

## ENTROPY-EXPANSIVE MAPS

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**Abstract.** Let  $f: X \rightarrow X$  be a uniformly continuous map of a metric space.  $f$  is called  $h$ -expansive if there is an  $\varepsilon > 0$  so that the set  $\Phi_\varepsilon(x) = \{y : d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0\}$  has zero topological entropy for each  $x \in X$ . For  $X$  compact, the topological entropy of such an  $f$  is equal to its estimate using  $\varepsilon$ :  $h(f) = h(f, \varepsilon)$ . If  $X$  is compact finite dimensional and  $\mu$  an invariant Borel measure, then  $h_\mu(f) = h_\mu(f, \mathcal{A})$  for any finite measurable partition  $\mathcal{A}$  of  $X$  into sets of diameter at most  $\varepsilon$ . A number of examples are given. No diffeomorphism of a compact manifold is known to be *not*  $h$ -expansive.

Let  $f: X \rightarrow X$  be a homeomorphism of a metric space. For  $\varepsilon > 0$  and  $x \in X$  define

$$\Gamma_\varepsilon(x) = \{y \in X : d(f^n(y), f^n(x)) \leq \varepsilon \text{ for all } n \in \mathbb{Z}\}.$$

$f$  is called *expansive* if for some  $\varepsilon$  these sets are as small as possible, i.e. if  $\Gamma_\varepsilon(x) = x$  for all  $x$ . We are concerned with entropy and shall call  $f$  *h-expansive* provided that for some  $\varepsilon > 0$  the  $\Gamma_\varepsilon(x)$  are negligible in terms of entropy, i.e. if the topological entropy  $h(f, \Gamma_\varepsilon(x)) = 0$  for all  $x$ .

We have two main results for  $h$ -expansive maps with  $X$  compact. First, the topological entropy satisfies  $h(f) = h(f, \varepsilon)$ . Second, assuming  $X$  is finite dimensional,  $h_\mu(f) = h_\mu(f, \mathcal{A})$  when  $\mu$  is an  $f$ -invariant normalized Borel measure on  $X$  and  $\mathcal{A}$  is a finite measurable partition of  $X$  into sets of diameter at most  $\varepsilon$ . Both these results are well known in case  $f$  is expansive (see [11] and [14] respectively). Arov [2] noted that the second statement was true for  $f$  an endomorphism of a torus and  $\mu$  Haar measure when he calculated  $h_\mu(f)$  for this case (see Example 1.2).

**1. Definitions and examples.** We now review the definition of topological entropy given in [4]. For  $X$  compact this definition was given independently by Dinaburg [7]; is related to the  $\varepsilon$ -entropy of Kolmogorov [12]. Topological entropy was defined first in [1].

Let  $f: X \rightarrow X$  be uniformly continuous on the metric space  $X$ . For  $E, F \subset X$  we say that  $E(n, \delta)$ -spans  $F$  (with respect to  $f$ ), if for each  $y \in F$  there is an  $x \in E$  so that  $d(f^k(x), f^k(y)) \leq \delta$  for all  $0 \leq k < n$ . We let  $r_n(F, \delta) = r_n(F, \delta, f)$  denote the minimum cardinality of a set which  $(n, \delta)$ -spans  $F$ . If  $K$  is compact, then the continuity of  $f$  guarantees  $r_n(K, \delta) < \infty$ . For compact  $K$  we define

$$\bar{r}_f(K, \delta) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(K, \delta)$$

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Received by the editors January 20, 1971.

AMS 1970 subject classifications. Primary 28A65, 54H20.

Key words and phrases. Entropy,  $h$ -expansive.

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and

$$h(f, K) = \lim_{\delta \rightarrow 0} \bar{r}_f(K, \delta)$$

(notice that  $\bar{r}_f(K, \delta)$  increases as  $\delta$  decreases). Finally let  $h(f) = \sup_K h(f, K)$  where  $K$  varies over all compact subsets of  $X$ . If  $X$  is compact, then  $h(f) = h(f, X)$  and we write  $h(f, \delta) = \bar{r}_f(X, \delta)$ .

