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HOMOLOGY INVARIANTS OF CYCLIC COVERINGS WITH APPLICATION TO LINKS

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Abstract. The main purpose of this paper is to study the homology of cyclic covering spaces of a codimension two link. The integral (rational) homology groups of an infinite cyclic cover of a finite complex can be considered as finitely generated modules over the integral (rational) group ring of the integers. We first describe the properties of the invariants of these modules for certain finite complexes related to the complementary space of links. We apply this result to the homology invariants of the infinite cyclic cover of a higher dimensional link. Further, we show that the homology invariants of the infinite cyclic cover detect geometric splittability of a link. Finally, we study the homology of finite unbranched and branched cyclic covers of a link.

1. **Introduction.** If K is a finite simplicial complex with nonzero first Betti number, then K has, among others, an infinite cyclic covering space. The integral (rational) homology groups of this cover are finitely generated modules over the integral (rational) group ring of the infinite cyclic group. Invariants of the module structure of the homology can be defined in each case, and as one would suspect, these invariants are intimately related. In §2 we develop the general relationship between the integral and rational invariants, and study the properties of the invariants for a certain class of finite complexes related to links. Some applications to linear graphs and the homology of subgroups of the fundamental group related to the infinite cyclic covers are also given.

In §3, we define the invariants of a higher-dimensional link, and show that these invariants, except for the 1-dimensional ones, are similar to those of a higher-dimensional knot.

In §4, we consider geometric splittability of a higher-dimensional link, and give some examples of nonsplittable links which are detectable by the homology invariants.

Finally, in §5, we study the homology groups of the k-fold unbranched and branched cyclic covers of links, particularly the case of q-simple links.

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2. **Polynomial invariants of infinite cyclic coverings.** Let K be an $(n+1)$-dimensional finite simplicial complex with homology groups

$$
(2.1)\qquad \begin{aligned} H_i(K; Z) &= Z, && i = 0, \\ &= \alpha Z, && i = 1, \\ &= \beta Z, && i = n+1, \\ &= 0, && \text{otherwise,} \end{aligned}
$$

where αZ (βZ) denotes the direct sum of α (β) copies of Z. Let $\tilde{K}$ be any infinite cyclic covering space of K with $J(t)$ (the infinite cyclic multiplicative group generated by t) as the group of covering translations. Let Λ denote the integral group ring of $J(t)$ and $\Gamma = \Lambda \otimes_Z Q$ the rational group ring of $J(t)$. Λ is a Noetherian unique factorization domain and Γ is a principal ideal domain.

For all q, the integral chain groups $C_q(\tilde{K}; Z)$ are finitely-generated Λ-modules, with generators in 1-1 correspondence with the q-simplexes of K. Since Λ is Noetherian, then $H_q(\tilde{K}; Z)$ is a finitely-generated Λ-module. Likewise, $H_q(\tilde{K}; Q) \cong H_q(\tilde{K}; Z) \otimes_Z Q$ is a finitely generated Γ-module. The integral (rational) invariants are defined by using the integral (rational) homology of $\tilde{K}$. They lie in $\Lambda(\Gamma)$, and are therefore polynomials in positive and negative powers of the variable t. The development of the rational invariants follows closely that of J. Levine for knots [13].

DEFINITION: *Presentation matrix of a module.* Consider an abelian group A which is finitely generated as a $\Lambda(\Gamma)$-module. An $m \times n$ matrix $M = (m_{ij}(t))$ with entries in $\Lambda(\Gamma)$ is said to present A as a module if there exists an exact sequence of $\Lambda(\Gamma)$-modules

$$
F_2 \xrightarrow{d} F_1 \longrightarrow A \longrightarrow 0
$$

where F_1 and F_2 are free on the entries $(x_1, \ldots, x_n)$ and $(r_1, \ldots, r_m)$ respectively, and $d(r_i) = \sum_{j=1}^{n} m_{ij}(t) x_j$.

DEFINITION: *Elementary ideals of a matrix.* Let M be an $(m \times n)$ matrix with entries in $\Lambda(\Gamma)$, and k an integer. The kth elementary ideal ε_k of M is the ideal in $\Lambda(\Gamma)$ generated by the determinants of the $(n-k+1) \times (n-k+1)$ submatrices of M, with the conventions

(i) $(n-k+1) < 1$ then $\varepsilon_k = \Lambda(\Gamma)$,

(ii) $(n-k+1) > m$ then $\varepsilon_k = 0$.

If the matrix M presents A as a $\Lambda(\Gamma)$-module, then it is well known that the elementary ideals of M are invariant of $\Lambda(\Gamma)$-isomorphism type of A. (See [22].) Since Λ is Noetherian and $H_q(\tilde{K}; Z)$ is finitely generated, then there exists a presentation for $H_q(\tilde{K}; Z)$ as a Λ-module, and likewise $H_q(\tilde{K}; Q)$ as a Γ-module.

Let $M_q = (m_{ij}(t))$ be a presentation matrix for $H_q(\tilde{K}; Z)$. Since $- \otimes_Z Q$ is a right-exact functor, we can take $M'_q = (m_{ij}(t) \otimes 1)$ to be a presentation matrix for $H_q(\tilde{K}; Q)$ as a Γ-module.

DEFINITION I. The kth integral invariant of $\tilde{K}$ in dimension q, $\Delta_k^q(t) \in \Lambda$, is the generator of the smallest principal Λ-ideal containing ε_k^q, the kth elementary ideal of M_q. This generator is defined up to units in Λ, and for each integer q there exists k_q an integer such that $\Delta_i^q(t)=1$ for $i>k_q$. In classical knot theory, the integral invariants of the knot complement are called the Alexander polynomials of the knot [7, pp. 119].

DEFINITION II. The kth rational invariant of $\tilde{K}$ in dimension q, $\Delta_k^{q\prime}(t) \in \Gamma$, is the generator of $\varepsilon_k^{q\prime}$, the kth elementary ideal of M_q'. $\varepsilon_k^{q\prime}$ is a principal ideal because Γ is a PID.

DEFINITION. $p(t) \in \Lambda$ is primitive if all its coefficients are relatively prime.

DEFINITION. $p(t), q(t) \in \Lambda(\Gamma)$ are associate in $\Lambda(\Gamma)$ if there exists a unit $h(t) \in \Lambda(\Gamma)$ such that $p(t)=h(t)q(t)$.

Following Levine [13], we have the following easily-proved lemma:

LEMMA 2.1. *Let $\lambda, \mu \in \Lambda$ and λ primitive. Then $\lambda|\mu$ in Λ if and only if $\lambda|\mu$ in Γ.*

Hence, it is clear that every associate class in Γ contains a primitive element of Λ, unique up to associate class in Λ. Since the generator of an ideal in Γ is defined only up to associate class in Γ, we can take the rational invariants of $\tilde{K}$ to be primitive elements of Λ.

As one might expect, the integral invariants of $\tilde{K}$ and the rational invariants of $\tilde{K}$ are intimately related. In fact, we have the following

LEMMA 2.2. *$\Delta_k^{q\prime}=r\Delta_k^q$ where $0\neq r \in Q$ is the unique rational such that $r\Delta_k^q$ is primitive. (If $\Delta_k^q=0$, take $r=1$.)*

Proof. See also Levine [11] and Neuwirth [17]. Regard Λ as embedded in Γ in the canonical way, that is, identify Λ with $\Lambda \otimes 1$ in $\Gamma=\Lambda \otimes_Z Q$. If $\lambda \in \Lambda$, let $[\lambda]$ denote the Λ-ideal generated by λ, and (λ) the Γ-ideal generated by λ. Let ε be a Λ-ideal. ε determines a Γ-ideal $\varepsilon'=(\lambda^*)$, λ^* the primitive generator of ε'. Let $[\lambda]$ be the smallest principal ideal containing ε, and $r \in Q$ be chosen such that $r\lambda$ is primitive. It suffices to prove that $r\lambda=\lambda^*$ (up to associate class in Λ). Since $\varepsilon \subset [\lambda]$, $\varepsilon'=(\lambda^*) \subset (\lambda)=(r\lambda)$. On the other hand, $\varepsilon \subset \varepsilon' \cap \Lambda=[\lambda^*]$, so $[\lambda] \subset [\lambda^*]$, hence $(r\lambda)=(\lambda) \subset (\lambda^*)$ and $\lambda^*=r\lambda$.

