TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 164, February 1972

A CHARACTERIZATION OF THE GROUP $U_3(4)$(1)

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Abstract. Let T be a Sylow 2-subgroup of the projective special unitary group $U_3(4)$, and let G be a finite group with Sylow 2-subgroups isomorphic to T. It is shown that if G is simple, then $G \cong U_3(4)$; if G has no proper normal subgroup of odd order or index, then $G \cong U_3(4)$ or T.

1. **Introduction.** We denote by $U_3(4)$ the projective special group of 3×3 unitary matrices with coefficients in the field of 4^2 elements. Let T be a Sylow 2-subgroup of $U_3(4)$. Our main result is

THEOREM 1. *Let G be a finite simple group whose Sylow 2-subgroups are isomorphic to T. Then $G \cong U_3(4)$.*

As a simple consequence we obtain

COROLLARY. *Let G be a finite group whose Sylow 2-subgroups are isomorphic to T. Suppose $O_{2'}(G) = G/O^{2'}(G) = 1$. Then $G \cong U_3(4)$ or $G \cong T$.*

Theorem 1 can be applied to complete the proof of the following result of Janko and Thompson [11].

THEOREM. *Let G be a finite nonabelian simple group with Sylow 2-subgroup S. Assume that*

(a) $SCN_3(S) = \varnothing$,

(b) *$C_G(x)$ is solvable whenever x is an involution in S such that $|S : C_S(x)| \leqq 2$.*

Then G is isomorphic to A_7, M_{11}, $L_3(3)$, $U_3(3)$, $U_3(4)$, or $L_2(q)$ for q odd.

When the classification of finite simple groups with wreathed Sylow 2-subgroups is finished (see [1]), it will combine with results of MacWilliams [12], Alperin-Brauer-Gorenstein [1], Gorenstein-Walter [10], and with Theorem 1 to provide a classification of finite simple groups in which every elementary abelian 2-subgroup has rank at most 2. If no new groups turn up in the wreathed case, then the only such groups are $L_2(q)$, $L_3(q)$, $U_3(q)$ for q odd; A_7, M_{11}, and $U_3(4)$.

Received by the editors March 22, 1971.

AMS 1969 *subject classifications*. Primary 2025, 2029.

Key words and phrases. Simple group, principal 2-block, generalized decomposition numbers.

(1) Most of this paper appeared in the author's doctoral dissertation at the University of Chicago. The author thanks his adviser, Professor J. G. Thompson, for his help and encouragement.

We note the following well-known facts about T:

(i) $|T|=2^6$;

(ii) $Z(T)=T'=\Phi(T)=\Omega_1(T)=\mho^1(T)$ is a four-group.

With a little extra effort we can prove the following slight strengthening of Theorem 1:

THEOREM 2. *Let G be a finite simple group. Suppose a Sylow 2-subgroup T of G satisfies* (i) *and* (ii). *Then $G \cong U_3(4)$.*

The proof of Theorem 2 is patterned after the characterization of M_{12} by Brauer and Fong [7]. Namely, we compute the generalized decomposition numbers for the principal 2-block of a group G satisfying the hypotheses of Theorem 2, and then use group-order formulas to conclude that G has an ordinary rational character of degree 12. From the resulting bound on $|G|$ it follows easily that G has a strongly embedded subgroup and so is isomorphic to $U_3(4)$ by a theorem of Bender [2].

2. **2-local structure.** We begin the proof of Theorem 2. Let G be a finite simple group with a Sylow 2-subgroup T satisfying (i) and (ii). Let t be a fixed element of T of order 4, and let $z=t^2$.

LEMMA 1. (a) *G has one class of involutions and one class of elements of order* 4.

(b) *Elements of order* 4 *are rational but not strongly real.*

(c) *$N_G(T)/O_{2'}(N_G(T)) \cong T\langle\beta\rangle$, where β is a fixed-point-free automorphism of T of order* 15.

(d) $C_G(z)/O_{2'}(C_G(z)) \cong T\langle\beta^3\rangle$.

(e) $|C_G(t)/O_{2'}(C_G(t))|=2^4$.