Let  $\Phi_\epsilon(x) = \bigcap_{n \geq 0} f^{-n} B_\epsilon(f^n(x)) = \{y : d(f^n(x), f^n(y)) \leq \epsilon \text{ for } n \geq 0\}$  and  $h_f^*(\epsilon) = \sup_{x \in X} h(f, \Phi_\epsilon(x))$ .  $f$  is called *h-expansive* if  $h_f^*(\epsilon) = 0$  for some  $\epsilon > 0$ . In case  $f$  is a homeomorphism we set

$$\Gamma_\epsilon(x) = \bigcap_{n \in \mathbb{Z}} f^{-n} B_\epsilon(f^n(x))$$

and

$$h_{f, \text{homeo}}^*(\epsilon) = \sup_{x \in X} h(f, \Gamma_\epsilon(x)).$$

REMARK. For  $f$  a homeomorphism,  $\Gamma_\epsilon(x) \subset \Phi_\epsilon(x)$  and so  $h_{f, \text{homeo}}^*(\epsilon) \leq h_f^*(\epsilon)$ . The definition of *h-expansiveness* for homeomorphisms mentioned in the introduction, namely  $h_{f, \text{homeo}}^*(\epsilon) = 0$ , is actually equivalent to the above one in case  $X$  is compact. For in 2.3 we prove  $h_f^*(\epsilon) = h_{f, \text{homeo}}^*(\epsilon)$  when  $X$  compact.

EXAMPLE 1.0. Expansive maps.

EXAMPLE 1.1. If  $f: R^n \rightarrow R^n$  is linear and  $d$  comes from a norm, then  $h_f^*(\epsilon) = 0$  for every  $\epsilon$ .

Proof.  $f$  decomposes into a direct sum of linear maps  $f = f_1 \oplus f_2: E_1 \oplus E_2 \rightarrow E_1 \oplus E_2$  where  $f_1$ 's eigenvalues have norm at most 1 and  $f_2$ 's have norm greater than 1. If  $u \in E_2, u \neq 0$ , then  $d(f_2^n(u), 0) \rightarrow \infty$  as  $n \rightarrow \infty$ . It follows that  $\Phi_\epsilon(0) \subset E_1$ . But  $h(f|E_1) = h(f_1) = 0$  by Theorem 15 of [4]. So  $h(f, \Phi_\epsilon(0)) = 0$ . But  $\Phi_\epsilon(x) = \Phi_\epsilon(0) + x$  and  $h(f, K+x) = h(f, K)$  for any compact set  $K$ .

EXAMPLE 1.2. An endomorphism  $f$  of a Lie group  $G$  is *h-expansive*.

Proof. Here we use a right invariant metric  $d$ . Then one checks  $\Phi_\epsilon(x) = \Phi_\epsilon(e)x$  and  $h(f, Kx) = h(f, K)$  for compact  $K$ . So it is enough to see  $h(f, \Phi_\epsilon(e)) = 0$  for some  $\epsilon$ . Now

$$\begin{array}{ccc} T_e G & \xrightarrow{df} & T_e G \\ \downarrow \text{exp} & & \downarrow \text{exp} \\ G & \xrightarrow{f} & G \end{array}$$

commutes and  $\text{exp}$  is a homeomorphism of a small neighborhood  $B_\alpha(0) \subset T_e G$  onto a neighborhood of some  $B_\epsilon(e)$ . Then  $\Phi_\epsilon(e, f) \subset \text{exp } \Phi_\alpha(0, df)$  and since  $f| \text{exp } \Phi_\alpha(0, df)$  is a quotient of  $df| \Phi_\alpha(0, df)$  one has

$$h(f, \Phi_\epsilon(e, f)) \leq h(df, \Phi_\alpha(0, df)) = 0.$$

EXAMPLE 1.3. Suppose  $f$  is *h-expansive* and  $T$  a uniformly continuous map so that  $(T \cdot f)^n = T_n \cdot f^n$  for  $n \geq 0$  where the  $T_n$  are isometries. Then  $T \cdot f$  is *h-expansive*.

**Proof.** One checks easily that  $\Phi_\epsilon(x, T \cdot f) = \Phi_\epsilon(x, f)$  and that a set which  $(n, \delta)$ -spans some  $F \subset X$  with respect to  $f$  also  $(n, \delta)$ -spans  $F$  with respect to  $T \cdot f$ . It follows that

$$h(T \cdot f, \Phi_\epsilon(x, T \cdot f)) \leq h(f, \Phi_\epsilon(x, f)) = 0.$$

**EXAMPLE 1.3\*.** Let  $G$  be a Lie group and for  $g, u \in G$  define  $L_g(u) = gu$  and  $R_g(u) = ug$ . If  $f$  is an endomorphism of  $G$  and  $g \in G$ , then the affine maps  $R_g \cdot f, f \cdot R_g, L_g \cdot f$  and  $f \cdot L_g$  are all  $h$ -expansive.

