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SPECTRAL CONCENTRATION AND VIRTUAL POLES. II

BY
JAMES S. HOWLAND[1]

Abstract. Spectral concentration at an isolated eigenvalue of finite multiplicity of the selfadjoint operator $H_\varepsilon = T_\varepsilon + A_\varepsilon B_\varepsilon$ is shown to arise from a pole of an analytic continuation of $A_\varepsilon(H_\varepsilon - z)^{-1}B_\varepsilon$. An application to quantum mechanical barrier penetration is given.

This article is concerned with the problem of isolated eigenvalues of finite multiplicity which disappear into a continuous spectrum on perturbation. We wish to determine conditions under which spectral concentration results from a virtual pole of the resolvent $R_\varepsilon(\zeta)$ of the perturbed operator H_ε; that is, from a pole of an analytic continuation of certain matrix elements of $R_\varepsilon(\zeta)$. In [5], we have reviewed previous work, and have discussed the case in which the unperturbed operator H_0 is of finite rank.

The present article is based on the factorization technique [6], [7], [8]. We proceed as follows. We do not divide the perturbed operator in the usual manner as

$$H_\varepsilon = H_0 + V_\varepsilon, \qquad \varepsilon > 0,$$

where V_ε is the perturbation, but rather as

$$H_\varepsilon = T_\varepsilon + A_\varepsilon B_\varepsilon, \qquad \varepsilon > 0,$$

where H_ε and T_ε are close in the sense that $Q_\varepsilon(z) = A_\varepsilon(T_\varepsilon - z)^{-1}B_\varepsilon$ is compact, and has an analytic continuation $Q_\varepsilon^+(z)$ from the upper half-plane across the axis at the eigenvalue λ_0 in question. The idea, which is implicit in [5], is that H_ε and T_ε are quite similar, where as H_ε and H_0 are not. In fact, in [5], we chose $A_\varepsilon B_\varepsilon = H_0$. Of course, the perturbation-theoretic philosophy here is that T_ε is a family of unperturbed operators, for which $(T_\varepsilon - z)^{-1}$ and other such quantities are computable, so that conditions may reasonably be placed on T_ε which cannot be placed on H_ε. Our main result (§2) states that, under suitable conditions, there is spectral concentration associated with poles of $[I + Q_\varepsilon^+(z)]^{-1}$ in the usual manner. We apply this (§3) to improve the results of [5] by eliminating a nondegeneracy assumption and (§4) to consider a problem of quantum mechanical barrier penetration, such as is usually discussed by means of the WKB method.

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This paper is independent of the previous article [5] of the same title.

Let $\sigma(T)$, $\rho(T)$ and $\mathscr{D}(T)$ denote the spectrum, resolvent set and domain of the operator T. $\boldsymbol{C}$ and $\boldsymbol{R}$ denote the complex and real numbers.

1. **Hypotheses and preliminaries.** We shall first recall some results of [6].

HYPOTHESIS I. *Let $\mathscr{H}$ and $\mathscr{H}'$ be separable Hilbert spaces. Let T be a selfadjoint operator on $\mathscr{H}$ with resolvent $G(z)=(T-z)^{-1}$, and let A and B be closed, densely defined operators from $\mathscr{H}$ to $\mathscr{H}'$. Assume that*

(a) *$\mathscr{D}(T)\subset\mathscr{D}(A)\cap\mathscr{D}(B)$ and $(Ax, By)=(Bx, Ay)$ for every $x, y\in\mathscr{D}(A)\cap\mathscr{D}(B)$.*

(b) *For every $z\in\rho(T)$, the operator $AG(z)B^*$, which is defined on $\mathscr{D}(B^*)$, has a compact extension $Q(z)$ to $\mathscr{H}'$.*

1.1. LEMMA [6], [7], [9]. *If Hypothesis* I *is satisfied, then*

(a) *There exists a selfadjoint extension H of $T+B^*A$ such that if $z\in\rho(T)$ and $I+Q(z)$ has a bounded inverse, then $z\in\rho(H)$ and*

(1.1) $R(z)=G(z)-[BG(\bar{z})]^*[I+Q(z)]^{-1}AG(z)$ *where* $R(z)=(H-z)^{-1}$.

(b) *$\mathscr{D}(H)\subset\mathscr{D}(A)$ and $AR(z)B^*$ has a compact extension $Q_1(z)$ from $\mathscr{D}(B^*)$ to $\mathscr{H}'$, which is such that*

(1.2) $I-Q_1(z)=[I+Q(z)]^{-1}$ *for every $z\in\rho(T)$ where the right side of* (1.2) *exists.*

(c) *T arises from H in the same manner; that is,*

(1.3) $G(z)=R(z)+[BR(\bar{z})]^*[I-Q_1(z)]^{-1}AR(z)$.

(*The signs change, since T extends $H-B^*A$.*)

HYPOTHESIS II. *Let Ω be a domain in $\boldsymbol{C}$ which intersects the real axis, and define $\Omega^\pm=\{z\in\Omega : \pm\operatorname{Im} z>0\}$. Then $Q(z)$ is analytic on $\Omega^\pm$. Assume that $Q(z)$ has an analytic continuation $Q^\pm(z)$ from $\Omega^\pm$ to all of Ω.*

Note that $Q^+(z)$ and $Q^-(z)$ need not agree on Ω.

It follows ([6, Lemma 1.5], [10] and [11, Theorem 1]) that $Q_1(z)$ has a meromorphic continuation from $\Omega^\pm$ to Ω given by

$$I-Q_1^\pm(z) = [I+Q^\pm(z)]^{-1}$$

and that the principal part of $Q_1^\pm(z)$ at each of its poles has finite rank.

1.2. THEOREM. *If Hypotheses* I *and* II *hold, then there exists a subspace $\mathscr{M}$ of $\mathscr{H}$ reducing both T and H, such that*

(a) *$T=H$ on $\mathscr{M}^\perp$,*

(b) *the parts of T and H in $\mathscr{M}$ are absolutely continuous, except that H has eigenvalues of finite multiplicity at the real poles of $Q_1^\pm(z)$.*

This result follows directly from Theorem 3.1 of [6], which also gives a formula for the multiplicity of the eigenvalues of H.

The following is the main hypothesis of this paper.

HYPOTHESIS III. *For $n\geqq 0$, let T_n, A_n and B_n satisfy Hypotheses* I *and* II *on a domain Ω independent of n. Assume that*

(a) *$Q^\pm(z, n)\to Q^\pm(z, 0)$ uniformly on Ω, in operator norm.*

(b) *$Q_1^\pm(z, 0)$ is analytic on Ω, except for a nontrivial pole at a real point λ_0 of Ω.*

(c) *There exists a dense subspace* $\mathscr{D}$ *of* $\mathscr{H}'$ *such that* $\mathscr{D}\subset\mathscr{D}(A_n^*)\cap\mathscr{D}(B_n^*)$ *for every* $n\geqq 0$, *and* $A_n^*x\to A_0^*x$ *and* $B_n^*x\to B_0^*x$ *strongly, for every* $x\in\mathscr{D}$.

(d) *For* $\operatorname{Im} z\neq 0$, $G_n(z)$, $A_nG_n(z)$ *and* $B_nG_n(z)$ *converge strongly to* $G_0(z)$, $A_0G_0(z)$ *and* $B_0G_0(z)$ *respectively.*

(e) λ_0 *is not an eigenvalue of* T_0.

