TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 163, January 1972

THE SIGN OF LOMMEL'S FUNCTION

BY
J. STEINIG

Abstract. Lommel's function $s_{\mu,\nu}(x)$ is a particular solution of the differential equation $x^2y''+xy'+(x^2-\nu^2)y=x^{\mu+1}$. It is shown here that $s_{\mu,\nu}(x)>0$ for $x>0$, if $\mu=\frac{1}{2}$ and $|\nu|<\frac{1}{2}$, or if $\mu>\frac{1}{2}$ and $|\nu|\leqq\mu$. This includes earlier results of R. G. Cooke's. The sign of $s_{\mu,\nu}(x)$ for other values of μ and ν is also discussed.

1. **Introduction.** In 1876, Lommel [6] considered the inhomogeneous differential equation

$$z^2y''+zy'+(z^2-\nu^2)y = z^{\mu+1}, \tag{1}$$

where μ and ν are complex parameters. He obtained two particular solutions: the Lommel functions of the first kind, $s_{\mu,\nu}(z)$, and of the second kind, $S_{\mu,\nu}(z)$. The homogeneous equation associated with (1) is

$$z^2y''+zy'+(z^2-\nu^2)y = 0, \tag{2}$$

Bessel's equation.

The function $s_{\mu,\nu}(z)$ is defined for all pairs μ, ν such that neither $\mu-\nu$ nor $\mu+\nu$ is an odd negative integer, and for all z with $-\pi<\arg z\leqq\pi$, by the series

$$s_{\mu,\nu}(z) = \tfrac{1}{4}z^{\mu+1}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2}z)^{2n}\Gamma((\mu-\nu+1)/2)\Gamma((\mu+\nu+1)/2)}{\Gamma((\mu-\nu+n+3)/2)\Gamma((\mu+\nu+n+3)/2)}. \tag{3}$$

The symmetry property

$$s_{\mu,\nu}(z) = s_{\mu,-\nu}(z) \tag{4}$$

is obvious from (3).

We shall consider $s_{\mu,\nu}(z)$ for μ and ν real, and positive z. Its importance arises from the formula [1, §3.20], [9, §10.74]

$$\int x^{\mu}C_\nu(x)\,dx = x[C_\nu(x)s'_{\mu,\nu}(x)-s_{\mu,\nu}(x)C'_\nu(x)], \tag{5}$$

in which $C_\nu(x)$ denotes any real solution of equation (2), that is,

$$C_\nu(x) = \alpha J_\nu(x)+\beta Y_\nu(x),$$

where ν, α and β are real, $x>0$, and $J_\nu(x)$ and $Y_\nu(x)$ denote the usual Bessel functions.

Presented to the Society, November 28, 1970; received by the editors December 22, 1970.

AMS 1970 *subject classifications*. Primary 33A70, 33A40; Secondary 26A33, 34C10, 44A20.

Key words and phrases. Lommel functions, Bessel functions, changes of sign, oscillation theorems, inhomogeneous Bessel equation.

A case of particular interest is that in which $\mu=\nu$. Then, unless ν is half of an odd negative integer, we have

(6) $$s_{\nu,\nu}(x) = 2^{\nu-1}\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})\boldsymbol{H}_\nu(x),$$

where $\boldsymbol{H}_\nu(x)$ is Struve's function of order ν [1, (3.127)]. Now it is known [9, §10.45] that $\boldsymbol{H}_\nu(x)>0$ for all $x>0$ if $\nu>\frac{1}{2}$, that $\boldsymbol{H}_{1/2}(x)=(2/\pi x)^{1/2}(1-\cos x)$, and that $\boldsymbol{H}_\nu(x)$ has an infinity of changes of sign if $\nu<\frac{1}{2}$. The corresponding problem for $s_{\mu,\nu}(x)$ is more difficult. R. G. Cooke [4] found conditions on μ and ν sufficient to ensure that $s_{\mu,\nu}(x)>0$ for all $x>0$. We may state his results as

THEOREM A. *If $\nu\geqq 0$ and $\mu\geqq\nu+1$, then $s_{\mu,\nu}(x)>0$ for $x>0$. If $\nu\geqq\frac{1}{2}$ and $\mu\geqq\nu$, then $s_{\mu,\nu}(x)>0$ for $x>0$, unless $\mu=\nu=\frac{1}{2}$, when $s_{\mu,\nu}(x)\geqq 0$ for $x>0$.*

The symmetry relation (4) gives the corresponding results for $\nu<0$.