**Proof.** If we set  $g_1 = g$  and  $g_{n+1} = f(g_n)g$ , one sees that  $(R_g \cdot f)^n = R_{g_n} \cdot f^n$ . As we use a right invariant metric,  $R_{g_n}$  is an isometry and 1.3 applies. Now  $(L_g \cdot f)(u) = gf(u) = (gf(u)g^{-1})g = (R_g \cdot f^*)(u)$  where  $f^*(u) = gf(u)g^{-1}$  is an endomorphism. We leave  $f \cdot R_g$  and  $f \cdot L_g$  to the reader.

**EXAMPLE 1.4.** Suppose  $H$  is a uniformly discrete subgroup of the Lie group  $G$ , i.e.  $G/H$  is compact and  $\pi: G \rightarrow G/H$  given by  $\pi(x) = xH$  is a covering. For  $f$  an endomorphism of  $G$  with  $f(H) \subset H$  and  $g \in G$  define  $f^*$  on  $G/H$  by  $f^*(uH) = gf(u)H$ . Then  $f^*$  is  $h$ -expansive.

**Proof.** For  $\delta$  small enough  $\pi$  maps  $B_\delta(x)$  isometrically onto  $B_\delta(xH)$  for every  $x$  and  $\pi\Phi_\delta(x, L_g \cdot f) = \Phi_\delta(xH, f^*)$ . Then

$$h(f^*, \Phi_\delta(xH, f^*)) \leq h(L_g \cdot f, \Phi_\delta(x, L_g \cdot f)) = 0.$$

So  $f^*$  is  $h$ -expansive (see [4] for some more details).

**EXAMPLE 1.5.** For the case of  $X$  compact define the nonwandering set

$$\Omega(f) = \left\{ x \in X : \text{for every neighborhood } U \text{ of } x, U \cap \bigcup_{n>0} f^n(U) \neq \emptyset \right\}.$$

Then  $f\Omega(f) \subset \Omega(f)$ . If  $f|_{\Omega(f)}$  is  $h$ -expansive, then so is  $f$ . An example of this is one of Smale's Axiom A diffeomorphisms [15], where  $f|_{\Omega(f)}$  is expansive.

**Proof.** Splice together the proof of Theorem 2.4 in [5] and that of 2.2 below for  $f|_{\Omega}$ ,  $a=0$  and  $x$  staying in a neighborhood of  $\Omega$  up to time  $n$ .

**EXAMPLE 1.6.** Suppose  $\Phi = \{\varphi_t: X \rightarrow X\}_{t \in \mathbb{R}}$  is a continuous flow on a compact metric space  $X$ . Suppose also that there are  $\epsilon > 0$  and  $s > 0$  so that

$$\begin{aligned} \Gamma_\epsilon(x, \Phi) &= \{y \in X : d(\varphi_t(y), \varphi_t(x)) \leq \epsilon \text{ for all } t \in \mathbb{R}\} \\ &\subset \varphi_{[-s, s]}(x) = \{\varphi_r(x) : |r| \leq s\}. \end{aligned}$$

Then each  $\varphi_t$  is  $h$ -expansive.

**Proof.** For any  $t \in \mathbb{R}$  there is a  $\delta$  so that  $d(x, y) \leq \delta$  implies  $d(\varphi_r(x), \varphi_r(y)) \leq \epsilon$  for all  $|r| \leq |t|$ . Then

$$\Gamma_\delta(x, \varphi_t) \subset \Gamma_\epsilon(x, \Phi) \subset \varphi_{[-s, s]}(x).$$

For  $\beta > 0$  choose  $\alpha > 0$  such that for all  $x \in X$  and all  $|r| \leq \alpha$  we have  $d(x, \varphi_r(x)) \leq \delta$  (here we use  $X$  compact). Let  $K$  be a set of numbers so that every point in  $[-s, s]$  is

within  $\alpha$  of one of them. Then  $\{\varphi_u(x) : u \in K\}$  ( $n, \delta$ )-spans  $\varphi_{[-s,s]}(x)$  with respect to  $\varphi_t$ . Hence

$$r_n(\varphi_{[-s,s]}(x), \delta, \varphi_t) \leq \text{card } K$$

and

$$h(\varphi_t, \Gamma_\delta(x, \varphi_t)) \leq h(\varphi_t, \varphi_{[-s,s]}(x)) = 0.$$

EXAMPLE 1.6\*. Let  $\Phi = \{\varphi_t\}$  be one of Smale's Axiom A flows [15]. Then  $\Phi|\Omega(\Phi)$  satisfies the condition of 1.6 [9]. By 1.5 and 1.6, each  $\varphi_t$  is  $h$ -expansive.