NOTATION. We shall write $G_n(z)=(T_n-z)^{-1}$, $R_n(z)=(H_n-z)^{-1}$, $T_n=\int\lambda\, dE_n(\lambda)$, $H_n=\int\lambda\, dE_n^{(1)}(\lambda)$, where H_n is the operator obtained from T_n, A_n and B_n according to Lemma 1.1. $Q(z,n)$ is the extension of $A_nG_n(z)B_n^*$, and $Q^{\pm}(z,n)$ its analytic continuation. Similarly for $Q_1^{\pm}(z,n)$.

By Theorem 1.2, λ_0 is an eigenvalue of H_0 of finite multiplicity m. If P_0 is the projection onto $\ker(H_0-\lambda_0)$, then [6, Theorem 3.1]

(1.4) $Q_1^{\pm}(z,0)=(z-\lambda_0)^{-1}AP_0[BP_0]^*+L_0^{\pm}(z)$ where $L_0^{\pm}(z)$ is analytic on Ω.

Hypothesis III(d) is used only in the following proof.

1.3. LEMMA. *If* $\operatorname{Im} z\neq 0$, *then* $R_n(z)\to R_0(z)$ *strongly.*

Proof. If $x\in\mathscr{D}$, then $[B_nG_n(\bar z)]^*x=G_n(z)B_n^*x$ converges to $G_0(z)B_0^*x=[B_0G_0(\bar z)]^*x$. Since the norms of $B_nG_n(z)$ are bounded, we have $[B_nG_n(\bar z)]^*\to[B_0G_0(\bar z)]^*$ strongly. The result now follows from formulas (1.1) and (1.2).

The next lemma holds for operator-valued functions on an arbitrary Banach space, and is due essentially to Steinberg [11].

1.4. LEMMA. *Let* $T_n(z)$ *be a sequence of analytic functions on a domain* Ω, *with values in the compact operators, such that* $T_n(z)\to T_0(z)$ *uniformly on* Ω. *Let* $z_0\in\Omega$. *Then there exists a neighborhood* D *of* z_0, *and analytic functions* $A_n(z)$, $U_n(z)$ *and* $F_n(z)$, $n\geqq 0$, *such that*

(a) $A_n(z)$ *and* $U_n(z)$ *are invertible for each* $z\in D$, *and* $U_0(z_0)=I$.

(b) $U(z)F(z)U^{-1}(z)$ *is reduced by* $\ker(I+T_0(0))$ *and vanishes on a complementary subspace of* $\ker(I+T(0))$.

(c) $A_n(z)(I+T_n(z))=I+F_n(z)$, $n\geqq 0$, *and*

(d) $A_n(z)$, $U_n(z)$ *and* $F_n(z)$ *converge to* $A_0(z)$, $U_0(z)$ *and* $F_0(z)$ *uniformly on* D.

The following theorem describes the singularities of $Q_1^{\pm}(z,n)$ near λ_0.

1.5. THEOREM. *If Hypothesis* III *is satisfied, then there exists a neighborhood* D *of* λ_0 *such that for n sufficiently large there are* (*counting multiplicities*) *no more than m poles of* $Q_1^{\pm}(z,n)$ *in* D, *all of which converge to* λ_0 *as n tends to infinity.*

Proof. By Lemma 1.4, $A_n(z)$, $U_n(z)$ and D may be chosen such that

$$A_n(z)(I+Q^{\pm}(z,n)) = I+F_n^{\pm}(z)$$

where $U_n(z)F_n^{\pm}(z)U_n^{-1}(z)$ is an operator on $\ker(I+Q^{\pm}(\lambda_0,0))$. It follows (cf. [11, Theorem 1]) that the poles of $Q_1^{+}(z,n)$ in D are the zeros of

$$\det(I+F_n^{\pm}(z)). \tag{1.5}$$

But (1.5) converges to $\det(I+F_0^{\pm}(z))$ uniformly on D, and by [6, Theorem 3.1], this function vanishes only at λ_0, and the order of its zero there is m. By Rouche's theorem, (1.5) must have zeros inside D of total multiplicity m, if n is large. If $\varepsilon>0$, the same argument applies to $D_\varepsilon=\{t : |z-\lambda_0|<\varepsilon\}$, so that for n large, all the zeros of (1.5) in D are also in D_ε; that is, the zeros of (1.5) in D all converge to λ_0.

2. **The main theorem.** The following is our principal result.

2.1. THEOREM. *Assume that Hypothesis* III *holds, and let* $\zeta_j^{\pm}(n)=\lambda_j^{\pm}(n)\mp i\Gamma_j^{\pm}(n)$ $(j=1,\dots,m)$ *be the zeros of* $\det(I+F_n^{\pm}(z))$ *in D, repeated according to multiplicity, and numbered in an arbitrary manner. Choose* $\delta_j^{\pm}(n)>0$ *such that* $\Gamma_j^{\pm}(n)=o(\delta_j^{\pm}(n))$ *and* $\delta_j^{\pm}(n)=o(1)$ *as* $n\to\infty, j=1,\dots,m$. *Define*

$$J_n = J_1^+(n)\cup\cdots\cup J_m^+(n)\cup J_1^-(n)\cup\cdots\cup J_m^-(n)$$

where $J_j^{\pm}(n)$ *is the open interval* $(\lambda_j^{\pm}(n)-\delta_j^{\pm}(n), \lambda_j^{\pm}(n)+\delta_j^{\pm}(n))$. *Then* $E_n^{(1)}[J_n]\to P_0$ *strongly.*

Note that the $\zeta_j^{\pm}(n)$*'s are the poles of* $Q_1^{\pm}(z,n)$ *and hence that* $\Gamma_j^{\pm}(n)\geqq 0$.

The proof of Theorem 2.1 will be divided into two parts. We shall study the asymptotic behavior of $Q_1^{\pm}(z,n)$ in Part I, and give the proof of convergence in Part II. *In Part* I *we shall, for simplicity, drop the* $\pm$ *superscripts.*

Part I. We shall first note two facts. In the first place, it must be shown that every subsequence of $E_n^{(1)}[J_n]$ has a subsequence converging to P_0. But since a subsequence would satisfy the same conditions as the original sequence, *it suffices to prove that some subsequence of* $E_n^{(1)}[J_n]$ *converges to* P_0.

Secondly, *we may always decrease the size of* J_n. For suppose that $J_n^{(0)}\subset J_n$ and $E_n^{(1)}[J_n^{(0)}]\to P_0$. Let $\varepsilon>0$. Then for large n, $J_n\subset(\lambda_0-\varepsilon, \lambda_0+\varepsilon)$, and

$$E_n^{(1)}[J_n^{(0)}] \leqq E_n^{(1)}[J_n] \leqq E_n^{(1)}(\lambda_0+\varepsilon)-E_n^{(1)}(\lambda_0-\varepsilon). \tag{2.1}$$

But $E_n^{(1)}(\lambda_0\pm\varepsilon)\to E_0^{(1)}(\lambda_0\pm\varepsilon)$ by Lemma 1.3 and [8, Theorem 1.15, p. 432]. Passing to the limit in (2.1), we obtain for each x

$$\begin{aligned}\|P_0x\|^2 &\leqq \liminf \|E_n^{(1)}[J_n]x\|^2 \leqq \limsup \|E_n^{(1)}[J_n]x\|^2\\ &\leqq (E_0^{(1)}(\lambda_0+\varepsilon)x-E_0^{(1)}(\lambda_0-\varepsilon)x, x).\end{aligned}$$

But the final term tends to $\|P_0x\|^2$ as $\varepsilon\to 0+$. Hence $\|E_n^{(1)}[J_n]x\|\to\|P_0x\|$, which implies that $E_n^{(1)}[J_n]\xrightarrow[s]{} P_0$, since these operators are orthogonal projections.