Proof. By the Z^*-theorem [8], no involution in $Z(T)$ is weakly closed in T. Since $Z(T)=\Omega_1(T)$ contains just three involutions, they must all be fused in G. Hence G has one class of involutions. Moreover, by a result of Burnside, $K= N_G(T)/TC_G(T)$ contains an element α of order 3 acting fixed-point-free on T. In particular, all involutions in T have the same number (20) of square roots in T.

As $|T/\Phi(T)|=2^4$, $|K| \mid 3^2\cdot 5\cdot 7$. We claim $|K|=15$, which will prove (c). Suppose $x\in K$ has order 7. Then $|C_T(x)|=2^3$ and x centralizes $Z(T)$, so stabilizes each set of 20 square roots of elements of $Z(T)^{\#}$. Therefore $|C_T(x)|\geqq 3.6$, a contradiction. Hence $7\nmid |K|$. Suppose $9 \mid |K|$; then K contains a Sylow 3-subgroup $\langle\alpha, \alpha_1\rangle$, where $\alpha_1^3=1$ and $|C_{T/\Phi(T)}(\alpha_1)|=4$. Then α_1 must centralize $Z(T)$, so $|C_T(\alpha_1)|=16$. Since α_1 commutes with α, it must fix the same number of square roots of each involution of T, and hence fixes 4 of each. Hence α_1 acts without fixed points on the remaining 16 square roots of each involution, which is absurd. Therefore $9\nmid |K|$.

Suppose $|K|=3$. Let $C=C_G(z)$. Obviously $Z(T)/\langle z\rangle$ is weakly closed in $T/\langle z\rangle$ with respect to $C/\langle z\rangle$, since $Z(T)=\Omega_1(T)$. By the Z^*-theorem, $Z(T)\leqq Z^*(C)$. Let $\bar{C}=C/O_{2'}(C)\cdot Z(T)$. As $|K|=3$, $\bar{C}$ has a Sylow 2-subgroup lying in the center of its normalizer; thus $\bar{C}$ has a normal 2-complement, so C does also. Moreover, we claim that $N_G(T)$ controls fusion of elements of T. This is clear for involutions. If

$t_1, t_2 \in T$ have order 4 and $t_1^g = t_2$ for some $g \in G$, then $t_1^2 = (t_2^2)^n$ for some $n \in N_G(T)$; hence $gn \in C_G(t_1^2)$ and $t_1^{gn} \in T$. As $C_G(t_1^2)$ has a normal 2-complement, t_1^{gn} is T-conjugate to t_1. Hence t_1^g is conjugate to t_1 in $N_G(T)$. Now by a theorem of Glauberman [9], G is a Suzuki group, which is absurd (e.g., $3 \mid |G|$). Therefore $|K| \neq 3$.

Hence $|K| = 15$, proving (c). We next prove (d). Let $C = C_G(z)$. As above, we have $Z(T) \leqq Z^*(C)$. Denote residues mod $Z(T)O_{2'}(C)$ by bars. Thus $|N_{\bar{C}}(\bar{T}) : C_{\bar{C}}(\bar{T})| = 5$. Let N be a minimal normal subgroup of $\bar{C}$. As $O_{2'}(\bar{C}) = 1$, $N \cap \bar{T} \neq 1$. But $N_{\bar{C}}(\bar{T})$ acts irreducibly on $\bar{T}$ so $\bar{T} \leqq N$. Now the main theorem of [14] implies that N is abelian, so $\bar{T} = N \lhd \bar{C}$. Thus $C = O_{2'}(C) \cdot N_C(T)$, which proves (d).

Next, β acts transitively on the elements of $(T/Z(T))^{\#}$. Hence the coset $tZ(T) = tT'$ contains representatives of all G-conjugacy classes of elements of order 4. Suppose that not all elements of $tZ(T)$ are fused in T, i.e. $|C_T(t)| > 2^4$. Then $|C_T(t)| = 2^5$ as $t \notin Z(T)$, and by applying β we conclude that $|C_T(x)| = 2^5$ if $x \in T - Z(T)$. This implies that T has $4 + 30$ conjugacy classes. Hence it has 16 linear characters and 18 ordinary characters of degree at least 2, so $|T| \geqq 16 + 4.18$, a contradiction. Therefore, all elements of $tZ(T)$ are fused, proving (a). Also, as $t^2 = z$, $C_G(t)/O_{2'}(C_G(t)) \cong C_{T\langle\beta^3\rangle}(t)$ by (c); this equals $C_T(t)$, proving (e).