*Problem.* Find some differentiable maps which are *not*  $h$ -expansive.

**2. Calculating topological entropy.**

*Assumption.* For the remainder of the paper  $X$  is compact.

LEMMA 2.1. *Suppose*  $0 = t_0 < t_1 < \dots < t_{r-1} < t_r = n$  and  $E_i(t_{i+1} - t_i, \alpha)$ -spans  $f^{t_i}(F)$  for  $0 \leq i < r$ . Then

$$r_n(F, 2\alpha) \leq \prod_{0 \leq i < r} \text{card } E_i.$$

**Proof.** For  $x_i \in E_i$  write

$$V(x_0, \dots, x_{r-1}) = \{x \in F : d(f^{t+t_i}(x), f^{t_i}(x_i)) \leq \alpha \text{ for } 0 \leq t < t_{i+1} - t_i, 0 \leq i < r\}.$$

If  $x, y \in V(x_0, \dots, x_{r-1})$ , then by the triangle inequality  $d(f^s(x), f^s(y)) \leq 2\alpha$  for  $0 \leq s < n$ . Since  $F = \bigcup V(x_0, \dots, x_{r-1})$  we get an  $(n, 2\alpha)$ -spanning set for  $F$  by taking one element from each nonempty  $V(x_0, \dots, x_{r-1})$ .

PROPOSITION 2.2. *Let*  $a = h_f^*(\epsilon)$  or  $h_{f, \text{homeo}}^*(\epsilon)$  (in case  $f$  is a homeomorphism). Then for every  $\delta > 0$  and  $\beta > 0$  there is a  $c$  such that

$$r_n\left(\bigcap_{k=0}^{n-1} f^{-k} B_\epsilon(f^k(x)), \delta\right) \leq ce^{(a+\beta)n}$$

for all  $x \in X$ .

**Proof.** We do the case where  $f$  is a homeomorphism and  $a = h_{f, \text{homeo}}^*(\epsilon)$ . The case where  $a = h_f^*(\epsilon)$  is slightly simpler and we leave the necessary modifications to the reader.

For each  $y \in X$  pick  $m(y)$  so that  $a + \beta \geq (1/m(y)) \log \text{card } E(y)$  where  $E(y)$  is a set which  $(m(y), \frac{1}{4}\delta)$ -spans  $\Gamma_\epsilon(y)$ . Then  $U(y) = \{w \in X : \exists z \in E(y) \text{ such that } d(f^k(w), f^k(z)) < \frac{1}{2}\delta \text{ for all } 0 \leq k < m(y)\}$  is an open neighborhood of the compact set  $\Gamma_\epsilon(y)$ . Let  $S_M = \bigcap_{|j| \leq M} f^{-j} B_\epsilon(f^j(y))$ . Then  $S_0 \supset S_1 \supset \dots$  is a decreasing chain of compact sets with intersection  $\Gamma_\epsilon(y)$ ; hence there is an integer  $N(y)$  so that  $S_{N(y)} \subset U(y)$ . Consider the compact sets  $W_\gamma = \bigcap_{|j| \leq N(y)} f^{-j} B_\gamma(f^j(y))$ . Then  $\bigcap_{\gamma > \epsilon} W_\gamma = W_\epsilon = S_{N(y)} \subset U(y)$ ; hence,  $W_\gamma \subset U(y)$  for some  $\gamma > \epsilon$ . Let  $V(y)$  be a neighborhood of  $y$  such that  $d(f^j(u), f^j(y)) < \gamma - \epsilon$  for  $|j| \leq N(y)$  when  $u \in V(y)$ . Then  $B_\epsilon(f^j(u)) \subset B_\gamma(f^j(y))$  and

$$\bigcap_{|j| \leq N(y)} f^{-j} B_\epsilon(f^j(u)) \subset U(y).$$

Let  $V(y_1), \dots, V(y_s)$  cover the compact space  $X$  and

$$N = \max \{N(y_1), \dots, N(y_s), m(y_1), \dots, m(y_s)\} + 1.$$

Consider now any  $x \in X$  and  $F_n = \bigcap_{j=0}^{n-1} f^{-j} B_\epsilon(f^j(x))$ . For any  $t \in [N, n-N]$ ,  $f^t(x)$  is in some  $V(y_i)$  and

$$f^t(F_n) = \bigcap_{k=-t}^{n-t-1} f^{-k} B_\epsilon(f^k(f^t(x))) \subset \bigcap_{|k| \leq N(y_1)} f^{-k} B_\epsilon(f^k(f^t(x))) \subset U(y_i).$$

Now  $E(y_i)$  ( $m(y_i), \frac{1}{2}\delta$ )-spans  $U(y_i)$ , so it does  $f^t(F_n)$  also.