These facts can now be used to reduce considerably the complexity of the problem. By passing to a subsequence, we may assume that the number N of *distinct* poles is independent of n. Let the poles be numbered so that $\lambda_1(n)\leqq\lambda_2(n)\leqq\cdots\leqq\lambda_N(n)$, and $\Gamma_k(n)<\Gamma_{k+1}(n)$ if $\lambda_k(n)=\lambda_{k+1}(n)$. By again passing to a subsequence, we may then assume that the order of each pole $\zeta_k(n)$ is independent of n.

Now the open set J_n has a finite number of connected components, and if $\lambda_k(n)$ and $\lambda_r(n)$ lie in the same component of J_n, then so do all of the $\lambda_j(n)$'s between

$\lambda_k(n)$ and $\lambda_r(n)$. Thus, for each n, the $\lambda_j(n)$'s are partitioned into sets

$$\{\lambda_1(n), \lambda_2(n), \ldots, \lambda_p(n)\}, \quad \{\lambda_{p+1}(n), \lambda_{p+2}(n), \ldots\}, \quad \ldots, \quad \{\ldots, \lambda_N(n)\}$$

so that the elements of each set are those $\lambda_j(n)$'s which lie in the same connected component of J_n. Since there are only a finite number of such partitions, we may assume by passing to a subsequence that the partition is independent of n. We shall call the sets of the corresponding partition of the poles *pole clusters*. Thus two poles $\zeta_k(n)$ and $\zeta_r(n)$ belong to the same pole cluster iff $\lambda_k(n)$ and $\lambda_r(n)$ belong to the same connected component of J_n.

Now consider the effect of decreasing J_n to J_n^* by replacing each $\delta_j(n)$ by $\delta_j^*(n) \leqq \delta_j(n)$ such that we still have $\Gamma_j(n) = o(\delta_j^*(n))$ $(j=1, \ldots, N)$. If we again pass to a subsequence to obtain pole clusters for J_n^*, it is clear that the new pole clusters are subdivisions of the old; for example $\{\zeta_1(n), \zeta_2(n), \ldots, \zeta_p(n)\}$ may break into several J_n^* pole clusters

$$\{\zeta_1(n), \zeta_2(n), \ldots, \zeta_s(n)\}, \quad \{\zeta_{s+1}(n), \ldots\}, \quad \ldots, \quad \{\ldots, \zeta_p(n)\}.$$

Since the number of poles is finite this process is eventually stable, and we obtain a J_n for which the pole clusters are *minimal* in the sense that no replacement of $\delta_j(n)$'s by $\delta_j^*(n)$'s as above will lead to a nontrivial subdivision of these pole clusters.

We shall therefore *assume that the pole clusters of J_n are minimal.* If $\{\zeta_1(n), \ldots, \zeta_p(n)\}$ is taken as a typical pole cluster, then the corresponding connected component is

$$J_1(n) \cup \cdots \cup J_p(n) = (c_n - \rho_n, c_n + \rho_n);$$

c_n will be called the *center* and ρ_n the *radius* of the pole cluster. Note that

$$\lambda_1(n) \leqq c_n \leqq \lambda_p(n). \tag{2.2}$$

We must have

$$\lambda_k(n) - c_n = o(\rho_n), \qquad k = 1, \ldots, p. \tag{2.3}$$

For if we replace $\delta_k(n)$ by $\delta_k^*(n)$ such that in addition to the previous conditions, $\delta_k^*(n) = o(\delta_k(n))$ $(k=1, \ldots, p)$, then by minimality $\lambda_1(n), \ldots, \lambda_p(n)$ are all in the same connected component $(c_n^* - \rho_n^*, c_n^* + \rho_n^*)$ of J_n^*. By (2.2), c_n is also in this component and we have

$$\lambda_k(n) - c_n = O(\rho_n^*) = o(\rho_n), \qquad k = 1, \ldots, p.$$

Clearly (2.3) holds for *any* $\lambda_k(n)$ if c_n is the center of its pole cluster.

Finally, if $c_n(1)$, $c_n(2)$, $\rho_n(1)$ and $\rho_n(2)$ are the centers and radii of two distinct pole clusters, then we may assume that

$$\rho_n(i) = o(|c_n(1) - c_n(2)|), \qquad i = 1, 2. \tag{2.4}$$

For since $(c_n(1) - \rho_n(1), c_n(1) + \rho_n(1))$ and $(c_n(2) - \rho_n(2), c_n(2) + \rho_n(2))$ do not intersect, we have

$$\rho_n(i) \leqq |c_n(1) - c_n(2)|, \qquad i = 1, 2,$$

while the above construction yields $c_n^*(i)$ and $\rho_n^*(i)$ $(i=1,2)$ such that $\rho_n^*(i) = o(\rho_n(i))$, $i=1,2$, and, by (2.2),

$$c_n^*(i)-c_n = O(\rho_n^*(i)), \qquad i = 1,2.$$

From the last three equations follows

$$\rho_n^*(i) = o(|c_n^*(1)-c_n^*(2)|), \qquad i = 1,2.$$

The interpretation of (2.3) is that the poles are asymptotically very close to the center of the interval $(c_n-\rho_n, c_n+\rho_n)$, while (2.4) says that the components of J_n are asymptotically very small compared with their relative distance apart.

Having made these reductions, we are ready to study the asymptotic form of the singularities of $Q_1^+(z,n)$. Let

$$\Delta_n(z) = (z-\zeta_1(n))^{p_1}(z-\zeta_2(n))^{p_2}\cdots(z-\zeta_N(n))^{p_N}$$

where p_i is the order of $\zeta_i(n)$. Then $\Delta_n(z)\to(z-\lambda_0)^N$ uniformly on ∂D. Write $Q_1^+(z,n)$ in a Laurent expansion

$$Q_1^+(z,n) = P_n(z)/\Delta_n(z)+L_n(z) \tag{2.5}$$

where $P_n(z)$ is a polynomial in z with coefficients of finite rank and $L_n(z)$ is analytic on D. *We assert that*

$$\lim P_n(z) = (z-\lambda_0)^{N-1}A_0P_0[B_0P_0]^* \tag{2.6}$$

and

$$\lim L_n(z) = L_0(z) \tag{2.7}$$

uniformly on D. For let $P_n(z)=C_{p-1}^{(n)}z^{p-1}+\cdots+C_0(n)$. Then $C_{p-1}(n)$ is the sum of the residues of $P_n(z)/\Delta_n(z)$ and hence

$$C_{p-1}(n) = \frac{1}{2\pi i}\int_{\partial D}\frac{P_n(z)}{\Delta_n(z)}\,dz = \frac{1}{2\pi i}\int_{\partial D} Q_1^+(z,n)\,dz$$

since $L_n(z)$ is analytic. But $Q_1^+(z,n)$ converges to $Q_1^+(z,0)$ uniformly on ∂D, so that $C_{p-1}(n)$ is a convergent sequence. Again $C_{p-2}(n)$ is the sum of the residues of $((z-\zeta_1(n))/\Delta_n(z))(P_n(z)-C_{p-1}(n)z^{p-1})$ so that

$$C_{p-2}(n) = \frac{1}{2\pi i}\int_{\partial D}\left\{Q_1^+(z,n)-\frac{(z-\zeta_1(n))C_{p-1}(n)z^{p-1}}{\Delta_n(z)}\right\}dz$$

where the integrand again converges uniformly on ∂D. Proceeding in this manner, we find that $P_n(z)$ is uniformly convergent on D. If D_1 is a slightly larger disc, concentric with D, then

$$L_n(z) = \frac{1}{2\pi i}\int_{\partial D_1}\frac{1}{\zeta-z}\left\{Q_1^+(\zeta,n)-\frac{P_n(\zeta)}{\Delta_n(\zeta)}\right\}d\zeta$$

so that $L_n(z)$ also converges uniformly on D. (2.6) and (2.7) now follow easily from (1.4).