Finally, (b) is clear from the fact that T contains three involutions, hence no subgroup isomorphic to D_8.

3. **Generalized decomposition numbers of $B_0(G)$.** For any group H, we denote the principal 2-block of H by $B_0(H)$. We first determine the Cartan matrices C^z and C^t of $B_0(C_G(z))$ and $B_0(C_G(t))$. Since $C_G(t)$ has a normal 2-complement, $B_0(C_G(t))$ contains just one Brauer character and $C^t = (16)$ with respect to the basic set $\{1\}$. Let λ be a fixed linear character of $C_G(z)$ with kernel $T \cdot O_{2'}(C_G(z))$. Let μ be the restriction of λ to the elements of $C_G(z)$ of odd order.

LEMMA 2. *$(C^z)_{ij} = 4(3 + \delta_{ij})$ with respect to the basic set $\{1, \mu, \mu^2, \mu^4, \mu^3\}$.*

Proof. We may assume $O_{2'}(C_G(z)) = 1$; then since $Z(T) \leqq Z(C_G(z))$, it suffices to show that $C_{ij} = 3 + \delta_{ij}$ where C is the Cartan matrix of $B_0(T\langle\beta^3\rangle/Z(T))$ with respect to the μ^i's considered as Brauer characters modulo $Z(T)$. (See [5], [6].) One checks directly that each λ^i, hence each μ^i, is in the principal 2-block; since the μ^i are the only Brauer characters of $T\langle\beta^3\rangle/Z(T)$, all ordinary characters of this group lie in the principal 2-block. There are five linear characters, and three faithful ones, which equal $\sum_{i=0}^{4} \mu^i$ on elements of odd order. The lemma follows easily.

Let $1 = \chi_0, \chi_1, \ldots, \chi_m$ be the ordinary characters in $B_0(G)$. Then there exist generalized decomposition numbers d_j^t and ${}_id_j^z$, $1 \leqq i \leqq 5$, $0 \leqq j \leqq m$, such that

$$\begin{aligned} \chi_j(t\rho) &= d_j^t \quad \text{and} \\ \chi_j(z\pi) &= {}_1d_j^z + {}_2d_j^z\mu(\pi) + {}_3d_j^z\mu^2(\pi) + {}_4d_j^z\mu^4(\pi) + {}_5d_j^z\mu^3(\pi) \end{aligned} \tag{3.1}$$

for all $\rho \in C_G(t)$ and $\pi \in C_G(z)$ of odd order. The ${}_id_j^z$ are automatically rational integers; since $\chi_j(t) = d_j^t$ and t is rational, the d_j^t are as well. We consider d^t and ${}_id^z$ to be columns of numbers indexed by $B_0(G)$, whose jth entries are d_j^t and ${}_id_j^z$

respectively. For any two columns A and B indexed by $B_0(G)$, put $(A, B)=\sum_{j=0}^{m} A_j\bar{B}_j$ (the bar denotes complex conjugation). By Lemma 2 and [3] we have

$$(3.2)\qquad \begin{gathered}(d^t, d^t) = 16; \qquad (d^t, {}_i d^z) = 0;\\ ({}_i d^z, {}_j d^z) = 4(3+\delta_{ij}) \quad \text{for } 1 \leqq i, j \leqq 5.\end{gathered}$$

The method of contribution [7] yields

$$(3.3)\qquad 4(d_j^t)^2+\sum_{i=1}^{5} ({}_i d_j^z)^2+3\sum_{h<i} ({}_h d_j^z-{}_i d_j^z)^2 < 64$$

for each j, $0\leqq j\leqq m$.

LEMMA 3. $\chi(z)\equiv\chi(t) \pmod 4$ *for any character* χ *of* G.

Proof. By Lemma 1, $|C_T(x)|=2^4$ for all $x\in T-Z(T)$. Since T has 2^4 linear characters all nonlinear characters of T vanish outside $Z(T)$. Let ψ be such a character not containing z in its kernel. Then $\psi(1)=4=-\psi(z)$ and so $(\chi|T, \psi)\in \boldsymbol{Z}$ implies $\chi(1)\equiv\chi(z) \pmod{16}$. Then summing χ on $C_T(t)$ yields $4\chi(z)+12\chi(t)\equiv 0 \pmod{16}$, proving the lemma.