We shall define integers  $0 = t_0 < t_1 < \dots < t_r = n$ . If  $n \leq N$ , let  $r = 1$  and  $t_1 = n$ . If  $n > N$ , take  $t_1 = N$  and pick  $V(y_{i_1})$  containing  $f^{t_1}(x)$ . Suppose we have chosen  $t_1, \dots, t_k$  and  $y_{i_1}, \dots, y_{i_k}$  (with  $t_k < n$ ). If  $t_k > n - N$ , then set  $r = k + 1$  and  $t_r = N$ . If  $t_k \leq n - N$ , then set  $t_{k+1} = t_k + m(y_{i_k}) < n$  and choose  $V(y_{i_{k+1}})$  containing  $f^{t_{k+1}}(x)$ . Eventually this process stops.

Let  $K$  be a set which  $(N, \frac{1}{2}\delta)$ -spans  $X$ . Then  $K$  ( $t_1 - t_0, \frac{1}{2}\delta$ )-spans  $F_n$  and also  $(t_r - t_{r-1}, \frac{1}{2}\delta)$ -spans  $f^{t_{r-1}}(F_n)$ . From the way the  $t_k$ 's and  $y_{i_k}$ 's were chosen we see that, for  $0 < k < r - 1$ ,  $E(y_{i_k})$  ( $t_{k+1} - t_k, \frac{1}{2}\delta$ )-spans  $f^{t_k}(F_n)$ . Lemma 2.1 applies to give

$$\begin{aligned} r_n(F_n, \delta) &\leq (\text{card } K)^2 \prod_{0 < k < r-1} \text{card } E(y_{i_k}) \\ &\leq (\text{card } K)^2 \prod_{0 < k < r-1} \exp((a + \beta)(n(y_{i_k}))) \leq (\text{card } K)^2 e^{(a + \beta)n}. \end{aligned}$$

**COROLLARY 2.3.** *If  $f$  is a homeomorphism, then  $h_{f, \text{homeo}}^*(\epsilon) = h_f^*(\epsilon)$ .*

**Proof.** Let  $a = h_{f, \text{homeo}}^*(\epsilon)$ . Fixing  $\beta, \delta$  the proposition gives us  $r_n(\Phi_\epsilon(x), \delta) \leq ce^{(a + \beta)n}$ . Hence  $\bar{r}_f(\Phi_\epsilon(x), \delta) \leq a + \beta$  and  $h(f, \Phi_\epsilon(x)) \leq a + \beta$ . As  $\beta > 0$  was arbitrary,  $h(f, \Phi_\epsilon(x)) \leq a$  and  $h_f^*(\epsilon) \leq a = h_{f, \text{homeo}}^*(\epsilon)$ . The reverse inequality we noted before.

**THEOREM 2.4.**  *$h(f) \leq h(f, \epsilon) + h_f^*(\epsilon)$ . In particular,  $h(f) = h(f, \epsilon)$  if  $\epsilon$  is an  $h$ -expansive constant for  $f$ .*

**Proof.** Let  $\delta > 0$  and  $\beta > 0$ . Let  $E_n(n, \epsilon)$ -span  $X$ , i.e.

$$X = \bigcup_{x \in E_n} \bigcap_{k=0}^{n-1} f^{-k} B_\epsilon(f^k(x)).$$

By Proposition 2.2 there is a constant  $c$  so that each of the sets in the above union can be  $(n, \delta)$ -spanned by using at most  $ce^{(a + \beta)n}$  elements (where  $a = h_f^*(\epsilon)$ ). Hence  $r_n(X, \delta) \leq \text{card } E_n ce^{(a + \beta)n} \leq r_n(X, \epsilon) ce^{(a + \beta)n}$ . It follows that  $h(f, \delta) \leq h(f, \epsilon) + a + \beta$ . Letting  $\beta \rightarrow 0$ ,  $h(f, \delta) \leq h(f, \epsilon) + a$ . Now letting  $\delta \rightarrow 0$  we get our result.