Hence if Δ_k^q is primitive (as is the case when $\Delta_k^q(1)=\pm 1$), then $\Delta_k^{q\prime}=\Delta_k^q$.

We will need the following technical lemma:

LEMMA 2.3. *If R is a Noetherian ring, A is a f.g. R-module and $\phi: A \to A$ is an epimorphism then ϕ is an isomorphism.*

Proof. Let $K_n=\operatorname{Ker} \phi^n$. Then $K_1 \subset K_2 \subset \cdots$ and since R is Noetherian and A is f.g., this ascending chain of submodules of A has a maximal element, say K_q. So $K_{q+i}=K_q$ $\forall i \geqq 0$, hence if $\phi^q(x) \neq 0$ for some $x \in A$ then $\phi^{q+i}(x) \neq 0$ $\forall i \geqq 0$. This means that $\operatorname{Ker} \phi=K_1=0$, because if $x \in K_1$ then $\exists y \in A$ such that $\phi^q(y)=x$ since ϕ^q is an epimorphism. But $0=\phi(x)=\phi^{q+1}(y)$ so $0=\phi^q(y)=x$.

We now have the following theorem describing the properties of the integral invariants:

THEOREM 2.4. *K^{n+1} as above, $n \geqq 2$ and $\{\Delta_i^q\}$ the integral invariants of $\tilde{K}$. Then*

(i) $$\Delta_{i+1}^q \mid \Delta_i^q \quad \textit{in } \Lambda,$$

(ii) $$\begin{aligned} \Delta_i^q(1) &= 0, & q &= 0, & i &= 1, \\ &= 0, & q &= 1, & i &\leqq \alpha-1, \\ &= 0, & q &= n+1, & i &\leqq \beta, \\ &= \pm 1, & & \textit{otherwise}. \end{aligned}$$

Proof. Adapting an argument of Milnor [15], consider the short exact sequence of chain complexes with integral coefficients

$$0 \longrightarrow C_*(\tilde{K}) \xrightarrow{(t-1)} C_*(\tilde{K}) \longrightarrow C_*(K) \longrightarrow 0.$$

This induces the long exact sequence of homology

$$\longrightarrow H_i(\tilde{K}) \xrightarrow{(t-1)} H_i(\tilde{K}) \longrightarrow H_i(K) \xrightarrow{\partial} H_{i-1}(\tilde{K}) \longrightarrow. \tag{2.2}$$

Think of $\boldsymbol{Z}$ as a Λ-module via the augmentation map

$$\begin{aligned} \Lambda &\xrightarrow{\varepsilon} \boldsymbol{Z} \\ t &\rightsquigarrow 1 \end{aligned}$$

If the matrix $M_q(t) = (m_{ij}(t))$ presents $H_q(\tilde{K})$ as a Λ-module, then $M_q(1)$ presents $H_q(\tilde{K}) \otimes_\Lambda \boldsymbol{Z}$ as an abelian group. Moreover, $\{\Delta_i^q(1)\}$ are invariants of $H_q(\tilde{K}) \otimes_\Lambda \boldsymbol{Z}$ as a $\boldsymbol{Z}$-module, and $H_q(\tilde{K}) \otimes_\Lambda \boldsymbol{Z}$ is isomorphic to the cokernel of the homomorphism $H_q(\tilde{K}) \xrightarrow{(t-1)} H_q(\tilde{K})$. Since $H_*(K)$ satisfies (2.1), (2.2) yields immediately that $H_q(\tilde{K}) \otimes_\Lambda \boldsymbol{Z} = 0$, $2 \leqq q \leqq n-1$ ($n \geqq 3$). At the top end of the exact sequence we have

$$\begin{aligned} 0 \longrightarrow H_{n+1}(\tilde{K}) \xrightarrow{(t-1)} H_{n+1}(\tilde{K}) \longrightarrow H_{n+1}(K) \\ \longrightarrow H_n(\tilde{K}) \xrightarrow{(t-1)} H_n(\tilde{K}) \longrightarrow 0. \end{aligned}$$

By Lemma 2.3, $(t-1)\colon H_n(\tilde{K}) \to H_n(\tilde{K})$ is an isomorphism, so $H_{n+1}(\tilde{K}) \otimes_\Lambda \boldsymbol{Z} \cong H_{n+1}(K) = \beta\boldsymbol{Z}$. At the bottom end of the exact sequence we have

$$\begin{tikzcd}[row sep=small] 0 \arrow[r] & H_1(\tilde{K}) \arrow[r, "(t-1)"] & H_1(\tilde{K}) \arrow[r] & H_1(K) \arrow[d, phantom, "\cong" rotate=-90] & & \\ & & & \alpha\boldsymbol{Z} & & \\ & \arrow[r] & H_0(\tilde{K}) \arrow[r, "0"] \arrow[d, phantom, "\cong" rotate=-90] & H_0(\tilde{K}) \arrow[r, "\cong"'] \arrow[d, phantom, "\cong" rotate=-90] & H_0(K) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & 0 \\ & & \boldsymbol{Z} & \boldsymbol{Z} & \boldsymbol{Z} & \end{tikzcd}$$

hence $H_1(\tilde{K}) \otimes_\Lambda \boldsymbol{Z} \cong (\alpha-1)\boldsymbol{Z}$. Now since $\tilde{K}$ is connected, $\Delta_1^0(t)=(t-1)$. This completes the proof of Theorem 2.4.

There is a better way of thinking of the rational invariants of $\tilde{K}$. Γ is a principal ideal domain, so by the structure theorem for finitely-generated modules over a PID, $H_q(\tilde{K}; Q) \cong_\Gamma F_q \oplus T_q$ where F_q is the direct sum of copies of Γ, and T_q is the direct sum of cyclic modules

$$T_q \cong \Gamma/\lambda_1^q \oplus \cdots \oplus \Gamma/\lambda_m^q, \qquad \lambda_i^q \in \Lambda,\ \lambda_{i+1}^q | \lambda_i^q \text{ in } \Lambda\ \forall i.$$

Here Γ/λ denotes the cokernel of the homomorphism

$$\begin{array}{c} \Gamma \longrightarrow \Gamma \\ 1 \rightsquigarrow \lambda \end{array} \qquad \text{(see Levine [13]).}$$

That is, if the Γ-rank of F_q is r_q, then $H_q(\tilde{K}; Q)$ is presented as a Γ-module by the diagonal matrix

$$\begin{pmatrix} 0 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & & & \\ & & & & \lambda_{r_q+1}^q & & \\ & & & & & \ddots & \\ & & & & & & \lambda_{r_q+m}^q \end{pmatrix}$$

and the rational invariants are given by the formula

$$\begin{aligned} \Delta_j^{q\prime} &= 0, && 1 \leqq j \leqq r_q, \\ &= \prod_{i=j}^{r_q+m} \lambda_i^q, && r_q+1 \leqq j \leqq r_q+m, \\ &= 1, && r_q+m+1 \leqq j. \end{aligned}$$

COROLLARY 2.5. *K as above and $n \geqq 2$, then $H_q(\tilde{K}; Q) \cong F_q \oplus T_q$ where r_q (Γ-rank of H_q)*

$$\begin{aligned} &= \alpha-1, && q = 1, \\ &= \beta, && q = n+1, \\ &= 0, && \textit{otherwise}. \end{aligned}$$

Moreover, the integral invariants $\Delta_j^q(t)=0$ for $q=1$, $1 \leqq j \leqq \alpha-1$ and $q=n+1$,

$1 \leqq j \leqq \beta$, *and* T_q *is the torsion module presented by the diagonal matrix* (λ_i^q) *with* $\lambda_i^q \in \Lambda$, $\lambda_{i+1}^q | \lambda_i^q$ *in* Λ, *and*

$$\begin{aligned} \lambda_i^q(1) &= 0, \qquad && q = 0, i = 1, \\ &= \pm 1, && \textit{otherwise}. \end{aligned}$$

Theorem 2.4 and Corollary 2.5 avoid the case where $n=1$. We will now study the rational invariants in this case.