Together with (3.1), Lemma 3 yields

$$(3.4)\qquad d_j^t \equiv \sum_{i=1}^{5} {}_i d_j^z \pmod 4, \qquad 0 \leqq j \leqq m.$$

Let σ be a Galois automorphism of some splitting field for G, such that $\mu^\sigma=\mu^2$. Then for any χ_j in $B_0(G)$, χ_j^σ is also in $B_0(G)$ so there exists an index k, $0\leqq k\leqq m$, such that $\chi_j^\sigma=\chi_k$. From (3.1) we obtain $d_j^t=d_k^t$, ${}_1d_j^z={}_1d_k^z$, ${}_2d_j^z={}_5d_k^z$, ${}_3d_j^z={}_2d_k^z$, ${}_4d_j^z={}_3d_k^z$, ${}_5d_j^z={}_4d_k^z$. We refer to this fact as "Galois symmetry."

Now, using (3.2), (3.3), (3.4), and Galois symmetry, we shall show that the generalized decomposition numbers for $B_0(G)$ are one of the possibilities (A) through (V) listed in Table I, up to a sign change in each row and a permutation of rows. In each case, the jth row consists of d_j^t and ${}_id_j^z$, $i=1, 2, 3, 4, 5$. We denote by v_j the 5-tuple $({}_1d_j^z, {}_2d_j^z, {}_3d_j^z, {}_4d_j^z, {}_5d_j^z)$.

TABLE I

Possible sets of generalized decomposition numbers of $B_0(G)$

(A) Z_1 or Z_2

d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
1	1	0	0	0	0	y_1
1	1	0	0	0	0	y_2
1	1	2	2	2	2	y_3
2	2	2	2	2	2	

(B) Z_1 or Z_2

d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
1	1	1	1	1	1	y_1
1	1	1	1	1	1	y_2
1	1	2	2	2	2	y_3
2	2	1	1	1	1	y_4
0	0	1	1	1	1	y_5

(C) Z_1 or Z_2

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	1	1	0	0	0	0	y_1
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	2	2	2	2	2	2	
	0	0	1	1	1	1	y_5
	0	0	1	1	1	1	y_6

(D) Z_1 or Z_2

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	1	1	0	0	0	0	
	1	1	0	0	0	0	
$4\times$	1	1	1	1	1	1	
	1	1	2	2	2	2	

(E) Z_1 or Z_2

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	1	1	0	0	0	0	
$6\times$	1	1	1	1	1	1	
	0	0	1	1	1	1	
	0	0	1	1	1	1	

(F) Z_3

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	1	1	0	0	0	0	
	0	0	1	1	1	1	
	0	0	1	1	1	1	

(G) Z_3

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	1	1	2	2	2	2	
	1	1	0	0	0	0	
	1	1	0	0	0	0	

(H) Z_4

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	0	2	2	1	2	1	
	0	2	1	2	1	2	
	-1	1	0	1	0	1	y_2
	-1	1	1	0	1	0	y_2
	-3	1	1	1	1	1	y_3
$3\times$	0	0	1	1	1	1	

(J) Z_4

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	-1	1	0	1	0	1	y_1
	-1	1	1	0	1	0	y_1
	-1	1	2	1	2	1	y_2
	-1	1	1	2	1	2	y_2
	2	2	1	1	1	1	y_3
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	1	1	0	0	0	0	y_6
	0	0	1	1	1	1	y_7

(K) Z_4

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	-1	1	0	1	0	1	y_1
	-1	1	1	0	1	0	y_1
	-1	1	0	1	0	1	y_2
	-1	1	1	0	1	0	y_2
	-2	2	2	2	2	2	
	1	1	2	2	2	2	y_4
	1	1	0	0	0	0	y_5
	1	1	0	0	0	0	y_6

(L) Z_4

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	-1	1	0	1	0	1	y_1
	-1	1	1	0	1	0	y_1
	-1	1	0	1	0	1	y_2
	-1	1	1	0	1	0	y_2
	-2	2	2	2	2	2	y_3