We now claim that we may assume that

$$\|P_n(z)\| \leqq C|\Delta_n(z)|(\operatorname{Im} z)^{-1}, \qquad \operatorname{Im} z > 0, z \in D, \tag{2.8}$$

where C is a constant independent of n. For by possibly decreasing D and dropping a finite number of terms of the sequence, we may assume that $U_n(z)$ and $U_n^{-1}(z)$ differ from I in norm by less than $(4d)^{-1}$ on D, where $d = \dim\ker(I+Q^+(\lambda_0, 0))$. Let Π be the projection on $\ker(I+Q^+(\lambda_0, 0))$, and choose $x_1, \ldots, x_d \in \mathscr{D}$ such that $\Pi x_1, \ldots, \Pi x_d$ is an orthonormal basis of $\Pi\mathscr{H}'$. Now it is easily seen that

$$E_n(z) = U_n(z)P_n(z)U_n^{-1}(z)$$

is an operator on $\Pi\mathscr{H}'$. The sets

$$\{\Pi U_n^{-1}(z)x_1, \ldots, \Pi U_n^{-1}(z)x_d\} \quad \text{and} \quad \{\Pi U_n^*(z)x_1, \ldots, \Pi U_n^*(z)x_d\}$$

are still bases of $\Pi\mathscr{H}'$, and the matrix of $E_n(z)$ between these bases is

$$\begin{aligned} &(E_n(z)\Pi U_n^{-1}(z)x_i, \Pi U_n^*(z)x_j) \\ &\qquad = (P_n(z)x_i, x_j) = \Delta_n(z)(Q_1^+(z, n)x_i, x_j) + \Delta_n(z)(L_n(z)x_i, x_j), \\ &\qquad\qquad i, j = 1, \ldots, d, \end{aligned} \tag{2.9}$$

since $\Pi E_n(z)\Pi = E_n(z)$. The second term is uniformly bounded, while the first term is equal to $\Delta_n(z)(R_n(z)B_n^*x_i, A_n^*x_j)$ which does not exceed

$$|\Delta_n(z)|\, |B_n^*x_i|\, |A_n^*x_j|(\operatorname{Im} z)^{-1}, \qquad \operatorname{Im} z > 0.$$

But $B_n^*x_i$ and $A_n^*x_j$ are convergent, so we may bound the matrix in (2.9) by $C_1|\Delta_n(z)|(\operatorname{Im} z)^{-1}$. A little consideration of the bases used shows that this implies that

$$\|E_n(z)\| \leqq C_2|\Delta_n(z)|(\operatorname{Im} z)^{-1}, \qquad \operatorname{Im} z > 0,$$

and (2.8) follows easily.

We are now ready to study the partial fraction expansion of $P_n(z)/\Delta_n(z)$. Consider the first pole cluster $\zeta_1(n), \ldots, \zeta_p(n)$, with center $c_1(n)$ and radius $\rho_1(n)$. By the binomial series

$$(z-\zeta_1(n))^{-1} = [z-c_1(n)-(\zeta_1(n)-c_1(n))]^{-1} = (z-c_1(n))^{-1}\{1+o(1)\}$$

uniformly on $|z-c_1(n)| \geqq \rho_1(n)$, by (2.4). The portion of the partial fraction expansion of $P_n(z)/\Delta_n(z)$ involving these poles can therefore be written as

$$\frac{P_n^{(1)}(z)}{(z-\zeta_1(n))^{m_1}\cdots(z-\zeta_p(n))^{m_p}} = \frac{P_n^{(1)}(z)\{1+o(1)\}}{(z-c_1(n))^{p_1}} \tag{2.10}$$

uniformly on $|z-c_1(n)| \geqq \rho_1(n)$, where $p_1 = m_1 + \cdots + m_p$. Expanding the right side of (2.10) in partial fractions, we obtain

$$\left\{\frac{A_{p_1}^{(1)}(n)}{(z-c_1(n))^{p_1}} + \cdots + \frac{A_1^{(1)}(n)}{(z-c_1(n))}\right\}\{1+o(1)\} \tag{2.11}$$

uniformly on $|z-c_1(n)|\geqq\rho_1(n)$. There are analogous expansions for the terms corresponding to the other pole clusters. *We wish to estimate the coefficients* $A_j^{(k)}(n)$ *in these expansions, and the key to this is* (2.8). Define

$$\alpha_j^{(k)}(n) = (-i)^j(\rho_k(n))^{-j+1}A_j^{(k)}(n)$$

for $1\leqq j\leqq p_k$, $k=1,\ldots,N$, where N is the number of pole clusters. Set $z=c_1(n)+i\rho_1(n)\sigma^{-1}$ in (2.8), where $\frac{1}{2}\leqq\sigma\leqq 1$. Then the above expansions are valid, and the term (2.11) becomes

$$(\rho_1(n))^{-1}\{\alpha_1^{(1)}(n)\sigma+\cdots+\alpha_{p_1}^{(1)}(n)\sigma^{p_1}\}\{1+o(1)\}$$

uniformly on $\frac{1}{2}\leqq\sigma\leqq 1$. For the same value of z, the terms of (2.8) for the other pole clusters are of the form

$$\frac{A_j^{(k)}(n)}{(z-c_k(n))^j} = \frac{(i)^j\alpha_j^{(k)}(n)}{\rho_1(n)}\frac{\rho_1(n)(\rho_k(n))^{j-1}}{(z-c_k(n))^j}.$$

However, by (2.4),

$$\frac{\rho_1(n)(\rho_k(n))^{j-1}}{(z-c_k(n))^j} \sim \frac{\rho_1(n)(\rho_k(n))^{j-1}}{(c_1(n)-c_k(n))^j} = o(1).$$

(2.8) therefore yields the equation

$$\{\alpha_1^{(1)}(n)+\cdots+\alpha_{p_1}^{(1)}(n)\sigma^{p_1-1}\}+\sum_{k=2}^{N}\sum_{j=1}^{p_1}\{\alpha_j^{(k)}(n)o(1)\} = O(1) \tag{2.12}$$

uniformly on $\frac{1}{2}\leqq\sigma\leqq 1$. There is an analogous equation for each of N pole clusters. From these equations we form a system of $p_1+\cdots+p_N$ linear equations for the $p_1+\cdots+p_N$ quantities $\alpha_j^{(k)}(n)$ in the following manner: evaluate the analogue of (2.12) for the kth pole cluster at p_k distinct points of $\frac{1}{2}\leqq\sigma\leqq 1$, yielding p_k equations. If this is done for each k, we obtain $p_1+\cdots+p_N$ equations. If we let α_n be the "unknown" vector, and β_n the inhomogeneous term, then these equations in matrix form become

$$(D+M_n)\alpha_n = \beta_n \tag{2.13}$$

where M_n consists of the $o(1)$ terms, and hence vanishes as $n\to\infty$, and $\beta_n=O(1)$. D is independent of n, and consists of N blocks $D_1,\ldots,D_N$ on the diagonal, where

$$D_k(\alpha_1^{(k)},\ldots,\alpha_{p_k}^{(k)}) = (\varphi_k(\sigma_1),\ldots,\varphi_k(\sigma_{p_k}));$$