Cooke's proof requires an expression for $s_{\mu,\nu}(x)$ as a fractional integral involving $J_\nu(x)$, and a previous result of his ([2] and [3]) which implies that if $\mu=0$ and $\nu>-1$, or $\mu=1-\nu$ and $\nu>\frac{1}{2}$, then

$$\int_0^\xi t^\mu J_\nu(t)\,dt > 0, \qquad \xi > 0.$$

We shall here consider the same problem. It will often be convenient to refer to a (μ,ν)-plane, and to associate with each pair μ, ν the point with those coordinates.

We shall see that $s_{\mu,\nu}(x)>0$ for $x>0$ if $\mu\geqq\frac{1}{2}$ and $|\nu|\leqq\mu$, except when $\mu=|\nu|=\frac{1}{2}$, in which case $s_{\mu,\nu}(x)\geqq 0$. These inequalities define a larger region in the (μ,ν)-plane than Cooke's. If $\mu<\frac{1}{2}$, or if $\mu=\frac{1}{2}$ and $|\nu|>\frac{1}{2}$, then $s_{\mu,\nu}(x)$ changes sign infinitely often on $(0,\infty)$. If $\mu>\frac{1}{2}$ and $\mu<|\nu|-1$, $s_{\mu,\nu}(x)$ has an odd number of changes of sign on $(0,\infty)$. Finally, if $\mu>\frac{1}{2}$ and $|\nu|-1<\mu<|\nu|$, $s_{\mu,\nu}(x)$ has an even number of changes of sign on $(0,\infty)$; but I have not been able to decide whether this number is always positive. I shall show, however, that there are points (μ,ν) in this region such that the corresponding $s_{\mu,\nu}(x)$ has changes of sign (an arbitrarily large number of them, in fact).

Our main tool is an oscillation theorem of Makai's [7] for second order differential equations. With it, we can determine the sign of the function

$$h(\xi) = \int_0^\xi t^\mu C_\nu(t)\,dt, \qquad \xi > 0,$$

for certain pairs μ, ν. Then, by using (5), we can deduce results on the sign of $s_{\mu,\nu}(x)$.

2. **Oscillation theorems.** Our starting point is

THEOREM B. *Let $y=y(x)$ be a solution of the differential equation $y''+\varphi(x)y=0$, where φ is continuous and increasing on (x_0,x_2). Let x_1 be the only zero of $y(x)$ on (x_0,x_2). Further, assume that*

(7) $$\lim_{x\to x_0+0} y(x) = \lim_{x\to x_2-0} y(x) = 0.$$

Then,

$$\int_{x_0}^{x_1} |y(x)|\,dx \geqq \int_{x_1}^{x_2} |y(x)|\,dx, \tag{8}$$

with strict inequality if φ *is strictly increasing.*

If φ *is decreasing on* (x_0, x_2), *then* (8) *is reversed.*

Theorem B is due essentially to E. Makai [7], [8, §1.82], but the endpoint conditions (7) have been introduced to avoid difficulties which may arise when φ is discontinuous at x_0, or at x_2. The monotonicity condition on φ could be relaxed somewhat [7, §2(d)], but we will not need this here.