Let $r_i = \Gamma$-rank of $H_i(\tilde{K}; Q)$, $i=1, 2$. We have as in Theorem 2.4 the exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_2(\tilde{K}) & \xrightarrow{(t-1)} & H_2(\tilde{K}) & \longrightarrow & H_2(K) & \longrightarrow & H_1(\tilde{K}) & \xrightarrow{(t-1)} H_1(\tilde{K}) \\ & \wr\| & & & & \wr\| & & & \\ & r_2\Gamma & & & & \beta Q & & & \\ & & \longrightarrow & H_1(K) & \longrightarrow & H_0(\tilde{K}) & \xrightarrow{0} & H_0(\tilde{K}) & \longrightarrow H_0(K). \\ & & & \wr\| & & \wr\| & & & \\ & & & \alpha Q & & Q & & & \end{array}$$

This reduces to

$$\begin{aligned} (2.3) \qquad 0 \longrightarrow (\beta - r_2)Q \longrightarrow H_1(\tilde{K}) &\xrightarrow{(t-1)} H_1(\tilde{K}) \\ &\longrightarrow (\alpha-1)Q \longrightarrow 0. \end{aligned}$$

Let $H_1(\tilde{K}; Q) \cong_\Gamma F_1 \oplus T_1$ as before. Consider the prime-power-order decomposition for the torsion module T_1. That is,

$$T_1 \cong \Gamma/(t-1)^{p_1} \oplus \Gamma/(t-1)^{p_2} \oplus \cdots \oplus \Gamma/(t-1)^{p_k} \oplus T_1'$$

where the prime order $(t-1)$ does not appear in T_1'. The homomorphism $(t-1)$ respects any Γ-splitting, and is injective on F_1 and an isomorphism on T_1'. Also

$$\begin{aligned} 0 \longrightarrow \Gamma/(t-1) \longrightarrow \Gamma/(t-1)^{p_i} &\xrightarrow{(t-1)} \Gamma/(t-1)^{p_i} \\ &\longrightarrow \Gamma/(t-1) \longrightarrow 0 \end{aligned}$$

is exact for all $1 \leqq i \leqq k$. Hence (2.3) yields the equations $k = \beta - r_2$ and $k + r_1 = \alpha - 1$, so $\beta - r_2 = \alpha - 1 - r_1$. In the above k represents the number of times a power of $(t-1)$ appears in the prime power decomposition of T_1. Hence we have $\chi(K) = r_2 - r_1$, where $\chi(K)$ denotes the Euler characteristic of K.

THEOREM 2.6. *Let* K^2 *be as above. Then*

(i) $\Delta_i^{1\prime} = 0$ *for* $i \leqq \alpha - \beta - 1$, *if* $\alpha - \beta - 1 \geqq 1$.

(ii) $\Delta_k^{1\prime}$ *is the first nonvanishing rational invariant of* $\tilde{K}$ (*i.e.* $\Delta_j^1 = 0$, $1 \leqq j \leqq k-1$, $\Delta_k^{1\prime} \neq 0$) *if and only if* $r_2 = \beta - \alpha + k$.

DEFINITION. Let P be a finite 1-dimensional subcomplex of S^3, such that $H_1(P; \boldsymbol{Z}) \neq 0$. Then P is called a linear graph in S^3.

Let P be a linear graph in S^3, and $X = S^3 - P$, $G = \Pi_1(X)$, and $G^* = \operatorname{Ker} \varphi$ where $\varphi: G \to J(t)$ is some epimorphism. Let $\tilde{X}$ be the infinite cyclic cover of X corresponding to G^*.

THEOREM 2.7. *Let α_0 be the number of components of P and β_0 be the 1-dimensional Betti number of P. Then*

(i) $\Delta_i^1(1) = 0$, $i \leqq \beta_0 - 1$, $= \pm 1$, $i \geqq \beta_0$,

(ii) $\Delta_i^1(t) = 0$, $i \leqq \beta_0 - \alpha_0$ *if* $(\beta_0 - \alpha_0) \geqq 1$,

(iii) $\Delta^1_{\beta_0 - \alpha_0 + m}(t)$ *is the first nonvanishing integral invariant of* $\tilde{X}$ *if and only if* $H_2(\tilde{X}; Q)$ *is a free* Γ*-module of rank* $m-1$.

REMARK. The properties (i) and (ii) are similar to Kinoshita's [10].

Proof. By Alexander duality, $H_*(X; Z)$ satisfies the condition (2.1) for $\alpha = \beta_0$ and $\beta = \alpha_0 - 1$. Further, X has a 2-dimensional subcomplex as its deformation retract. Hence (ii) and (iii) are immediate consequences of Lemma 2.2 and Theorem 2.6.

As in Theorem 2.4, we have an exact sequence

$$H_1(\tilde{X}; Z) \xrightarrow{(t-1)} H_1(\tilde{X}; Z) \longrightarrow (\beta_0 - 1)Z \longrightarrow 0,$$

so $\operatorname{Coker}(t-1) \cong (\beta_0 - 1)Z$. This yields (i).

DEFINITION. A linear graph P in S^3 is said to be nonsplittable if given any embedded S^2 in S^3 such that $S^2 \cap P = \varnothing$, then P is contained entirely within one of the complementary domains.

COROLLARY 2.8. *Let P be a nonsplittable linear graph in S^3. Then $H_2(G^*; \boldsymbol{Z}) = 0$ iff $\Delta^1_{\beta_0 - \alpha_0 + 1} \neq 0$.*

Proof. The proof follows [20]. By Theorem 2.7(iii) we have $H_2(\tilde{X}; Q) = 0$. Since $\tilde{X}$ is 2-dimensional, this means that $H_2(\tilde{X}; \boldsymbol{Z}) = 0$. Now P is nonsplittable, so by [18] X is aspherical and $\tilde{X} = K(G^*, 1)$ the Eilenberg-Mac Lane space. Hence $H_2(G^*; \boldsymbol{Z}) \cong H_2(K(G, 1); \boldsymbol{Z}) = 0$.

This corollary is analogous to a result by Crowell and Cochran [6]. The splittable case will be studied in §4.

Let P be any tamely embedded linear graph as before, $G = \Pi_1(S^3 - P)$ and $\varphi_1: G \to Z$ an epimorphism. Let $\tilde{X}_1$ be the infinite cyclic covering space associated with Kernel φ_1 and $\{\Delta_i^1\}$ the integral invariants of $\tilde{X}_1$. Let $X = S^3 - P$ and $G' = [G, G]$.

THEOREM 2.9. *Let $\tilde{X}$ be the universal abelian covering of X. If $\Delta^1_{\beta_0 - \alpha_0 + 1} \neq 0$, then $H_2(\tilde{X}; Z) \cong 0$.*

Proof. By Theorem 2.7 we have $H_2(\tilde{X}_1; Z) \cong 0$. The epimorphism $\varphi_1: G \to Z$ factors through $H_1(X; Z) \cong G/G' \cong \beta_0 Z$. Hence G' is a normal subgroup of $G_1 = \operatorname{Kernel} \varphi_1$, and $G' \supset [G_1, G_1]$.

Now $(G/G')/(G_1/G')\cong G/G_1\cong Z$, hence $0\to G_1/G'\to\beta_0 Z\to Z\to 0$ is split exact and $G_1/G'\cong(\beta_0-1)Z$. Let $\varphi_2\colon G_1\to Z$ be the epimorphism indicated below:

$$\begin{tikzcd} G_1 \arrow[r] \arrow[dr, "\varphi_2"'] & G_1/G' \cong (\beta_0-1)Z \arrow[d] & (1,\dots,1) \arrow[d] \\ & Z & (1,0,\dots,0) \end{tikzcd}$$

Let G_2 be the kernel of $\varphi_2\colon G_1\to Z$. As before, we have $G_2/G'\cong(\beta_0-2)Z$. Let $\tilde{X}_2$ be the infinite cyclic covering space of $\tilde{X}_1$ associated with Kernel φ_2. Let $J(t)$ denote the infinite cyclic group of covering translations of $\tilde{X}_2$. As before, we have the long exact homology sequence

$$0\longrightarrow H_2(\tilde{X}_2;Z)\xrightarrow[\cong]{t-1}H_2(\tilde{X}_2;Z)\longrightarrow 0.$$

Since $\tilde{X}_2$ is 2-dimensional, then as before $H_2(\tilde{X}_2;Z)\cong 0$. The process can be repeated β_0 times, obtaining a stack of covering spaces

$$\tilde{X}_{\beta_0}\to\cdots\to\tilde{X}_2\to\tilde{X}_1\to X,$$

each $\tilde{X}_i$ being an infinite cyclic covering space of $\tilde{X}_{i-1}$, and $\tilde{X}_{\beta_0}$ being the universal abelian covering of X. Hence $H_2(\tilde{X};Z)\cong 0$.