$\varphi_k(\sigma)=\alpha_1^{(k)}+\cdots+\alpha_{p_k}^{(k)}\sigma^{p_k-1}$, and $\sigma_1,\ldots,\sigma_{p_k}$ are distinct points. But if a polynomial of degree p_k-1 vanishes at p_k distinct points it is zero. Therefore $D_1,\ldots,D_N$ and hence D are invertible. For large n we thus obtain

$$\alpha_n = (D+M_n)^{-1}\beta_n = O(1)$$

by the Neumann series. This means that the $\alpha_j^{(k)}(n)$'s are all bounded, and hence that

$$A_j^{(k)}(n) = O([\rho_k(n)]^{-j+1}) \tag{2.14}$$

for $j=1,\ldots,p_k$, and $k=1,\ldots,N$. Moreover, since $\rho_k(n)$ could be replaced by $\rho_k^*(n)=o(\rho_k(n))$, we have for $j\geqq 2$,

$$A_j^{(k)}(n) = O([\rho_k^*(n)]^{-j+1}) = o([\rho_k(n)]^{-j+1}). \tag{2.15}$$

Let $\gamma_k^+(n)$ be the upper half of $|z-c_k(n)|=\rho_k(n)$, $\gamma_k^-(n)$ the lower half, both positively oriented, and $\gamma_k(n)=\gamma_k^+(n)+\gamma_k^-(n)$. By elementary estimates, using (2.4) and (2.15), we find that for any analytic function $\Phi(z)$ on D

$$\int_{\gamma_1^+(n)} \Phi(z)(z-c_k(n))^{-j}A_j^{(k)}(n)\{1+o(1)\}\,dz = o(1) \tag{2.16}$$

unless $k=j=1$, in which case, using (2.14),

$$\begin{aligned}\int_{\gamma_1^+(n)} &\Phi(z)(z-c_1(n))^{-1}A_1^{(1)}(n)\{1+o(1)\}\,dz\\ &= i\pi\Phi(c_1(n))A_1^{(1)}(n)+o(1)\\ &= \frac{1}{2}\int_{\gamma_1(n)} \Phi(z)(z-c_1(n))^{-1}A_1^{(1)}(n)\{1+o(1)\}\,dz\end{aligned} \tag{2.17}$$

since $\Phi(z)-\Phi(c_1(n))=o(1)$ uniformly on $\gamma_1^+(n)$.

Part II. Let us first observe that the reductions to minimal pole clusters could have been made for the poles of $Q_1^+(z,n)$ and $Q_1^-(z,n)$ simultaneously. We shall assume that this has been done. Of course, all the poles in a given cluster may or may not lie on the same side of the axis. Thus

$$J_n = \bigcup_{k=1}^{N} (c_k(n)-\rho_k(n),\ c_k(n)+\rho_k(n)).$$

Define

$$J_n^{\pm} = \pm(\gamma_1^{\pm}(n)+\cdots+\gamma_N^{\pm}(n))$$

where $\gamma_k^-(n)$ is the positively oriented *lower* half of $|z-c_k(n)|=\rho_k(n)$. Let $\varphi,\psi\in C_c^\infty(\boldsymbol{R})$ be functions which agree with polynomials on $D\cap\boldsymbol{R}$, and let $x,y\in\mathscr{D}$. It is easy to see that

$$\begin{aligned}(E_n^{(1)}&[J_n]\varphi(H_n)B_n^*x,\ \psi(H_n)A_n^*y)\\ &= \frac{1}{2\pi i}\int_{J_n^+} \varphi(z)\bar{\psi}(z)(Q_1^+(z,n)x,y)\,dz-\frac{1}{2\pi i}\int_{J_n^-} \varphi(z)\bar{\psi}(z)(Q_1^-(z,n)x,y)\,dz\end{aligned} \tag{2.18}$$

where $\varphi(z)$ and $\bar{\psi}(z)$ are the polynomials equal to $\varphi(\lambda)$ and $\bar{\psi}(\lambda)$ for real $\lambda\in D$. The terms in (2.18) corresponding to $L_n^{\pm}(z)$ are $o(1)$ since $L_n^{\pm}(z)$ is uniformly bounded. Hence, by (2.16) and (2.17) we find that the first integral of (2.18) is equal to

$$\frac{1}{2}\sum_{k=1}^{N}\varphi(c_k(n))\bar{\psi}(c_k(n))(A_1^{(k)}(n)x,y)+o(1) = \frac{1}{4\pi i}\int_{\partial D}\varphi(z)\bar{\psi}(z)\frac{(P_n^+(z)x,y)}{\Delta_n^+(z)}\,dz+o(1).$$

By (2.6), this converges to

$$\frac{1}{4\pi i}\int_{\partial D}\varphi(z)\bar{\psi}(z)\frac{(P_0B_0^*x,\ A_0^*y)}{z-\lambda_0}\,dz = \frac{\varphi(\lambda_0)\bar{\psi}(\lambda_0)}{2}(P_0B_0^*x,\ A_0^*y).$$

By a similar analysis, the second term of (2.18) converges to the same quantity, so that we finally obtain

$$\text{(2.19)} \qquad \lim_{n\to\infty} (E_n^{(1)}[J_n]\varphi(H_n)B_n^* x, \psi(H_n)A_n^* y) = (P_0\varphi(H_0)B_0^* x, \psi(H_0)A_0^* y)$$

for $x, y \in \mathscr{D}$, and φ, ψ as described.

Let $P_n = E_n^{(1)}[J_n]$, $x_n = \varphi(H_n)B_n^* x$ and $y_n = \psi(H_n)A_n^* y$. By Lemma 1.3 above and Lemma 5.1 of [4], $\varphi(H_n) \to \varphi(H_0)$ strongly, so that $x_n \to x_0$ strongly, and similarly $y_n \to y_0$ strongly. Hence,

$$(P_n x_0, y_0) - (P_0 x_0, y_0) = (P_n(x_0 - x_n), y_0) + (P_n x_n, y_0 - y_n) + (P_n x_n, y_n) - (P_0 x_0, y_0)$$

tends to zero, by (2.19). Moreover [6, §1] elements of the form x_0, and those of the form y_0 are dense in the reducing subspace $\mathscr{M}_0$, of T_0 and H_0 referred to in Theorem 1.2, so that

$$\text{(2.20)} \qquad (P_n x, y) \to (P_0 x, y), \qquad x, y \in \mathscr{M}_0.$$

Let $y \in \mathscr{M}_0^\perp$. Then for every $\varepsilon > 0$

$$|P_n y| \leqq |E_n^{(1)}(\lambda_0 + \varepsilon)y - E_n^{(1)}(\lambda_0 - \varepsilon)y|.$$

By [8, Theorem 1.15, p. 432]

$$\limsup |P_n y| \leqq |E_0^{(1)}(\lambda_0+\varepsilon)y - E_0^{(1)}(\lambda_0-\varepsilon)y| = |E_0(\lambda_0+\varepsilon)y - E_0(\lambda_0-\varepsilon)y|.$$

But this last expression vanishes as $\varepsilon \to 0+$, since λ_0 is not an eigenvalue of T_0. Thus

$$\text{(2.21)} \qquad P_n y \to P_0 y$$

strongly, for $y \in \mathscr{M}_0^\perp$. Finally, let $u = x + y$, $x \in \mathscr{M}_0$, $y \in \mathscr{M}_0^\perp$, be an arbitrary element of $\mathscr{H}$. Then

$$|P_n u|^2 = (P_n x, x) + (P_n y, x+y) + (x, P_n y).$$

The last two terms vanish by (2.21), so that by (2.20)

$$|P_n u|^2 \to |P_0 x|^2 = |P_0 u|^2$$

which implies that $P_n \to P_0$.