From Theorem B we deduce

THEOREM 1. *Let* $y=y(x)$ *be a solution of the differential equation*

$$(r(x)y')'+p(x)y = 0, \tag{9}$$

where r and p are continuous, pr is increasing and r is positive, on (x_0, x_2). *Assume further that* $\int_{x_0}^{x} (r(u))^{-1}\,du$ *converges. Let*

$$\lim_{x\to x_0+0} y(x) = \lim_{x\to x_2-0} y(x) = 0, \tag{10}$$

and let x_1 *be the only zero of y on* (x_0, x_2). *Then,*

$$\int_{x_0}^{x_1} \frac{|y(x)|}{r(x)}\,dx \geqq \int_{x_1}^{x_2} \frac{|y(x)|}{r(x)}\,dx, \tag{11}$$

with strict inequality if pr is strictly increasing. And if pr is decreasing (11) *is reversed.*

Proof. We transform the independent variable in (9) by setting

$$t = f(x) = \int_{x_0}^{x} \frac{du}{r(u)}.$$

Equation (9) can then be written as

$$d^2y/dt^2+r(g(t))p(g(t))y = 0, \tag{12}$$

where g denotes the inverse of the (increasing) function f [5, p. 235]. Inequality (11) now follows by applying Theorem B to (12), and then changing the independent variable back to x.

In the sequel, we shall consider the equation

$$(x^{1-2\mu}y')'+x^{-2\mu-1}(x^2+\mu^2-\nu^2)y = 0, \tag{13}$$

which is of the form (9), with

$$(p(x)r(x))' = 2x^{-4\mu-1}[(1-2\mu)x^2+2\mu(\nu^2-\mu^2)]. \tag{14}$$

The integrand in (5), $x^\mu C_\nu(x)$, is a solution of equation (13) [9, §4.31, (19)-(20)]. We shall often choose $\mu=\frac{1}{2}$; (13) then becomes

$$y''+(1+(1-4\nu^2)/4x^2)y = 0. \tag{15}$$

3. **Lommel functions.** The two Lommel functions $s_{\mu,\nu}(x)$ and $S_{\mu,\nu}(x)$ are related by the identity

$$(16)\qquad \begin{aligned} s_{\mu,\nu}(x) &= S_{\mu,\nu}(x)+2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\\ &\quad\cdot\left[\cos\frac{(\mu-\nu)\pi}{2}\,Y_\nu(x)-\sin\frac{(\mu-\nu)\pi}{2}\,J_\nu(x)\right], \end{aligned}$$

whenever $s_{\mu,\nu}(x)$ is defined [9, §10.71].

Now it can be shown [9, §10.75] that $S_{\mu,\nu}(x)\sim x^{\mu-1}$, as $x\to+\infty$. On combining this with the relations

$$J_\nu(x) = (2/\pi x)^{1/2}[\cos(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+O(1/x)]$$

and

$$Y_\nu(x) = (2/\pi x)^{1/2}[\sin(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+O(1/x)]$$

[9, §7.22], we see that as $x\to+\infty$, the dominant term on the right-hand side of (16) is

$$2^{\mu-1}\left(\frac{2}{\pi x}\right)^{1/2}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)\sin(x-\tfrac{1}{2}\mu\pi-\tfrac{1}{4}\pi)\quad\text{or}\quad x^{\mu-1},$$

according as $\mu<\frac{1}{2}$ or $\mu>\frac{1}{2}$. It follows that $s_{\mu,\nu}(x)$ has an infinity of changes of sign if $\mu<\frac{1}{2}$, but is positive for all sufficiently large x, if $\mu>\frac{1}{2}$. We now proceed to make this simple observation more precise. We begin by establishing

THEOREM 2. *If $\mu<\frac{1}{2}$, or if $\mu=\frac{1}{2}$ and $|\nu|>\frac{1}{2}$, then $s_{\mu,\nu}(x)$ has an infinity of changes of sign on* $(0,\infty)$.

