s+n}[a_t,a_s]^{x_{s,t}-x_{n,2n}}\Bigg).$$

Since $G'_{a,n}$ is a free abelian group, generated by $\{[a_t,a_s]\}_{1\leqq s<t\leqq 2n}$ and freely generated by

$$\{[a_t,a_s]\}_{1\leqq s<t\leqq 2n;(s,t)\neq(n,2n)},$$

it follows that each exponent in the right side of equation $(\#)$ must be 0.

Question. Which endomorphisms of $H_1(S)$ are induced by maps? According to the above calculations, if $E=(e_{ij})$ is the matrix of a map-induced endomorphism

f_* of $H_1(S)$, given in terms of a prestandard basis for S, then E must be "nearly symplectic," in the sense that for some integer λ, $E'JE=\lambda J$. For genus $(S)=1$, this is no restriction, consonant with the fact that $H_1(S)=\Pi_1(S)$. But if genus $(S)\geqq 2$, and f is not (homotopic to) a homeomorphism, then, using Euler characteristic arguments, the fact that $\Pi_1(S)$ is Hopfian, and Lemma 3.2 of [6], it can be shown that there is a homeomorphism $h: S \to S$ such that $f \circ h$ annihilates at least one of a_i, a_{n+i} for each $i=1, \ldots, n$. It thus follows that $\lambda \neq \pm 1 \Rightarrow \lambda=0$. But is it true that any $2n \times 2n$ integer matrix E such that $E'JE=0$ is the matrix of a map $f: S \to S$?

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