(M)

	d^t	$_1d^z$	$_2d^z$	$_3d^z$	$_4d^z$	$_5d^z$	$\pm\chi(1)$
	1	1	0	0	0	0	1
				Z_5			x_1
				Z_5			x_2
				Z_5			x_3
				Z_6			x_4
	-1	3	3	3	3	3	y_1

	d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
	1	1	1	1	1	1	
	1	1	1	1	1	1	
	1	1	0	0	0	0	y_6
	0	0	1	1	1	1	y_7
	0	0	1	1	1	1	y_8

	d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
	1	1	0	0	0	0	y_2
	1	1	0	0	0	0	y_3
	0	0	1	1	1	1	y_4

(N)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_5			x_2
				Z_6			x_3
				Z_6			x_4
	-2	2	2	2	2	2	x_5
	1	1	0	0	0	0	x_6
	1	1	2	2	2	2	x_7
	1	1	0	0	0	0	x_8

(P)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_5			x_2
				Z_6			x_3
				Z_6			x_4
	-2	2	2	2	2	2	x_5
	1	1	0	0	0	0	x_6
	1	1	1	1	1	1	x_7
	1	1	1	1	1	1	x_8
	0	0	1	1	1	1	x_9
	0	0	1	1	1	1	x_{10}

(Q)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_6			x_2
				Z_6			x_3
				Z_6			x_4
	-3	1	1	1	1	1	x_5
	1	1	2	2	2	2	x_6
	1	1	0	0	0	0	x_7
	0	0	1	1	1	1	x_8

(R)	1	1	0	0	0	0	1
				Z_5			x_1
				Z_6			x_2
				Z_6			x_3
				Z_6			x_4
	-3	1	1	1	1	1	x_5
	1	1	1	1	1	1	x_6
	1	1	1	1	1	1	x_7
$3\times$	0	0	1	1	1	1	

(S)							
				Z_7			
				Z_6			x_2
				Z_6			x_3
	-2	0	1	0	1	0	x_4
	-2	0	0	1	0	1	x_4
	1	1	2	2	2	2	x_5
	1	1	1	1	1	1	x_6
	1	1	1	1	1	1	x_7
	0	0	1	1	1	1	x_8

(T)							
				Z_7			
				Z_6			x_2
				Z_5			x_3
	-1	1	2	1	2	1	x_4
	-1	1	1	2	1	2	x_4
	1	1	2	2	2	2	x_5
	2	2	1	1	1	1	x_6

(U)

d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
Z_7						
Z_6						x_2
Z_5						x_3
-1	1	2	1	2	1	x_4
-1	1	1	2	1	2	x_4
2	2	2	2	2	2	x_5
1	1	0	0	0	0	x_6
0	0	1	1	1	1	x_7

(V)

	d^t	${}_1d^z$	${}_2d^z$	${}_3d^z$	${}_4d^z$	${}_5d^z$	$\pm\chi(1)$
	Z_7						
	Z_6						x_2
	Z_5						x_3
	-1	1	2	1	2	1	x_4
	-1	1	1	2	1	2	x_4
$4\times$	1	1	1	1	1	1	$x_5^{(i)}$ $(1\leqq i\leqq 4)$
	1	1	0	0	0	0	x_6
	0	0	1	1	1	1	x_7

where

Z_1:	1	1	0	0	0	0	1
	-1	1	1	1	1	-1	x_1
	-1	1	1	1	-1	1	x_1
	-1	1	1	-1	1	1	x_1
	-1	1	-1	1	1	1	x_1
	-2	2	2	2	2	2	x_2
Z_2:	1	1	0	0	0	0	1
	-1	1	0	0	0	2	x_1
	-1	1	0	0	2	0	x_1
	-1	1	0	2	0	0	x_1
	-1	1	2	0	0	0	x_1
	-2	2	2	2	2	2	x_2
Z_3:	1	1	0	0	0	0	1
	1	1	2	0	1	1	x_1
	1	1	0	1	1	2	x_1
	1	1	1	1	2	0	x_1
	1	1	1	2	0	1	x_1
	-1	1	1	1	0	0	x_2
	-1	1	1	0	0	1	x_2
	-1	1	0	0	1	1	x_2
	-1	1	0	1	1	0	x_2
	-2	2	2	2	2	2	x_3