If  $h_f^*(\epsilon) = 0$ , then  $h(f) \leq h(f, \epsilon)$ . But  $h(f) \geq h(f, \epsilon)$  from the definition of  $h(f)$ ; hence  $h(f) = h(f, \epsilon)$ .

**COROLLARY 2.5.** *If  $h(f) = h(f, \epsilon) + h_f^*(\epsilon)$ , then  $(1/n) \log r_n(X, \epsilon) \rightarrow h(f, \epsilon)$ . In particular, if  $h_f^*(\epsilon) = 0$ , then  $(1/n) \log r_n(f, \epsilon) \rightarrow h(f)$ .*

**Proof.** Otherwise there is an increasing sequence of integers  $\{n_k\}$  so that  $(1/n_k) \log r_{n_k}(X, \epsilon) \rightarrow b < h(f, \epsilon)$ . Let  $a = h_f^*(\epsilon)$ . Then  $h(f) > a + b$  and, for  $\gamma > 0$

small enough,  $h(f, \gamma) > a + b$ . Choose  $\beta > 0$  so that  $h(f, \gamma) > a + b + \beta$ . For some  $c$ , as in the proof of the theorem, we have  $r_{n_k}(X, \frac{1}{2}\gamma) \leq r_{n_k}(X, \epsilon)c \exp((a + \beta)n_k)$ . So

$$\limsup_{k \rightarrow \infty} \frac{1}{n_k} \log r_{n_k}(X, \frac{1}{2}\gamma) \leq b + a + \beta < h(f, \gamma).$$

Choose  $R$  so that  $(1/R) \log r_R(X, \frac{1}{2}\gamma) = \alpha < h(f, \gamma)$ . This means there is an  $(R, \frac{1}{2}\gamma)$ -spanning set for  $X$  with  $e^{R\alpha}$  elements. By Lemma 2.1 (using  $t_k = kR$ ) one gets  $r_{Rp}(X, \gamma) \leq (e^{R\alpha})^p$ . For  $0 \leq q \leq R$ ,

$$r_{Rp+q}(X, \gamma) \leq r_{R(p+1)}(X, \gamma) \leq e^{R\alpha(p+1)}.$$

Hence

$$h(f, \gamma) = \limsup_{n = Rp+q \rightarrow \infty} \frac{1}{n} \log r_n(X, \gamma) \leq \limsup_{p \rightarrow \infty} \frac{(p+1)R\alpha}{Rp} = \alpha.$$

But we chose  $\alpha < h(f, \gamma)$ , a contradiction.

If  $h_f^*(\epsilon) = 0$ , then  $h(f) = h(f, \epsilon)$  by the theorem, and so the first statement applies.

REMARKS. For expansive homeomorphisms the second part of 2.4 was proved in [11] and the second part of 2.5 in [6]. In the original definition of topological entropy using open covers [1] certain limits existed whose analogues might not exist when one uses spanning sets. 2.5 is a technical result giving us conditions which insure that these limits exist. It has an application in counting periodic orbits of the Axiom A diffeomorphisms and flows of Smale (see [6]).

3. **Measures.** We continue to assume  $f: X \rightarrow X$  is continuous and  $X$  a compact metric space.  $\mu$  denotes a Borel measure on  $X$  with  $\mu(X) = 1$  which is  $f$ -invariant, i.e.  $\mu(f^{-1}(E)) = \mu(E)$  for Borel sets  $E$ .

We call  $A = \{A_1, \dots, A_r\}$  a (finite) *Borel partition* provided the  $A_i$  are pairwise disjoint Borel sets whose union is  $X$ . (Note that any finite  $\mu$ -measurable partition is  $\mu$ -equivalent to a Borel partition.) We write

$$H_\mu(A) = \sum_{i=1}^r -\mu(A_i) \log \mu(A_i).$$

If  $A, B$  are two Borel partitions, so is  $A \vee B = \{A \cap B : A \in A, B \in B\}$ . Setting  $A^n = A \vee f^{-1}A \vee \dots \vee f^{-(n-1)}A$ , one defines the entropies (of Kolmogorov and Sinai, see [3])

$$h_\mu(f, A) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(A^n) \quad \text{and} \quad h_\mu(f) = \sup_A h_\mu(f, A).$$

An important device for calculating  $h_\mu(f)$  in some examples is Goodwyn's theorem [8]:  $h_\mu(f) \leq h(f)$ . We shall use his ideas to prove a stronger statement for the case of  $X$  finite dimensional:  $h_\mu(f) \leq h_\mu(f, A) + h_f^*(\epsilon)$  where  $A$  is a finite Borel partition with  $\text{diam } A = \max \{\text{diam } A : A \in A\} \leq \epsilon$ . This reduces to Goodwyn's theorem when we take  $A = \{X\}$  and  $\epsilon = \text{diam } X$ . If  $\epsilon$  is an  $h$ -expansive constant for  $f$ , it gives  $h_\mu(f) = h_\mu(f, A)$ .