COROLLARY 2.10. *If P is a nonsplittable linear graph in S^3 and $\Delta^1_{\beta_0-\alpha_0+1}\neq 0$, where $\Delta^1_{\beta_0-\alpha_0+1}$ is the invariant associated with some infinite cyclic cover of X, then $H_2(G';Z)=0$.*

Proof. Since X is aspherical, then as before $\tilde{X}=K(G',1)$ and the result follows from Theorem 2.9.

3. **Polynomial invariants of higher dimensional links.**

DEFINITION. An *n-link $L=K_1\cup\cdots\cup K_\mu$ of multiplicity μ* is the disjoint union of μ oriented and smoothly embedded n-spheres K_i in S^{n+2}. Two n-links L and L' of multiplicity μ belong to the *same link type* if there exists an orientation preserving homeomorphism f of S^{n+2} onto itself such that $f(K_i)=K_i'$ and $f|K_i$ is orientation preserving, $i=1,2,\dots,\mu$.

Let X be the complement of an open tubular neighborhood of L in S^{n+2} and $\tilde{X}$ the covering space of X belonging to Ker φ, where $\varphi\colon \Pi_1(X)\to J(t)$ is an epimorphism given by

$$\varphi(g) = t^{\operatorname{link}(g,L)} \quad \text{for } g\in\Pi_1(X), \tag{3.1}$$

link $(g,L)=\sum_i$ link (g,K_i). X is a finite complex and $\tilde{X}$ is an infinite cyclic cover of X. Hence, as in §2, the integral invariants $\{\Delta_i^q(t)\}$ and the rational invariants

$\{\Delta_i^{q\prime}(t)\}$ of $\tilde{X}$ are well defined. It is easily seen that these are link type invariants. If $n=1$, $\Delta_1^1(t)$ is known as the reduced Alexander polynomial of L and its properties have been studied. For the case $n\geqq 2$, we have the following theorem:

THEOREM 3.1. *Let L be an n-link of multiplicity μ and $n\geqq 2$. Then*

(i)
$$\Delta_{i+1}^q(t)\,|\,\Delta_i^q(t) \quad \textit{in } \Lambda,$$

(ii)
$$\begin{aligned} \Delta_i^q(1) &= 0, & & q = 0, i = 1, \\ &= 0, & & q = 1, i \leqq \mu-1, \\ &= 0, & & q = n+1, i \leqq \mu-1, \\ &= \pm 1, & & \textit{otherwise}, \end{aligned}$$

(iii)
$$\begin{aligned} \Delta_i^1(t) &= 0, & & i < \mu, \\ &= \Delta_{i-\mu+1}^n(t^{-1}), & & i \geqq \mu, \\ \Delta_i^q(t) &= \Delta_i^{n+1-q}(t^{-1}), & & 2 \leqq q \leqq n-1. \end{aligned}$$

Proof. X has the same homotopy type as an $(n+1)$-dimensional complex, and by Alexander duality it satisfies the condition (2.1) for $\alpha=\mu$ and $\beta=\mu-1$. Hence Theorem 2.4 yields (i) and (ii).

Now $\partial\tilde{X}$ is the disjoint union of μ copies of $S^n\times R^1$. Hence it follows easily that $H_i(\tilde{X})\simeq H_i(\tilde{X}, \partial\tilde{X})$ for $2\leqq i\leqq n-1$. Further, $H_n(\partial\tilde{X})$ is the direct sum of μ copies of $\Lambda/(t-1)$ and $\Delta_1^n(1)=\pm 1$. Hence, in the exact sequence of Λ-modules

$$\cdots \longrightarrow H_n(\partial\tilde{X}) \xrightarrow{i_*} H_n(\tilde{X}) \longrightarrow H_n(\tilde{X}, \partial\tilde{X}) \longrightarrow 0,$$

i_* is trivial, which implies $H_n(\tilde{X}) \cong_\Lambda H_n(\tilde{X}, \partial\tilde{X})$.

Theorem 2.4 and Corollary 2.5 show that $H_i(\tilde{X})$ is a torsion Λ-module for $2\leqq i\leqq n$ and $\mu-1$ is the maximal number of linearly independent elements in the Λ-module $H_1(\tilde{X})$. Hence Corollary 4.8 and Lemma 4.10 of [4] yield (iii).

DEFINITION. An n-link $L=K_1\cup\cdots\cup K_\mu$ of multiplicity μ is called *null-cobordant* if there exist μ mutually disjoint $(n+1)$-balls $D_1^{n+1},\dots, D_\mu^{n+1}$ in D^{n+3} with $\partial D_i^{n+1}=K_i$, $i=1, 2,\dots,\mu$, where $\partial D^{n+3}=S^{n+2}$.

K. Murasugi showed that for a null-cobordant 1-link of multiplicity μ

$$\begin{aligned} \Delta_i^1(t) &= 0, & & i < \mu, \\ &= f(t)f(t^{-1}), & & i = \mu, \end{aligned}$$

where $f(t)\in\Lambda$ and $f(1)=\pm 1$ [16]. For higher-dimensional links we have

THEOREM 3.2. *If L is a null-cobordant $(2m-1)$-link of multiplicity μ and $m\geqq 2$, then $\Delta_1^m(t)=f(t)f(t^{-1})$ for some $f(t)\in\Lambda$ with $f(1)=\pm 1$.*

Proof. The proof is analogous to the case of null-cobordant knots [12]. By using the Thom-Pontrjagin construction as in Levine [14], one can show that every

n-link bounds a smooth connected $(n+1)$-manifold V in S^{n+2}. If $n=2m-1$, we define a Seifert matrix A of V as in [12]. The argument in §2 of [13] together with the fact that $H_q(\tilde{X}; Q)$ is a torsion Γ-module for $2 \leqq q \leqq n$ shows that $tA+(-1)^m A'$ is a presentation matrix for $H_m(\tilde{X}; Q)$. Finally, by modifying the proof of Lemma 2 of [12] it can be shown that if L is null-cobordant, then A is null-cobordant in the sense of [12], which yields the theorem.

4. Detecting geometric linking.

DEFINITION. An n-link $L=K_1 \cup \cdots \cup K_\mu$ of multiplicity μ is said to be geometrically k-splittable $(0 \leqq k \leqq \mu-1)$ if $\exists B = B_1^{n+2} \cup \cdots \cup B_k^{n+2}$, a collection of k mutually disjoint sub-balls of S^{n+2} satisfying

(i) $B_i^{n+2} \cap L \neq \varnothing$ $1 \leqq i \leqq k$,

(ii) $\partial B \cap L = \varnothing$,

(iii) $S^{n+2} - B \cap L \neq \varnothing$,

(iv) k is maximal with respect to the above properties.

If $k=\mu-1$, we say that L is *completely splittable*, and if $k=0$ we say that L is *nonsplittable*. If $L=K_1 \cup \cdots \cup K_\mu$ is k-splittable, then we say that

$$\begin{aligned} L_i &= L \cap B_i^{n+2}, \qquad\qquad 1 \leqq i \leqq k, \\ L_{k+1} &= L \cap (S^{n+2} - B) \end{aligned}$$

are the *nonsplittable components* of L.

Suppose that $L=K_1 \cup K_2$ is an n-link of multiplicity 2. Let $X=S^{n+2}-L$, and $\tilde{X}$ be the infinite cyclic covering space of X as in §3. Let $X_i = S^{n+2}-K_i$ and $\tilde{X}_i$ be the infinite cyclic cover of X_i, $i=1, 2$.