Proof. For $\mu<\frac{1}{2}$ and ν unrestricted, the result follows as above. For $\mu=\frac{1}{2}$ and $|\nu|>\frac{1}{2}$, we apply Theorem B to the particular solution $y(x)=x^{1/2}J_\nu(x)$ of equation (15). It is clear from (14) that pr is increasing on $(0,\infty)$ for such pairs μ, ν. Now let $j_{\nu,k}$ denote the kth positive zero of $J_\nu(x)$. In (7), we may take $x_0=j_{\nu,k}$, for any k. And we may also take $x_0=0$, since $J_\nu(x)=O(x^\nu)$ as $x\to 0+$. Since $J_\nu(x)>0$ for $0<x<j_{\nu,1}$ if $\nu>-1$, it follows from (8) that

$$\int_0^\xi x^{1/2}J_\nu(x)\,dx > 0,\qquad \xi>0,\ \nu>\tfrac{1}{2}.$$

Together with (5), this shows that $J_\nu(\xi)s'_{1/2,\nu}(\xi)-s_{1/2,\nu}(\xi)J'_\nu(\xi)>0$, for the right-hand side of (5) vanishes at $x=0$ if $\beta=0$ and $\mu+\nu+1>0$, by (3). In particular,

$$s_{1/2,\nu}(j_{\nu,k})J'_\nu(j_{\nu,k}) < 0,\qquad k=1,2,\ldots,\ \nu>\tfrac{1}{2}.$$

But $\operatorname{sgn} J'_\nu(j_{\nu,k})=(-1)^k$ for $\nu\geqq 0$; thus $s_{1/2,\nu}(x)$ must have an odd number of changes of sign between consecutive positive zeros of $J_\nu(x)$, if $\nu>\frac{1}{2}$. Because of (4), this is also true if $\nu<-\frac{1}{2}$, and the proof is complete.

Next, we turn to the case $\mu\geqq\frac{1}{2}$, and prove

THEOREM 3. *If* $\mu \geqq \frac{1}{2}$ *and* $|\nu| \leqq \mu$, *then* $s_{\mu,\nu}(x) > 0$ *for all* $x > 0$, *except if* $\mu = |\nu| = \frac{1}{2}$, *when* $s_{\mu,\nu}(x) \geqq 0$.

Proof. For $\mu = |\nu| = \frac{1}{2}$ we have, from (6),

$$s_{1/2,1/2}(x) = s_{1/2,-1/2}(x) = x^{-1/2}(1-\cos x). \tag{17}$$

For $\mu = \frac{1}{2}$ and $|\nu| < \frac{1}{2}$ we shall use Theorem B and (5). From (3), and familiar facts about the behavior of $J_\nu(x)$ and $Y_\nu(x)$ as $x \to 0+$, it is easily seen that we may take 0 as lower limit of integration in (5), and that the right-hand side of (5) vanishes at $x=0$, if $\mu > |\nu| - 1$. Hence, for $\mu > |\nu| - 1$ and $\xi > 0$, we have

$$\int_0^\xi x^\mu C_\nu(x)\, dx = \xi[C_\nu(\xi)s'_{\mu,\nu}(\xi) - s_{\mu,\nu}(\xi)C'_\nu(\xi)].$$

The particular choice $\xi = c_{\nu,k}$, the kth positive zero of $C_\nu(x)$, yields

$$\int_0^{c_{\nu,k}} x^\mu C_\nu(x)\, dx = -c_{\nu,k}s_{\mu,\nu}(c_{\nu,k})C'_\nu(c_{\nu,k}). \tag{18}$$

Now $y(x) = x^\mu C_\nu(x)$ is a solution of (13). Since $\lim_{x\to 0+} x^\mu C_\nu(x) = 0$ if $\mu > |\nu|$, (10) is satisfied by $x_0 = 0$. Moreover, it is clear from (14) that pr is strictly decreasing on $(0, \infty)$, if $\mu = \frac{1}{2}$ and $|\nu| < \frac{1}{2}$.