LEMMA 3.1. *If  $a_1, \dots, a_n \geq 0$  and  $s = \sum_{i=1}^n a_i \leq 1$ , then*

$$\sum -\mu(a_i) \log \mu(a_i) \leq s(\log n - \log s).$$

**Proof.** This is a well-known case of Jensen's inequality [13, pp. 11–12].

LEMMA 3.2. *Let  $A_1, A_2, \dots$  be finite Borel partitions of  $X$  with  $\text{diam } A_m \rightarrow 0$ . Then  $h_\mu(f, A_m) \rightarrow h_\mu(f)$ .*

**Proof.** This is a slight variation of a well-known result of Rohlin. Looking at 6.3, 8.6 and 9.5 of [16], one sees that our lemma is implied by the following statement:

Given a Borel partition  $\beta = \{B_1, \dots, B_n\}$  and  $\epsilon > 0$ , then for large  $m$  we can find a partition  $\alpha = \{C_1, \dots, C_n\}$  coarser than  $A_m$  (i.e. each  $C_i$  is the union of members of  $A_m$ ) so that  $\mu(B_i \triangle C_i) < \epsilon$  for  $1 \leq i \leq n$ .

We now prove this statement. Since  $\mu$  is a Borel measure, one can choose compact sets  $K_i \subset B_i$  with  $\mu(B_i \setminus K_i) < \epsilon/n$ . Choose  $\delta > 0$  so that  $d(K_i, K_j) > \delta$  for  $i \neq j$  and suppose  $\text{diam } A_m < \delta$ . Form  $\alpha = \{C_1, \dots, C_n\}$  coarser than  $A_m$  by putting  $A \in \mathcal{A}$  into

- (a)  $C_i$  if  $A \cap K_i \neq \emptyset$  or
- (b)  $C_k$  if  $A \cap K_i = \emptyset$  for all  $i$ .

This makes sense, for if  $x \in A \cap K_i$  and  $y \in A \cap K_j$ , then

$$d(K_i, K_j) \leq d(x, y) \leq \text{diam } A < \delta$$

and so  $i=j$ .

Clearly  $C_i \supset K_i$ . Hence  $\mu(B_i \setminus C_i) \leq \mu(B_i \setminus K_i) < \epsilon/n$ . Since  $C_i \setminus B_i \subset \bigcup_{j \neq i} (B_j \setminus C_j)$ ,  $\mu(C_i \setminus B_i) < (n-1)\epsilon/n$ . Thus  $\mu(B_i \triangle C_i) < \epsilon$ .

REMARK. We used 3.2 in [4] but stated there (in the introduction) instead a stronger form—which we cannot prove.

Suppose now that  $\mathfrak{B}$  is any finite cover of  $X$ . For  $E \subset X$  let

$$F(E, \mathfrak{B}) = \{B \in \mathfrak{B} : B \cap E \neq \emptyset\}.$$

We give a very slight modification of Proposition 2 of [8].

LEMMA 3.3. *Let  $\mathfrak{B}$  be a finite cover of  $X$  by closed sets such that each point  $x \in X$  lies in at most  $m$  elements of  $\mathfrak{B}$ . There is a  $\delta > 0$  so that  $\text{card } F(E, \mathfrak{B}^n) \leq r_n(\delta, E)m^n$  for all  $E \subset X, n \geq 0$ .*

**Proof.** For each  $x \in X$  choose a neighborhood  $U_x$  intersecting at most  $m$  elements of  $\mathfrak{B}$ . Let  $U_{x_1}, \dots, U_{x_r}$  cover  $X$  and  $\delta > 0$  be a Lebesgue number for this open cover. For each  $n$  let  $K_n$  be a set which  $(n, \delta)$ -spans  $E$  and has  $r_n(E, \delta)$  elements. For each  $\beta \in F(E, \mathfrak{B}^n)$  pick  $p(\beta) \in E \cap \beta$  and  $q(\beta) \in K_n$  so that  $d(f^t(q(\beta)), f^t(p(\beta))) \leq \delta$  for  $0 \leq t \leq n$ . If  $\beta = \bigcap_{i=0}^{n-1} B_{i_t}$ ,  $B_{i_t} \in \mathfrak{B}$ , then  $f^t p(\beta) \in B_\delta(f^t q(\beta)) \cap B_{i_t} \neq \emptyset$ . Since  $B_\delta(f^t q(\beta))$  lies inside some  $U_{x_j}$ , for a given  $q(\beta)$  there are at most  $m$  possibilities for  $B_{i_t}$ . It follows that, for  $z \in K_n$ ,  $\text{card } q^{-1}(z) \leq m^n$ . Hence  $\text{card } F(E, \mathfrak{B}^n) \leq (\text{card } K_n)m^n$ .