THEOREM 4.1. *If L as above is completely splittable, then*

$$\begin{aligned} H_i(\tilde{X}; \mathbf{Z}) &\cong_\Lambda H_i(\tilde{X}_1; \mathbf{Z}) \oplus H_i(\tilde{X}_2; \mathbf{Z}), \qquad 2 \leqq i \leqq n, \\ H_1(\tilde{X}; \mathbf{Z}) &\cong_\Lambda H_1(\tilde{X}_1; \mathbf{Z}) \oplus H_1(\tilde{X}_2; \mathbf{Z}) \oplus \Lambda, \\ H_{n+1}(\tilde{X}; \mathbf{Z}) &\cong_\Lambda H_{n+1}(\tilde{X}_1; \mathbf{Z}) \oplus H_{n+1}(\tilde{X}_2; \mathbf{Z}) \oplus \Lambda \cong \Lambda. \end{aligned}$$

Proof. Let T_i be the closed tubular neighborhood of K_i in S^{n+2}, $i=1, 2$. Then $S^{n+2}-K_2 \simeq X_2 \vee B_1^{n+2}$ where B_1^{n+2} is the $n+2$-ball given by the splitting, $B_1^{n+2} \supset K_1$. That is, we can isolate any B^{n+2} missing K_2 in S^{n+2} in a wedge product decomposition for X_2. Let C be a collar on ∂B_1^{n+2} contained in $S^{n+2}-B_1^{n+2}$ and missing T_2. Let ∂C be the exterior boundary of C. Connect ∂T_2 to ∂C by an arc whose interior is contained in $S^{n+2}-(B_1^{n+2} \cup C \cup T_2)$. Collapse away from ∂T_2 in a tubular neighborhood of the arc until ∂C is reached. This frees a top-dimensional disc on ∂C, and C can be then collapsed to an arc connecting B_1^{n+1} to ∂C (Figure 1). Clearly the complement in S^{n+2} of the configuration of Figure 1 is diffeomorphic to X_2.

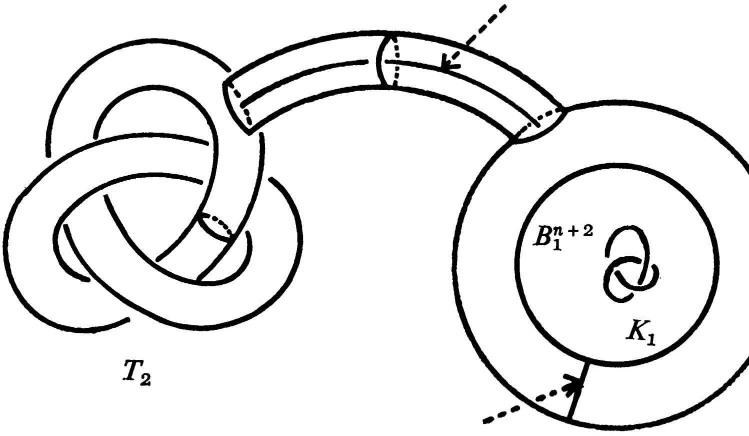

FIGURE 1

We also have that

$$B_1^{n+2}-K_1 \simeq X_1 \vee S^{n+1}.$$

This homotopy equivalence is obtained in exactly the same way as the previous one, collapsing away from one boundary component of $\partial(B_1^{n+2}-T_1)$ and isolating the other in a wedge product decomposition.

Hence if L is splittable, then $X=S^{n+2}-L\simeq X_1\vee X_2\vee S^{n+1}$. Let $*$ denote the base point of the wedge product. Then (up to homotopy equivalence) $\tilde{X}$ consists of one copy of $\tilde{X}_1$ and one copy of $\tilde{X}_2$, identified along $\tilde{*}$, the points lying over $*$, together with Z copies of S^{n+1}, one stuck on at each point of identification of $\tilde{X}_1$ and $\tilde{X}_2$. Split $\tilde{X}$ into two pieces, $W_1=\tilde{X}_1\cup_{\tilde{*}}\bigcup_{i\in Z}S_i^{n+1}$ and $W_2=\tilde{X}_2$. Then $\tilde{X}=W_1\cup_{\tilde{*}}W_2$. The Mayer-Vietoris sequence for the triad $(\tilde{X}, W_1, W_2)$ yields $\cdots\to H_i(\tilde{*})\to H_i(W_1)\oplus H_i(W_2)\to H_i(\tilde{X})\to\cdots$. Hence $H_i(\tilde{X})\cong H_i(\tilde{X}_1)\oplus H_i(\tilde{X}_2)$ for $2\leqq i\leqq n$. With augmented homology $\#$

$$0\to H_1(W_1)\oplus H_1(W_2)\to H_1(\tilde{X})\to H_0^{\#}(\tilde{*})\to 0. \tag{1}$$

Now $H_0^{\#}(\tilde{*})\cong\Lambda$ because we have the exact sequence

$$0\longrightarrow\Lambda\xrightarrow{\;t-1\;}\Lambda\xrightarrow{\;\varepsilon\;}Z\longrightarrow 0,$$

where $C_0(\tilde{*})\cong\Lambda$ and ε is the augmentation map. So (1) splits, and $H_1(\tilde{X})\cong_\Lambda H_1(\tilde{X}_1)\oplus H_1(\tilde{X}_2)\oplus\Lambda$. Likewise, $H_{n+1}(W_1)\cong_\Lambda H_{n+1}(\tilde{X}_1)\oplus\Lambda$. So

$$H_{n+1}(\tilde{X})\cong_\Lambda H_{n+1}(W_1)\oplus H_{n+1}(W_2)\cong_\Lambda H_{n+1}(\tilde{X}_1)\oplus H_{n+1}(\tilde{X}_2)\oplus\Lambda.$$

This completes the proof of Theorem 4.1.

Let L be an n-link of multiplicity μ which is geometrically k-splittable. Let L_i, $1\leqq i\leqq k+1$, be the nonsplittable components of L. Let $X=S^{n+2}-L$, $X_i=S^{n+2}-L_i$, and $\tilde{X}_i$ denote the infinite cyclic cover of X_i, $1 \leqq i \leqq k+1$. As a corollary to the proof of Theorem 4.1, we have

COROLLARY 4.2. *If L as above is geometrically k-splittable, then*

$$H_j(\tilde{X}; \mathbf{Z}) \cong_\Lambda \bigoplus_{i=1}^{k+1} H_j(\tilde{X}_i; \mathbf{Z}), \qquad 2 \leqq j \leqq n,$$

$$H_1(\tilde{X}; \mathbf{Z}) \cong_\Lambda \bigoplus_{i=1}^{k+1} H_1(\tilde{X}_i; \mathbf{Z}) \oplus k\Lambda,$$

$$H_{n+1}(\tilde{X}; \mathbf{Z}) \cong_\Lambda \bigoplus_{i=1}^{k+1} H_{n+1}(\tilde{X}_i; \mathbf{Z}) \oplus k\Lambda.$$

COROLLARY 4.3. *If L is an n-link of multiplicity μ which is k-splittable then $\Delta_i^1 = 0$, $1 \leqq i \leqq k$.*

Let P be a linear graph in S^3 which is k-splittable, and let P_i, $1 \leqq i \leqq k+1$, be the nonsplittable components of P. As before let $X = S^3 - P$, $X_i = S^3 - P_i$, and $\tilde{X}_i$ denote the infinite cyclic cover of X_i. Let $G = \Pi_1(X)$ and $G_i = \Pi_1(X_i)$. Then

$$G = G_1 * G_2 * \cdots * G_{k+1}$$

and $\exists$ a natural monomorphism $\psi_i \colon G_i \to G$, $1 \leqq i \leqq k+1$. Suppose that $\varphi \colon G \to J(t)$ is an epimorphism such that $\varphi_i = \varphi\psi_i \colon G_i \to J(t)$ is an epimorphism $1 \leqq i \leqq k+1$. Let $G^* = \operatorname{Ker} \varphi$ and $\tilde{X}$ be the infinite cyclic covering space of X associated with G^*. Likewise, let G_i^* be the kernel of φ_i, and $\tilde{X}_i$ the infinite cyclic cover of X_i associated with G_i^*.