Thus, the hypotheses of Theorem B are satisfied by $y(x) = x^{1/2}C_\nu(x)$ on each of the intervals $(0, c_{\nu,2})$ and $(c_{\nu,k}, c_{\nu,k+2})$, $k \geqq 1$, if $|\nu| < \frac{1}{2}$. It follows that

$$\left\{\int_0^{c_{\nu,k}} x^{1/2}C_\nu(x)\, dx\right\}\left\{\int_0^{c_{\nu,k+1}} x^{1/2}C_\nu(x)\, dx\right\} < 0, \qquad k \geqq 1,\ |\nu| < \tfrac{1}{2}. \tag{19}$$

Since $C'_\nu(c_{\nu,k})C'_\nu(c_{\nu,k+1}) < 0$ [9, §15.21], it follows from (18) and (19) that

$$s_{1/2,\nu}(c_{\nu,k}) > 0, \qquad k \geqq 1,\ |\nu| < \tfrac{1}{2}. \tag{20}$$

But $x^{1/2}C_\nu(x)$ is an *arbitrary* nonnull solution of (15). Therefore, (20) implies that $s_{1/2,\nu}(x) > 0$ for all $x > 0$ if $|\nu| < \frac{1}{2}$, since any $x > 0$ is a zero of some nonnull solution of (15). Together with (17), this proves Theorem 3 for $\mu = \frac{1}{2}$. And if we use the integrals

$$s_{\mu+\sigma,\nu+\sigma}(x) = 2x^\sigma \frac{\Gamma((\mu+\nu+2\sigma+1)/2)}{\Gamma(\sigma)\Gamma((\mu+\nu+1)/2)} \int_0^{\pi/2} s_{\mu,\nu}(x \sin\theta) \sin^{\nu+1}\theta \cos^{2\sigma-1}\theta\, d\theta,$$

$$s_{\mu+\sigma,\nu-\sigma}(x) = 2x^\sigma \frac{\Gamma((\mu+\nu+2\sigma+1)/2)}{\Gamma(\sigma)\Gamma((\mu-\nu+1)/2)} \int_0^{\pi/2} s_{\mu,\nu}(x \sin\theta) \sin^{1-\nu}\theta \cos^{2\sigma-1}\theta\, d\theta,$$

both valid for $\sigma > 0$ and $\mu \pm \nu > -1$ [1, §3.20], the theorem's truth for $\mu > \frac{1}{2}$ is an immediate consequence of its truth for $\mu = \frac{1}{2}$.

An intermediate situation between those of the last two theorems is described by

THEOREM 4. *Let* $\mu > \frac{1}{2}$. *If* $\mu < |\nu| - 1$, *then* $s_{\mu,\nu}(x)$ *has an odd number of changes of sign on* $(0, \infty)$. *If* $|\nu| - 1 < \mu < |\nu|$, *then* $s_{\mu,\nu}(x)$ *has an even number of changes of sign (perhaps none) on* $(0, \infty)$.

Proof. From (3) we have $s_{\mu,\nu}(x) \sim x^{\mu+1}/((\mu+1)^2-\nu^2)$, as $x \to 0+$. On the other hand, it follows from our discussion of (16) that when $\mu > \frac{1}{2}$, $s_{\mu,\nu}(x) \sim x^{\mu-1}$, as $x \to +\infty$. The conclusion is now obvious.

I have not been able to decide whether the even number alluded to in Theorem 4 is in fact always positive. However, I can show that if the point (μ, ν), with $\mu > \frac{1}{2}$ and $|\nu|-1 < \mu < |\nu|$, is close enough to the line $\mu = \frac{1}{2}$, or to one of the lines $\mu = |\nu|-1$, then the corresponding $s_{\mu,\nu}(x)$ changes sign. In fact, by choosing the point (μ, ν) close enough to the line $\mu = \frac{1}{2}$ in the region $\mu > \frac{1}{2}$, $|\nu|-1 < \mu < |\nu|$, we can find Lommel functions with an arbitrarily large number of changes of sign.

This last statement can be verified as follows. We may assume $\nu > 0$. By applying Theorem 1 to (13), we see that if

$$(1-2\mu)x^2 + 2\mu(\nu^2-\mu^2) > 0 \quad \text{for } 0 < x < \bar{x}, \tag{21}$$

and if j is the largest zero of $J_\nu(x)$ in $(0, \bar{x}]$, then

$$\int_0^x t^{3\mu-1} J_\nu(t)\, dt > 0 \quad \text{for } 0 < x \leqq j,$$

since (10) is satisfied by $y(x) = x^\mu J_\nu(x)$ and $x_0 = 0$, if $\mu + \nu > 0$. For $\mu \geqq \frac{1}{2}$, this implies that

$$\int_0^x t^\mu J_\nu(t)\, dt > 0 \quad \text{for } 0 < x \leqq j, \tag{22}$$

by the second mean value theorem.