3. **Unperturbed operators of finite rank.** We are now in a position to improve the results in §5 of [5]. Let H_0 be a selfadjoint operator of finite rank r, T_1 a selfadjoint operator, and define $H_\varepsilon = H_0 + \varepsilon T_1$ for $\varepsilon \geqq 0$. If we take $T_\varepsilon = \varepsilon T_1$, and make the factorization $H_0 = H_0 P$, where P is the orthogonal projection on the range of H_0, then $G_\varepsilon(z) = \varepsilon^{-1} G_1(z/\varepsilon)$ so that $Q_\varepsilon(z) = \varepsilon^{-1}\Phi(z/\varepsilon)$ where $\Phi(z) = H_0 G_1(z) P$ is an operator on the finite-dimensional range of H_0. Thus, $I + Q_\varepsilon(z)$ is the Weinstein-Aronszajn matrix

$$W(z, \varepsilon) = I + H_0 G_\varepsilon(z) P = I + \varepsilon^{-1}\Phi(z/\varepsilon)$$

for the perturbation of εT_1 by H_0. We assume as usual that the smallest subspace reducing T_1 and containing the range of H_0 is the whole space $\mathscr{H}$.

If $\delta>0$ and $0<\alpha<\pi/2$, define $\Sigma(\alpha, \delta)$ to be the union of the two "sectors" $\{z : |z|>\delta \text{ and } |\arg z|<\alpha\}$ and $\{z : |z|>\delta \text{ and } |\arg z-\pi|<\alpha\}$. Our *Basic Hypothesis* [5, §1] is that $\Phi(z)$ has a continuation $\Phi_+(z)$ from $\operatorname{Im} z>0$ to some $\Sigma(\alpha_0, \delta_0)$. In order that our theory apply, we must show that as $\varepsilon\to 0+$,

$$\varepsilon^{-1}\Phi_+(z/\varepsilon)\to -z^{-1}H_0 = H_0G_0(z)P$$

uniformly on some neighborhood Ω of the nonzero eigenvalues $c_1, \ldots, c_r$ of H_0. This is shown to hold in Theorem 2 of [5], under the additional assumption that

$$\Phi_+(z)-\Phi_-(z) = O(z^{-2}) \tag{3.1}$$

uniformly on $\Sigma(\alpha_0, \delta_0)$, where $\Phi_-(z)$ is the continuation of $\Phi(z)$ to $\Sigma(\alpha_0, \delta_0)$ from $\operatorname{Im} z<0$. Ω is taken to be $\Sigma(\alpha_0, \delta_1)$ where $0<\delta_1<\min\{|c_1|, \ldots, |c_r|\}$. The other conditions of Hypothesis III are easily verified. This yields

3.1. THEOREM. *Under the hypotheses of Theorem* 2 *of* [5], *there is spectral concentration on the union of the intervals* $(\lambda_k(\varepsilon)-\delta_k(\varepsilon), \lambda_k(\varepsilon)+\delta_k(\varepsilon))$, *where* $z_k(\varepsilon)=\lambda_k(\varepsilon)-i\Gamma_k(\varepsilon)$ $(k=1, \ldots, r)$, $\Gamma_k(\varepsilon)=o(\delta_k(\varepsilon))$, *and* $\delta_k(\varepsilon)=o(1)$, *as* $\varepsilon\to 0+$.

This generalizes Theorem 5 of [5], in which we assumed that the perturbation εT_1 removed the degeneracy of H_0, and also that the left side of (3.1) was $O(z^{-3})$.

4. **Barrier penetration problems for ordinary differential operators.** We now wish to apply the theory to some problems of quantum mechanical "barrier penetration." According to Beck and Nussenzweig, "complex eigenvalues" were first introduced into quantum mechanics in connection with such problems, in the famous paper [2] of Gamov on alpha-particle decay. The usual textbook approach is via the WKB method.

As usually stated, the problem contains no natural small parameter, or unperturbed operator, so we will have to formulate it in a way in which concentration becomes meaningful. We shall therefore consider an operator

$$H_0 = -d^2/dx^2+q(x)+\beta^2$$

where $q(x)\in L_1$. H_0 will have continuous spectrum $[\beta^2, \infty)$ and perhaps some eigenvalues in $(-\infty, \beta^2)$. We shall assume that H_0 has an eigenvalue λ_0, $0<\lambda_0<\beta^2$. If we now "cut off" the potential $q(x)+\beta^2$ at a large distance R from the origin to obtain the operator

$$H_R = -d^2/dx^2+[q(x)+\beta^2]\chi_{[0,R]}(x)$$

then we find that H_R has continuous spectrum $[0, \infty)$ ($\chi_{[0,R]}$ is the characteristic function of $[0, R]$). The eigenvalue λ_0 has disappeared. However, if the barrier width R becomes large, we can expect concentration of the spectrum of H_R at λ_0. If we choose T_0 and T_R to be the operators obtained by setting $q(x)=0$ above, then

T_0 and T_R are computable and our method applies to show that there is concentration due to a virtual pole.

We shall consider both the one-dimensional Schroedinger operator on $(-\infty, +\infty)$ and the radial equation on $(0, \infty)$.

EXAMPLE 1. *The radial equation*, $l=0$. Let $q(x) \in L_1(0, \infty)$ be real valued and not identically zero, and $\beta>0$. Define

$$H_0 = -d^2/dx^2+q(x)+\beta^2 \tag{4.1}$$

and

$$T_0 = -d^2/dx^2+\beta^2. \tag{4.2}$$

For $R>0$, let $\chi_{[0,R]}(x)$ denote the characteristic function of $[0, R]$ and define

$$H_R = -d^2/dx^2+[\beta^2+q(x)]\chi_{[0,R]}(x)$$

and

$$T_R = -d^2/dx^2+\beta^2\chi_{[0,R]}(x).$$

All of these operators are given the boundary condition $u(0)=0$, and operate on $L_2(0, \infty)$. We shall choose A_0, B_0, A_R and B_R to be multiplication operators:

$$A_0 = |q(x)|^{1/2}, \qquad B_0 = \operatorname{sgn} q(x)|q(x)|^{1/2}, \tag{4.3}$$

$$A_R = |q(x)|^{1/2}\chi_{[0,R]}(x), \qquad B_R = \operatorname{sgn} q(x)|q(x)|^{1/2}\chi_{[0,R]}(x). \tag{4.4}$$

Note that if $q(x)$ has compact support, then for R sufficiently large $A_R=A_0$ and $B_R=B_0$.