THEOREM 4.4.

$$H_1(G^*; \mathbf{Z}) \cong_\Lambda \bigoplus_{i=1}^{k+1} H_1(G_i^*; \mathbf{Z}) \oplus k\Lambda,$$

$$H_2(G^*; \mathbf{Z}) \cong_\Lambda \bigoplus_{i=1}^{k+1} H_2(G_i^*; \mathbf{Z}) \ (\textit{free abelian}),$$

$$H_p(G^*; \mathbf{Z}) = 0, \qquad p \geqq 3.$$

Proof. By the proof to Theorem 4.1, we know that $X \simeq [\bigvee_{i=1}^{k+1} X_i] \vee [\bigvee_{j=1}^{k} S_j^2]$ and that $\tilde{X}$ can be constructed from the $\tilde{X}_i$ by identifying lifts of the wedge point $*$ at equivalent levels in the various $\tilde{X}_i$ and then tacking on a copy of $[\bigvee_{j=1}^{k} S_j^2]$ to each such vertex. For the purposes of calculating $H_p(G^*; \mathbf{Z})$, the sphere packet $[\bigvee_{j=1}^{k} S_j^2]$ is irrelevant, so delete it, and consider $X = \bigvee_{i=1}^{k+1} X_i$. Now by [18], each X_i is aspherical, and $\tilde{X}_i \cap \tilde{X}_j \subset \tilde{X}$ is aspherical, so $\tilde{X}$ will also be aspherical. This can be seen by considering the universal cover $\tilde{X}^{\sim} \xrightarrow{P} \tilde{X}$ of $\tilde{X}$. Now $P^{-1}(\tilde{X}_i)$ will be the disjoint union of lots of copies of $\tilde{X}_i^{\sim}$, the universal cover of $\tilde{X}_i$, since $\Pi_1(\tilde{X}_i) \xrightarrow{i_*} \Pi_1(\tilde{X})$ is injective. The Mayer-Vietoris sequence in $\tilde{X}^{\sim}$ shows that it is aspherical. Hence $\tilde{X} = K(G^*, 1)$ and the theorem follows.

Let χ_i denote the Euler characteristic of P_i. Let $\{{}_i\Delta_k^1\}$ denote the integral invariants of $H_1(\tilde{X}_i; \mathbf{Z})$, $i = 1, 2, \ldots, k+1$.

COROLLARY 4.5. *$H_2(G^*; Z)=0$ iff ${}_i\Delta^1_{1-x_i}(t)\neq 0$ for $1\leqq i\leqq m$.*

Proof. Corollary 2.8 and Theorem 4.4 yield this result.

Examples of isotopy linking (nonsplittability) [1], [21], [23] and homotopy linking [2], [8] of codimension 2 spheres are well known, the former usually being demonstrated by studying Π_1 and the latter by proving that an unknotted sphere is not homotopic to 0 in the complement of the other sphere. We will now exhibit two examples of 2-links of multiplicity 2 in S^4 which are not geometrically splittable and detectable as such by considering their homology invariants.

EXAMPLE 1. *Spun link.* In R^4 we take coordinates x_1, x_2, x_3, x_4. Let R^3_+ be the half-space described by $x_4=0$ and $x_3\geqq 0$ and R^2_0 the plane defined by $x_4=0$ and $x_3=0$. We consider two disjoint arcs in R^3_+ with endpoints in R^2_0 as shown in Figure 2.

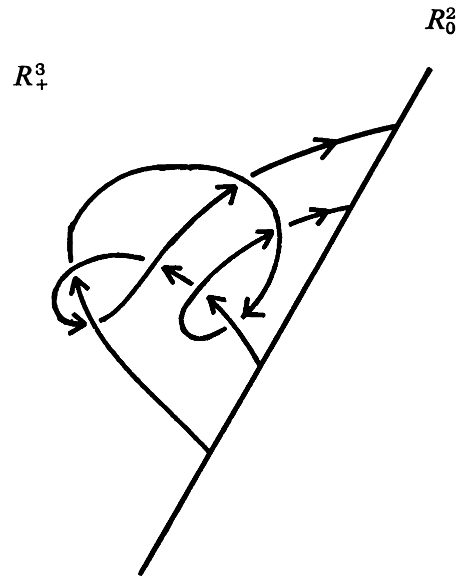

FIGURE 2

Following Artin [1] we rotate R^3_+ about R^2_0. Then the rotation of two arcs sweeps out two disjoint 2-spheres in R^4 and we obtain a 2-link L of multiplicity 2. By using Artin's theorem [1] it is easy to show that $\Pi_1(S^4-L)$ is isomorphic to the fundamental group of the complement of the linear graph shown in Figure 2 of [9]. Therefore, if $\tilde{X}$ is the infinite cyclic covering space of X defined by an epimorphism φ given by (3.1), then $[t^2+t+1, 2]$ is a presentation matrix for $H_1(\tilde{X}; Z)$. Hence L is not geometrically splittable by Theorem 4.1.

EXAMPLE 2. *Van Kampen's link of unknotted spheres* [21], [23]. This time we spin the two arcs of Figure 3 to produce a 2-link of multiplicity 2, in which each of the components of the link is an unknotted S^2.

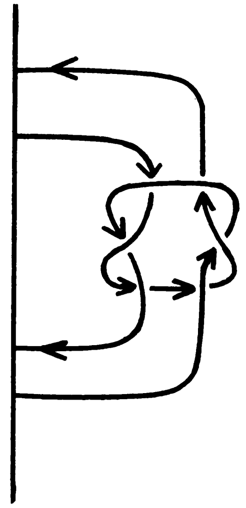

FIGURE 3

This time we obtain as a presentation matrix for $H_1(\tilde{X}; Z)$ the (1×2) matrix $(t^2-t+1, 0)$ so the link is clearly nonsplittable.

Related examples of nonsplittable n-links can be produced by surgery or by $(n-1)$-spinning [3], [8], [19] the configurations in Figures 2 and 3.

5. **Finite cyclic coverings.** For a classical knot $k: S^1 \to S^3$ it is well known that if A is a Seifert matrix for k, then $A+A'$ is a presentation matrix for the 2-fold cyclic covering space of S^3 branched along k. We will obtain a similar result for q-simple n-links.

As before, let L be an n-link of multiplicity μ and $X=S^{n+2}-L$.

DEFINITION. L is called q-simple $(q\geqq 0)$ if $\Pi_i(X)=\Pi_i(C_{\mu,n})$ for $i\leqq q$, where $C_{\mu,n}=(\bigvee_{i=1}^{\mu} S_i^1)\vee(\bigvee_{j=1}^{\mu-1} S_j^{n+1})$. Note that every link is 0-simple. The trivial link is q-simple for all q. As is shown in [19], it is possible for an n-link to be $(n-1)$-simple and still be nontrivial.

Let $\tilde{X}$ denote the infinite cyclic cover of X associated with Kernel φ, where φ is given by (3.1).

LEMMA 5.1. *If an n-link L of multiplicity μ is q-simple $q\geqq 1$ then*

(i) $H_1(\tilde{X}; Z) \cong_\Lambda (\mu-1)\Lambda$,

(ii) *if* $2\leqq q\leqq n$ *then* $H_i(\tilde{X}; Z)=0$, $2\leqq i\leqq q$, *and*

(iii) *if* $q\geqq n+1$ *then* $H_{n+1}(\tilde{X}; Z)\cong(\mu-1)\Lambda$.

Proof. Consider the map $f: C_\mu=\bigvee_{j=1}^{\mu} S_j^1 \to X$ which induces an isomorphism on Π_1. Let M be the mapping cylinder of f. $M\simeq X$, and $\Pi_1(M, C_\mu)=0$ and $\Pi_i(M, C_\mu)=\Pi_i(X)$, $i\geqq 2$. Let $\tilde{M}$ be the infinite cyclic cover of M associated with the epimorphism $\varphi: \Pi_1(X)\to J(t)$. We then have induced infinite cyclic covers $\tilde{X}$ of X and $\tilde{C}_\mu$ of C_μ. Moreover, $H_1(\tilde{C}_\mu; Z) \cong_\Lambda (\mu-1)\Lambda$ and $H_i(\tilde{C}_\mu; Z)=0$, $i\geqq 2$.

Moreover, the inclusion $i_*\colon \Pi_1(\tilde{C}_\mu) \xrightarrow{\cong} \Pi_1(\tilde{X})$ is an isomorphism so

$$i_*\colon H_1(\tilde{C}_\mu) \xrightarrow{\cong} H_1(\tilde{X})$$

and (i) is proved. If $2 \leqq q \leqq n$ then $\Pi_i(\tilde{M}; \tilde{C}_\mu)=0$ for $i \leqq q$ and so $H_i(\tilde{M}, \tilde{C}_\mu)=0$, $i \leqq q$, and (ii) is proved.