Thus, if $(0, \bar{x}]$ contains at least two positive zeros of $J_\nu(t)$, then (22) holds with $j = j_{\nu,2}$, and hence also with $j = j_{\nu,3}$. As before, if $\mu > \nu - 1$, this will imply that $s_{\mu,\nu}(x)$ changes sign on $(j_{\nu,1}, j_{\nu,2})$ and on $(j_{\nu,2}, j_{\nu,3})$. Now for $\mu > \frac{1}{2}$, (21) holds if and only if $\mu < |\nu|$ and $|\bar{x}| \leqq (2\mu(\nu^2-\mu^2)/(2\mu-1))^{1/2}$. Therefore, $s_{\mu,\nu}(x)$ will change sign if $\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$, and

$$j_{\nu,2} \leqq (2\mu(\nu^2-\mu^2)/(2\mu-1))^{1/2}. \tag{23}$$

Now if we choose some $\nu > \frac{1}{2}$, and keep it fixed, then the left-hand side of (23) is fixed. But the right-hand side tends to $+\infty$, as $\mu \to \frac{1}{2}+0$. Hence for each ν, $\frac{1}{2} < \nu < \frac{3}{2}$, we can find a μ ($\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$) such that $s_{\mu,\nu}(x)$ has at least two changes of sign. In order to produce a Lommel function with at least $2n$ changes of sign, it suffices to satisfy the inequality obtained by writing $j_{\nu,2n}$ for $j_{\nu,2}$ in (23).

We now turn our attention to points near the line $\mu = \nu - 1$, in the region $\mu > \frac{1}{2}$, $\nu - 1 < \mu < \nu$. Here, we use (16). We fix $\nu \geqq \frac{3}{2}$, and choose $x^* > 0$ such that $J_\nu(x^*) < 0$. Then, we let $\mu \to (\nu-1)+0$. Now $S_{\mu,\nu}(x^*)$ tends to a limit, as $\mu \to \nu - 1$ [9, §10.73]. And

$$\lim_{\mu \to \nu-1} \left[\cos\frac{(\mu-\nu)\pi}{2}\, Y_\nu(x^*) - \sin\frac{(\mu-\nu)\pi}{2}\, J_\nu(x^*)\right] = J_\nu(x^*).$$

Hence, for $x=x^*$, the right-hand side of (16) tends to $-\infty$ as $\mu \to (\nu-1)+0$, because of the factor $\Gamma((\mu-\nu+1)/2)$. Therefore, $s_{\mu,\nu}(x^*)<0$ for an appropriate choice of μ.

References

1. A. W. Babister, *Transcendental functions satisfying nonhomogeneous linear differential equations*, Macmillan, New York, 1967. MR **34** #6158.
2. R. G. Cooke, *Gibbs's phenomenon in Fourier-Bessel series and integrals*, Proc. London Math. Soc. (2) **27** (1928), 171–192.
3. ———, *A monotonic property of Bessel functions*, J. London Math. Soc. **12** (1937), 180–185.
4. ———, *On the sign of Lommel's function*, J. London Math. Soc. **7** (1932), 281–283.
5. W. Leighton, *Ordinary differential equations*, 3rd ed., Wadsworth, Belmont, Calif., 1970.
6. E. Lommel, *Ueber eine mit den Besselschen Functionen verwandte Function*, Math. Ann. **9** (1876), 425–444.
7. E. Makai, *On a monotonic property of certain Sturm-Liouville functions*, Acta Math. Acad. Sci. Hungar. **3** (1952), 165–172. MR **14**, 872.
8. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1959. MR **21** #5029.
9. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, Cambridge; Macmillan, New York, 1944. MR **6**, 64.

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Current address: Institut Mathématique, Université de Genève, Genève, Switzerland