DEFINITION. For  $A, B$  two Borel partitions let

$$b(A, B) = \max_{A \in \mathcal{A}} \text{card } F(A, B).$$

LEMMA 3.4.  $h_\mu(f, A \vee B) \leq h_\mu(f, A) + \liminf_{n \rightarrow \infty} \frac{1}{n} \log b(A^n, B^n).$

**Proof.** Since  $(A \vee B)^n = A^n \vee B^n,$

$$H_\mu((A \vee B)^n) = \sum_{\alpha \in A^n} \sum_{\beta \in F(\alpha, B^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta).$$

By Lemma 3.1

$$\sum_{\beta \in F(\alpha, B^n)} -\mu(\alpha \cap \beta) \log \mu(\alpha \cap \beta) \leq \mu(\alpha)(\log b(\alpha, B^n) - \log \mu(\alpha))$$

and so

$$H_\mu((A \vee B)^n) \leq \log b(A^n, B^n) + H_\mu(A^n).$$

Divide by  $n$  and let  $n \rightarrow \infty.$

**THEOREM 3.5.** *Assume  $X$  is finite dimensional. Let  $A$  be a Borel partition of  $X$  with  $\text{diam } A \leq \epsilon.$  Then  $h_\mu(f) \leq h_\mu(f, A) + h_f^*(\epsilon)$  for any normalized  $f$ -invariant Borel measure  $\mu.$  If  $\epsilon$  is an  $h$ -expansive constant for  $f,$  then  $h_\mu(f) = h_\mu(f, A).$*

**Proof.** Say  $\dim X = m - 1.$  Then for each  $\gamma > 0$  we can find a finite closed cover  $\mathfrak{B} = \mathfrak{B}(\gamma)$  with diameter  $< \gamma$  and no point of  $X$  in more than  $m$  elements of  $\mathfrak{B}$  (see [10]). Let  $M$  be a fixed positive integer.

Let  $B = \{B_1^*, \dots, B_r^*\}$  be a Borel partition of  $X$  where  $B_i^* \subset B_i$  and  $\mathfrak{B} = \{B_1, \dots, B_r\}.$  We consider  $f^M$  with respect to the partition  $B \vee A_f^M.$  If  $\alpha \in (A_f^M)^{nM}$  and  $x \in \alpha,$  then  $\alpha \subset \bigcap_{s=0}^{nM-1} f^{-s} B_\epsilon(f^s(x)).$  Let  $\delta > 0$  be as in Lemma 3.3 and  $\beta > 0$  arbitrary. By 2.2 we have

$$r_n(\alpha, \delta, f^M) \leq r_{Mn}(\alpha, \delta, f) \leq ce^{(a+\beta)nM}$$

where  $a = h_f^*(\epsilon).$  Using Lemma 3.3 we get (the first inequality is obvious)

$$\text{card } F(\alpha, B_f^{nM}) \leq \text{card } F(\alpha, \mathfrak{B}_f^{nM}) \leq ce^{(a+\beta)nM} m^n$$

Applying Lemma 3.4,

$$h(f^M, B \vee A_f^M) \leq h_\mu(f^M, A_f^M) + M(a + \beta) + \log m.$$

Letting  $\gamma \rightarrow 0,$   $\text{diam } B \vee A_f^M \leq \text{diam } B \leq \text{diam } \mathfrak{B}(\gamma) \rightarrow 0$  and so by Lemma 3.2

$$h_\mu(f^M) \leq h_\mu(f^M, A_f^M) + M(a + \beta) + \log m.$$

Now  $h_\mu(f^M) = Mh_\mu(f)$  and  $h_\mu(f^M, A_f^M) = Mh_\mu(f, A).$  So

$$h_\mu(f) \leq h_\mu(f, A) + a + \beta + \frac{1}{M} \log m.$$

Letting  $\beta \rightarrow 0$  and then  $M \rightarrow \infty,$  we get our result.

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