If $q \geqq n+1$, then consider M to be the mapping cylinder of the map $f\colon C_{\mu,n} \to X$ which induces isomorphism on Π_i, $i \leqq q$. We have from the following Hurewicz exact ladder (substituting $\tilde{X}$ for $\tilde{M}$)

$$\begin{array}{ccccccccc}
\longrightarrow \Pi_{n+2}(\tilde{X}) & \longrightarrow & \Pi_{n+2}(\tilde{X}, \tilde{C}_{\mu,n}) & \longrightarrow & \Pi_{n+1}(\tilde{C}_{\mu,n}) & \xrightarrow{\cong} & \Pi_{n+1}(\tilde{X}) \longrightarrow \\
\downarrow & & \downarrow h & & \downarrow & & \downarrow \\
\longrightarrow H_{n+2}(\tilde{X}) & \longrightarrow & H_{n+2}(\tilde{X}, \tilde{C}_{\mu,n}) & \longrightarrow & H_{n+1}(\tilde{C}_{\mu,n}) & \xrightarrow{i_*} & H_{n+1}(\tilde{X}) \longrightarrow.
\end{array}$$

Since h is an epimorphism then i_* is an isomorphism and (iii) is proved.

THEOREM 5.2. *Suppose that L is a q-simple n-link of multiplicity μ. Let X_k^u be the k-fold unbranched cyclic covering space of $X=S^{n+2}-L$. Then*

(i) $H_1(X_k^u; \mathbf{Z}) \cong_{\mathbf{Z}} \operatorname{Cok}(t^k-1) \oplus \mathbf{Z}$ *where* $(t^k-1)\colon H_1(\tilde{X}; \mathbf{Z}) \to H_1(\tilde{X}; \mathbf{Z})$,

(ii) *if* $q \geqq 1$ *then* $H_1(X_k^u; \mathbf{Z}) \cong [k(\mu-1)+1]\mathbf{Z}$,

(iii) *if* $2 \leqq q \leqq n$ *then* $H_i(X_k^u; \mathbf{Z})=0$, $2 \leqq i \leqq q$,

(iv) *if* $1 \leqq q \leqq n$ *then* $H_{q+1}(X_k^u; \mathbf{Z}) \cong \operatorname{Cok}(t^k-1)$ *where* $(t^k-1)\colon H_{q+1}(\tilde{X}; \mathbf{Z}) \to H_{q+1}(\tilde{X}; \mathbf{Z})$,

(v) *if* $q \geqq n+1$ *then* $H_{n+1}(X_k^u; \mathbf{Z}) \cong k(\mu-1)\mathbf{Z}$.

Proof. The short exact sequence of chain complexes

$$0 \longrightarrow C_*(\tilde{X}; \mathbf{Z}) \xrightarrow{(t^k-1)} C_*(\tilde{X}; \mathbf{Z}) \longrightarrow C_*(X_k^u; \mathbf{Z}) \longrightarrow 0$$

induces the long exact sequence of homology

$$\cdots \longrightarrow H_i(\tilde{X}; \mathbf{Z}) \xrightarrow{(t^k-1)} H_i(\tilde{X}; \mathbf{Z}) \longrightarrow H_i(X_k^u; \mathbf{Z}) \longrightarrow \cdots.$$

This gives us

$$\begin{array}{ccccccccc}
H_1(\tilde{X}; \mathbf{Z}) & \xrightarrow{(t^k-1)} & H_1(\tilde{X}; \mathbf{Z}) & \longrightarrow & H_1(X_k^u; \mathbf{Z}) & \longrightarrow & H_0(\tilde{X}; \mathbf{Z}) & \xrightarrow{0} & 0 \\
& & & & & & \wr\| & & \\
& & & & & & \mathbf{Z} & &
\end{array}$$

from which (i) immediately follows.

Now if $q \geqq 1$, then we have

$$0 \longrightarrow (\mu-1)\Lambda \xrightarrow{(t^k-1)} (\mu-1)\Lambda \longrightarrow H_1(X_k^u; \mathbf{Z}) \longrightarrow \mathbf{Z} \longrightarrow 0$$

which yields

$$0 \to k(\mu-1)\boldsymbol{Z} \to H_1(X_k^u; \boldsymbol{Z}) \to \boldsymbol{Z} \to 0$$

and so (ii) is proved. Since $H_i(\tilde{X}; \boldsymbol{Z})=0$ for $2 \leqq i \leqq q$ the exact sequence yields immediately (iii) and (iv) when $q \geqq 2$. If $q=1$, then $(t^k-1)\colon H_1(\tilde{X}; \boldsymbol{Z}) \to H_1(\tilde{X}; \boldsymbol{Z})$ is injective and (iv) is proved. The proof for (v) is similar to that of (ii) and left to the reader.

Suppose now that $M(t)$ is a Λ-presentation matrix for $H_{q+1}(\tilde{X}; \boldsymbol{Z})$, and that $M(t)$ has m columns. Let $[t^k-1]$ denote the $m \times m$ diagonal matrix with t^k-1 down the diagonal, and

$$\begin{pmatrix} M \\ [t^k-1] \end{pmatrix}$$

the matrix obtained from $M(t)$ by adjoining $[t^k-1]$ to the bottom.

Corollary 5.3. *With the hypotheses of Theorem* 5.2, *we have*

(i) *if* $q \geqq 1$ *then*

$$\begin{pmatrix} M(t) \\ [t^k-1] \end{pmatrix}$$

is a presentation matrix for $H_{q+1}(X_k^u; \boldsymbol{Z})$ *as a* Λ*-module*,

(ii) *if* $1 \leqq q \leqq n$ *and* $k=2$ *then* $M(-1)$ *is a presentation matrix for* $H_{q+1}(X_2^u; \boldsymbol{Z})$ *as an abelian group*,

(iii) *if* $q \geqq 0$, $k=2$ *and* $\mu=1$ *then*

$$\left(\begin{array}{c|c} & 0 \\ M(-1) & \vdots \\ & 0 \\ \hline 0 \cdots 0 & 0 \end{array}\right)$$

is a presentation matrix for $H_1(X_2^u; \boldsymbol{Z})$ *as an abelian group.*

Proof. (i) follows immediately from Lemma 8 of [3]. Now if $k=2$, then $(t^2-1) =(t-1)(t+1)$ as a composite of homomorphisms. If $1 \leqq q < n+1$, then the arguments of §2 apply to show

$$H_{q+1}(\tilde{X}; \boldsymbol{Z}) \xrightarrow[\cong]{(t-1)} H_{q+1}(\tilde{X}; \boldsymbol{Z}).$$

So Cok $(t^2-1) \cong_\Lambda$ Cok $(t+1)$, and Cok $(t+1)$ is presented as a Λ-module by the matrix

$$\begin{pmatrix} M(t) \\ [t+1] \end{pmatrix}$$

hence as an abelian group by the matrix $(M(-1))$. If $k=2$ and $\mu=1$ (we have a knot in this case), then again the arguments of §2 yield

$$H_1(\tilde{X}; \mathbf{Z}) \xrightarrow[\cong]{(t-1)} H_1(\tilde{X}; \mathbf{Z})$$

and (iii) is proved.

A more interesting situation to study is that of the branched cyclic covering.

THEOREM 5.4. *Let L be an n-link of multiplicity μ, and X_k^u and X_k^b denote the k-fold unbranched and branched cyclic covers, respectively. Then*

(i) $0 \longrightarrow \mu\mathbf{Z} \longrightarrow H_1(X_k^u; \mathbf{Z}) \xrightarrow{i_*} H_1(X_k^b; \mathbf{Z}) \longrightarrow 0$ *is exact, and is split exact if either L is completely splittable or if L is* 1*-simple,*

(ii) $i_*: H_i(X_k^u; \mathbf{Z}) \xrightarrow{\cong} H_i(X_k^b; \mathbf{Z})$ *if* $2 \leqq i \leqq n$,

(iii) $0 \longrightarrow (\mu-1)\mathbf{Z} \longrightarrow H_{n+1}(X_k^u; \mathbf{Z}) \xrightarrow{i_*} H_{n+1}(X_k^b; \mathbf{Z}) \longrightarrow 0$ *is exact, and is split exact if L is completely splittable, or if L is* $(n+1)$*-simple.*

Proof. By excision,

$$\begin{aligned} H_i(X_k^b, X_k^u; \mathbf{Z}) &\cong \mu\mathbf{Z}, \qquad i = 2, n+2, \\ &\cong 0, \qquad \text{otherwise.} \end{aligned}$$

Now from the exact sequence of the pair we have

$$\begin{tikzcd} {} \arrow[r] & H_2(X_k^b) \arrow[r] & H_2(X_k^b, X_k^u) \arrow[r, "\partial"] \arrow[d, phantom, "\wr\Vert"] & H_1(X_k^u) \arrow[r] & H_1(X_k^b) \arrow[r] & 0 \\ & & \mu\mathbf{Z} & & & \end{tikzcd}$$