4.1. THEOREM. *If λ_0 is an eigenvalue of H_0, $0<\lambda_0<\beta^2$, and $q(x) \in L_1(0, \infty)$, then as $R \to \infty$, the above operators satisfy the conditions of Theorem* 2.1.

We shall proceed to verify the conditions of Hypothesis III. First of all, $\mathscr{D}(T_R) =\mathscr{D}(T_0)=\mathscr{D}(d^2/dx^2)$, since T_R and T_0 differ from $-d^2/dx^2$ by bounded operators. But $\mathscr{D}(d^2/dx^2)$ contains only bounded functions (cf. [8, p. 301]). Thus $\mathscr{D}(A_R) =\mathscr{D}(B_R)\supseteq\mathscr{D}(T_R)$ for $R\geqq 0$, and Hypothesis I(a) clearly holds, since A_R and B_R are selfadjoint and commute.

Secondly, $\sigma(T_0)=[\beta^2, \infty)$, so that λ_0 cannot be an eigenvalue of T_0. Moreover, $\mathscr{D}(A_0)\subseteq\mathscr{D}(A_R)$ and $A_Rf \to A_0f$ strongly for $f \in \mathscr{D}(A_0)$ (Hypotheses III(c) and (e)). Since T_R-T_0 is bounded and converges strongly to zero, as $R \to \infty$, we have for $\operatorname{Im} z\neq 0$

$$\begin{aligned}\|G_R(z)f-G_0(z)f\| &= \|G_R(z)(T_0-T_R)G_0(z)f\| \\ &\leqq |\operatorname{Im} z|^{-1}\|(T_0-T_R)G_0(z)f\|\end{aligned}$$

for each $f \in L_2(0, \infty)$. If we show that for some nonreal z_0

$$\|A_RG_0(z_0)\| \leqq C \tag{4.5}$$

where C is independent of R, then for each $f \in L_2(0, \infty)$

$$A_R G_R(z_0) f - A_0 G_0(z_0) f = A_R G_0(z_0)(T_0 - T_R) G_R(z_0) f + (A_R - A_0) G_0(z_0) f$$

vanishes strongly as $R \to \infty$, for $G_0(z_0) f \in \mathscr{D}(A_R) \subseteq \mathscr{D}(A_0)$, and $A_R - A_0$ and $T_R - T_0$ vanish strongly. Hypothesis III(d) then follows, since $B_R = WA_R$, where $W = \operatorname{sgn} q(x)$ is bounded. To prove (4.5), note that [1, p. 1329] $(T_0 - z)^{-1}$ is an integral operator with Green's function kernel

$$\begin{aligned} G_0(x, y; k) &= -\kappa^{-1} \sinh \kappa x \, e^{-\kappa y}, \qquad y > x, \\ &= -\kappa^{-1} \sinh \kappa y \, e^{-\kappa x}, \qquad y < x. \end{aligned}$$

Therefore, if $k = i\eta$, $\eta > 0$, then $\kappa = (\beta^2 + \eta^2)^{1/2} > 0$ and

$$|G_0(x, y; k)| \leqq (2\kappa)^{-1} \exp(-\kappa |x - y|). \tag{4.6}$$

It follows that the Schmidt norm of the kernel

$$|q(x)|^{1/2} \chi_{[0,R]}(x) G_0(x, y; k)$$

does not exceed

$$(2\kappa)^{-1} \int_0^\infty \int_0^\infty |q(x)| \exp(-\kappa |x - y|) \, dx \, dy \leqq \kappa^{-2} \int_0^\infty |q(x)| \, dx$$

which is finite, since $q \in L_1(0, \infty)$. This proves (4.5). This estimate also shows that $A_0 G_0(-\eta^2)$ is compact, and hence by formula (1.1), that $R_0(z) - G_0(z)$ is compact. Therefore, $\sigma_e(H_0) = \sigma_e(T_0) = [\beta^2, \infty)$, where σ_e denotes essential spectrum, so that λ_0 is an *isolated* eigenvalue of finite (in fact, simple) multiplicity. This fact is well known, and can be deduced in a number of ways (cf. [3, §4] and [1, pp. 1593–1604]).

In general, if $\operatorname{Re} \kappa > 0$, (4.6) may be replaced by

$$|G_0(x, y; k)| \leqq (2|\kappa|)^{-1} \exp(-\operatorname{Re} \kappa |x - y|)$$

and it follows that $A_0 G_0(z) B_0$ is analytic in Schmidt norm in the z-plane, cut along the interval $[\beta^2, \infty) = \sigma(T_0)$.

Let $\beta^2 > \lambda_0 \in \sigma(T_0)$, and let P be the projection on $\ker(H_0 - \lambda_0)$. The range of P is contained in $\mathscr{D}(H_0) = \mathscr{D}(T_0) \subseteq \mathscr{D}(A_0) = \mathscr{D}(B_0)$, and the residue of $A_0 R_0(z) B_0$ is $AP[BP]^*$ [6, Theorem 3.1]. If $AP[BP]^* = 0$, then multiplying by $\operatorname{sgn} q(x)$, we obtain $BP[BP]^* = 0$, and hence $BP = 0$. Thus if $\varphi \in \ker(H_0 - \lambda_0)$, we have $B\varphi = 0$, or, after multiplying by $|q(x)|^{1/2}$, $q(x)\varphi(x) = 0$ a.e. But $\varphi(x)$ is continuously differentiable, so if $q(x) \neq 0$ on a set of positive measure, then $\varphi(x)$ vanishes on a set of positive measure. But such a set has an accumulation point x_0, and hence $\varphi(x_0) = \varphi'(x_0) = 0$. It follows that $\varphi(x)$ vanishes identically, since φ satisfies

$$-\varphi''(x) + q(x)\varphi(x) = (\lambda_0 - \beta^2)\varphi(x).$$

Therefore, if λ_0 is a nontrivial eigenvalue of H_0, $A_0 R_0(z) B_0$ has a *nontrivial* pole at λ_0, unless $q(x)$ vanishes identically.

It remains to consider $Q_R(z) = A_R G_R(z) B_R$. For this, we shall require the Green's function kernel for $G_R(z) = (T_R - z)^{-1}$. This is given by [1, p. 1329].

$$(4.7) \qquad \begin{aligned} G_R(x, y; k) &= [W(k)]^{-1}\psi_+(x, k)\varphi(y, k), \qquad y < x, \\ &= [W(k)]^{-1}\psi_+(y, k)\varphi(x, k), \qquad y > x, \end{aligned}$$

where $z = k^2$, Im $k > 0$. $\psi_+(x, k)$ and $\varphi(x, k)$ are the solutions of

$$(4.8) \qquad -u''(x) + \beta^2 \chi_{[0,R]}(x) u(x) = k^2 u(x)$$

such that $\varphi(0, k) = 0$ and $\psi_+(x, k)$ is square-integrable at infinity, while

$$W(k) = \begin{vmatrix} \varphi(x, k) & \psi_+(x, k) \\ \varphi'(x, k) & \psi'_+(x, k) \end{vmatrix}$$

is the Wronskian of ψ_+ and φ, which is independent of x. Elementary calculations yield

$$\begin{aligned} \psi_+(x, k) &= ae^{-\kappa x} + be^{\kappa x}, \qquad 0 \leqq x \leqq R, \\ &= e^{ikx}, \qquad R \leqq x, \end{aligned}$$

where $\kappa^2 = \beta^2 - k^2$, Im $\kappa > 0$ and

$$(4.9) \qquad a = (2\kappa)^{-1}(\kappa - ik)e^{\kappa R}e^{ikR}$$

and

$$(4.10) \qquad b = (2\kappa)^{-1}(\kappa + ik)e^{-\kappa R}e^{ikR}$$

while

$$\begin{aligned} \varphi(x, k) &= \sinh \kappa x, \qquad 0 \leqq x \leqq R, \\ &= ce^{ikx} + de^{-ikx}, \qquad R \leqq x, \end{aligned}$$

where

$$c = (4ik)^{-1}e^{-ikR}[(\kappa + ik)e^{\kappa R} + (\kappa - ik)e^{-\kappa R}]$$

and

$$d = -(4ik)^{-1}e^{ikR}[(\kappa - ik)e^{\kappa R} + (\kappa + ik)e^{-\kappa R}].$$