Now ∂ is injective, as is easily seen by considering the pair $(X_k^u, \partial X_k^u)$ covering the pair $(X, \partial X)$. Let $p: (X_k^u, \partial X_k^u) \to (X, \partial X)$ be the covering map. Then we have the commutative diagram

$$\begin{tikzcd} H_1(\partial X_k^u) \arrow[r, "\tilde{\imath}_*"] \arrow[d, "p_*"] & H_1(X_k^u) \arrow[d, "p_*"] \\ H_1(\partial X) \arrow[r, "i_*"] & H_1(X) \end{tikzcd}$$

Now if $n>1$, then $H_1(\partial X_k^u) \cong \mu\mathbf{Z} \cong H_1(\partial X)$ and $p_*: H_1(\partial X_k^u) \to H_1(\partial X)$ is multiplication by k, hence injective. Now $i_*: H_1(\partial X) \to H_1(X)$ is an isomorphism in this case, so $\tilde{\imath}_*$ is injective. Clearly then ∂ is injective. If $n=1$, the above argument goes through by considering the direct summand of $H_1(\partial X)$ generated by the meridian curves.

Now Crowell [5] proves that for a knot $(\mu=1)$ then $0 \to \mathbf{Z} \to H_1(X_k^u) \to H_1(X_k^b) \to 0$ splits. As in §4, let $\{L_i\}_{i=1}^p$ be the nonsplittable components of L, and $X_i = S^{n+2} - L_i$, and $X_{k,i}^u$ and $X_{k,i}^b$ denote the k-fold unbranched and branched cyclic covers, respectively, of X_i. Then $X \simeq (\bigvee_{j=1}^{p-1} S_j^{n+1}) \vee \{\bigvee_{i=1}^{p} X_i\}$. Let $S = \bigvee_{j=1}^{p-1} S_j^{n+1}$. Then $X_k^u \simeq (\bigcup X_{k,i}^u) \cup_{\tilde{*}} (\bigcup_{j=0}^{k-1} S_j)$, that is the union of one copy each of $X_{k,i}^u$ and k copies $\{S_j\}_{j=0}^{k-1}$ of S, identified along k points $\tilde{*}$, the lift of the base point $*$ in the wedge product decomposition for X. Now the Mayer-Vietoris sequence for X_k^u gives us

(a) $H_i(X_k^u) \simeq \bigoplus_{i=1}^{p} H_i(X_{k,i}^u)$ for $2 \leq i \leq n$, and

(b) $0 \to \bigoplus_{i=1}^{p} H_1(X_{k,i}^u) \to H_1(X_k^u) \to \mathbf{Z} \to 0$, and

(c) $H_{n+1}(X_k^u) = \bigoplus_{i=1}^{p} H_{n+1}(X_{k,i}^u) \oplus k(p-1)\mathbf{Z}$.

Likewise, we get (a) and (b) relating the homology of X_k^b and $\{X_{k,i}^b\}$. Moreover we have that the inclusion i: $X_k^u \to X_k^b$ induces the following exact diagram, when the link L is completely splittable:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mu\mathbf{Z} \arrow[r] \arrow[d] & \mu\mathbf{Z} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i=1}^{\mu} H_1(X_{k,i}^u) \arrow[r] \arrow[d, "i_*"] & H_1(X_k^u) \arrow[r] \arrow[d, "i_*"] & \mathbf{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^{\mu} H_1(X_{k,i}^b) \arrow[r] \arrow[d] & H_1(X_k^b) \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The left-hand column is the direct sum of μ split exact sequences, so the middle column splits.

If L is 1-simple, then as in the proof of Lemma 5.1 we have the map i: $C_\mu \to X$ inducing an isomorphism on Π_1, and if M is the mapping cylinder of i, and M_k the k-fold cover of M, then we have the k-fold cover $C_{\mu,k}$ of C_μ induced. Moreover,

$$H_1(C_{\mu,k}) \underset{\cong}{\xrightarrow{i_*}} H_1(M_k) \cong H_1(X_k^u),$$

and the elements of the bouquet C_μ lift to $C_{\mu,k}$ to generate $\mu\mathbf{Z}$ as a direct summand of $H_1(X_k^u)$. This copy of $\mu\mathbf{Z}$ is exactly that generated by the meridians in ∂X_g^u, so (i) is proved.

Now X_k^b is an orientable $(n+2)$-manifold, and we have from the exact sequence of the pair (X_k^b, X_k^u),

$$\begin{array}{ccccc} 0 \longrightarrow & H_{n+2}(X_k^b) & \xrightarrow{j_*} & H_{n+2}(X_k^b, X_k^u). \\ & \Big\downarrow \cong & & \Big\downarrow \cong \\ & Z & \xrightarrow{(1,\ldots,1)} & \mu Z \end{array}$$

That is, the map j_* can be represented by the $(1\times\mu)$ matrix $(1,\ldots,1)$. Hence

$$0 \longrightarrow (\mu-1)Z \xrightarrow{\partial} H_{n+1}(X_k^u) \xrightarrow{i_*} H_{n+1}(X_k^b) \longrightarrow 0 \tag{1}$$

is exact. Moreover, if L is completely splittable, then each of the splitting spheres $\{S_j^{n+1}\}$ in X lifts to a splitting sphere in X_k^u, so as in §4 we can isolate $\bigvee_{j=1}^{\mu-1} S_j^{n+1}$ in a wedge product decomposition for X_k^u. It is also clear that the direct summand $(\mu-1)Z$ of $H_{n+1}(X_k^u)$ from the wedge product decomposition is precisely $\operatorname{Im}\partial$ in (1) above.

If L is $(n+1)$-simple, the argument follows that of (i) above and is left to the reader. This completes the proof of Theorem 5.4.

Note that the sequence $0\to\mu Z\to H_1(X_k^u; Z)\to H_1(X_k^b)\to 0$ can fail to be split exact if L is not completely splittable. We have the following example for $n=1$, $\mu=2$, $k=2$ due to S. Kinoshita:

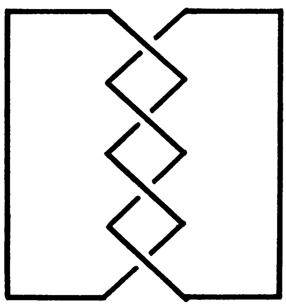

FIGURE 4

For the link of Figure 4, we have

$$\begin{array}{ccccccc} 0 \longrightarrow Z\oplus Z \longrightarrow & H_1(X_2^u) & \longrightarrow & H_1(X_2^b) & \longrightarrow 0 \\ & \wr\| & & \wr\| & \\ & Z\oplus Z\oplus Z_2 & & Z_4 & \end{array}$$

which does not split.

Combining previous results, we have

COROLLARY 5.5. *L a q-simple n-link. Then*

(i) *If* $1 \leqq q \leqq n-1$ *then* $H_1(X_k^b; \boldsymbol{Z}) = [(k-1)(\mu-1)]\boldsymbol{Z}$, $H_i(X_k^b; \boldsymbol{Z}) = 0$, $2 \leqq i \leqq q$, *and* $H_{q+1}(X_k^b; \boldsymbol{Z}) \cong H_{q+1}(X_k^u; \boldsymbol{Z}) \cong \operatorname{Cok}(t^k - 1)$ *where* $(t^k-1): H_{q+1}(\tilde{X}; \boldsymbol{Z}) \to H_{q+1}(\tilde{X}; \boldsymbol{Z})$.

(ii) *If* $1 \leqq q \leqq n-1$, $k=2$ *and* $M(t)$ *presents* $H_{q+1}(\tilde{X}; \boldsymbol{Z})$ *as a* Λ*-module then* $M(-1)$ *presents* $H_{q+1}(X_2^b; \boldsymbol{Z})$ *as an abelian group.*

COROLLARY 5.6. *Suppose that* L *is a* q*-simple* $(2q+1)$*-link of multiplicity* μ, *where* $q \geqq 1$ *if* $\mu \geqq 2$. *If* A *is a Seifert matrix for* L, *then* $A + (-1)^q A'$ *is a presentation matrix for* $H_{q+1}(X_2^b; \boldsymbol{Z})$.

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