Computing the Wronskian for $x = 0$, we obtain $W(k) = -\kappa(a + b)$.

Observe now that (4.7) and the formulas following it provide a continuation of $G_R(x, y; k)$ across the interval $(0, \beta)$ to the lower half of the k-plane. Let Ω be a bounded neighborhood of λ_0, $0 < \lambda_0 < \beta$, which is bounded away from $k = \beta$ and $k = 0$. Since κ is real and positive on $(-\infty, \beta)$ (and pure imaginary on (β, ∞)), we may choose Ω such that Re κ is positive and bounded away from zero on Ω. Let $k \in \Omega$ and $0 \leqq x \leqq R$; then

$$[W(k)]^{-1}\psi_+(x, k) = O(e^{-\kappa x}).$$

This may be seen by multiplying both $W(k)$ and $\psi_+(x, k)$ by $e^{-\kappa R}e^{-ikR}$. Then $e^{-\kappa R}e^{-ikR}W(k)$ is bounded, and bounded away from zero, and $e^{-\kappa R}e^{-ikR}\psi_+(x, k) = O(e^{-\kappa x})$. Since $\sinh \kappa x = O(e^{\kappa x})$, we have that for $0 \leqq y < x \leqq R$ and $k \in \Omega$,

$$G_R(x, y; k) = O(e^{-\kappa x}e^{\kappa y}) = O(1),$$

and similarly for $0 \leqq x < y \leqq R$. Hence

$$\sup_{0 \leqq x, y \leqq R} |G_R(x, y; k)| \leqq C, \qquad k \in \Omega,$$

where C is independent of R and $k \in \Omega$.

Now, for $k \in \Omega$, define $Q^+(k^2)$ to be the operator with kernel

$$\chi_{[0,R]}(x)|q(x)|^{1/2}G_R(x, y; k)|q(y)|^{1/2} \operatorname{sgn} q(y)\chi_{[0,R]}(y).$$

This kernel is analytic in k, and we have

$$\begin{aligned} \|Q^+(k^2)\|_2^2 &= \int_0^R \int_0^R |q(x)|\, |G_R(x, y)|^2 |q(y)|\, dx\, dy \\ &\leqq C^2\left(\int_0^\infty |q(x)|\, dx\right)^2 \end{aligned}$$

for the Schmidt norm of $Q^+(k^2)$. It follows that $Q^+(k^2)$ is a compact, analytic extension of $Q(k^2)$ to a neighborhood of λ_0. Thus Hypothesis II holds. Hypothesis I(b) is verified similarly. For Hypothesis III(a), note that $\chi_{[0,R]}(x)G_R(x, y; k)\chi_{[0,R]}(y)$ converges pointwise to $G_0(x, y; k)$ and is uniformly bounded. By the above estimates $Q_R^+(k^2) \to Q_0^+(k^2)$ in Schmidt norm. This completes the verification of the conditions.

EXAMPLE 2. *The one-dimensional Schroedinger equation.* Let $q(x) \in L_1(-\infty, +\infty)$ be real valued and not identically zero, and $\beta > 0$. Define H_0 and T_0 by (4.1) and (4.2), and let

$$H_R = -d^2/dx^2 + [\beta^2 + q(x)]\chi_{[-R,+R]}(x)$$

and

$$T_R = -d^2/dx^2 + \beta^2\chi_{[-R,+R]}(x)$$

for $R > 0$. In this case, there are no boundary conditions. A_0, B_0, A_R and B_R are given by (4.3)–(4.4).

4.2. THEOREM. *If λ_0 is an eigenvalue of H_0, $0 < \lambda_0 < \beta^2$, and $q(x) \in L_1(-\infty, +\infty)$, then as $R \to \infty$, the above operators satisfy the conditions of Theorem* 2.1.

The proof is essentially the same as that of Theorem 4.1. We shall simply note the appropriate formulas for G_0 and G_R.

Let $\zeta = k^2$, $\operatorname{Im} k > 0$ and $\kappa^2 = \beta^2 - k^2$, $\operatorname{Re} \kappa > 0$. Then

$$G_0(x, y; k) = -\kappa^{-1} \exp(-\kappa|x-y|)$$

while

$$\begin{aligned} G_R(x, y; k) &= [W(k)]^{-1}\psi_+(x, k)\psi_-(y, k), \qquad y < x, \\ &= [W(k)]^{-1}\psi_+(y, k)\psi_-(x, k), \qquad y > x, \end{aligned}$$

where $W(k)$ is the Wronskian of ψ_+ and ψ_- and $\psi_\pm(x, k)$ satisfies

$$-\psi''_\pm(x)+\beta^2\chi_{[-R,+R]}(x)\psi_\pm(x) = k^2\psi_\pm(x)$$

and is square-integrable at $\pm\infty$. By symmetry, $\psi_-(x)=\psi_+(-x)$, and ψ_+ is the same as in the preceding example:

$$\begin{aligned}\psi_+(x, k) &= ae^{-\kappa x}+be^{\kappa x}, & -R \leqq x \leqq +R,\\ &= e^{ikx}, & R \leqq x,\end{aligned}$$

where a and b are given by (4.9) and (4.10). We do not need $\psi_+(x, k)$ for $x \leqq -R$. We may now compute

$$G_R(x, y; k) = [2\kappa(b^2-a^2)]^{-1}[ae^{-\kappa x}+be^{+\kappa x}][ae^{+\kappa y}+be^{-\kappa y}]$$

for $-R \leqq y < x \leqq R$. It now follows easily that $G_R(x, y; k)$ is bounded on $[-R, R] \times [-R, R]$ uniformly in R, and that $G_R(x, y; k) \to G_0(x, y; k)$ as $R \to +\infty$.

Notes Added in Proof. (1) In Theorem 2.1, the poles of $Q_1^-(z, n)$ are the complex conjugates of the poles of $Q_1^+(z, n)$, at least when either A or B is bounded, and perhaps in general. Thus, in this case, $J_k^+(n)$ and $J_k^-(n)$ are the same if $\delta_k^+(n) = \delta_k^-(n)$.

(2) The obscure formula just following (2.13) signifies that the action of D_k on the vector $(\alpha_1^{(k)}, \ldots, \alpha_{p_k}^{(k)})$ produces a vector whose components are the values of the polynomial φ_k at the points $\sigma_1, \ldots, \sigma_{p_k}$.

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University of Virginia, Charlottesville, Virginia 22903