Z_4:	1	1	0	0	0	0	1
	1	1	2	1	0	1	x_1
	1	1	1	0	1	2	x_1
	1	1	0	1	2	1	x_1
	1	1	1	2	1	0	x_1
Z_5:	1	0	1	0	0	0	
	1	0	0	1	0	0	
	1	0	0	0	1	0	
	1	0	0	0	0	1	
Z_6:	0	1	1	1	1	0	
	0	1	1	1	0	1	
	0	1	1	0	1	1	
	0	1	0	1	1	1	
Z_7:	1	1	0	0	0	0	1
	-1	1	1	1	0	0	x_1
	-1	1	1	0	0	1	x_1
	-1	1	0	0	1	1	x_1
	-1	1	0	1	1	0	x_1

Define the following columns of rational integers indexed by $B_0(G)$: ${}_0A={}_1d^z-{}_2d^z$, ${}_1A={}_2d^z-{}_3d^z$, ${}_2A={}_3d^z-{}_4d^z$, ${}_3A={}_4d^z-{}_5d^z$, ${}_4A={}_5d^z-{}_2d^z$. Thus for any j, $\sum_{i=1}^4 {}_iA_j=0$, and by Galois symmetry there exists j' with ${}_0A_{j'}={}_0A_j+{}_1A_j$, ${}_iA_j={}_{i+1}A_{j'}$ $(i=1, 2, 3)$, ${}_4A_j={}_1A_{j'}$. From (3.2) we get $({}_iA, {}_iA)=8$ $(0\leqq i\leqq 4)$; $({}_1A, {}_4A)=({}_iA, {}_{i+1}A)=({}_0A, {}_4A)=-4$ $(0\leqq i\leqq 3)$; and $({}_0A, {}_2A)=({}_0A, {}_3A)=({}_1A, {}_3A)=({}_2A, {}_4A)=0$. We always take $\chi_0=1_G$; thus ${}_0A_0=1$, ${}_iA_0=0$ for $i>0$.

We consider first the case when some entry of some ${}_iA$, $i>0$, is ± 2. By Galois symmetry we may assume $i=1$, and since we are allowing permutations of rows and sign changes in each row, we may assume ${}_1A_1=2$. If ${}_3A_1=2$, then $({}_iA, {}_iA)=8$ and $\sum_{i=1}^{4} {}_iA_1=0$ imply ${}_2A_1={}_4A_1=-2$. By Galois symmetry, we may assume ${}_1A_2={}_3A_2=-{}_2A_2=-{}_4A_2=-2$, contradicting $({}_1A, {}_3A)=0$. Therefore ${}_3A_1\neq 2$. Again by Galois symmetry, we may assume ${}_iA_j={}_jA_i$ for $1\leqq i, j\leqq 4$; ${}_1d_1^z={}_1d_2^z={}_1d_3^z={}_1d_4^z$; $d_1^t=d_2^t=d_3^t=d_4^t$; and ${}_0A_j={}_0A_{j-1}+{}_1A_{j-1}$ for $2\leqq j\leqq 4$. It follows easily from $({}_1A, {}_3A)=0$ that ${}_3A_1=0$. We consider the several possibilities for ${}_0A_1$ separately. Note that always $\chi_j(z)\neq 0$, for otherwise, $(d^t, d^t)=16$, $d_j^t=\chi_j(t)\equiv\chi_j(z) \pmod 4$ imply $\chi_j(t)=0$, whence χ_j has defect zero, contradicting $\chi_j\in B_0(G)$.

Case 1. ${}_0A_1>0$. Then ${}_0A_2={}_0A_1+{}_1A_1\geqq 2$, contradicting $({}_0A, {}_0A)=8$.

Case 2. ${}_0A_1=0$. By an argument like that in Case 1, we find ${}_1A_2={}_2A_1=-1$ or -2.

(a) Suppose ${}_1A_2=-2$. $({}_1A, {}_1A)=8$ yields ${}_1A_j=0$ for $j>2$. From $(d^t, d^t)=({}_1d^z, {}_1d^z)=16$, (3.3), and (3.4), we find $d_1^t=-1$ and $v_1=(1, 1, -1, 1, 1)$. For $j>4$ we have ${}_2d_j^z={}_3d_j^z={}_4d_j^z={}_5d_j^z$, so ${}_1d_j^z\equiv d_j^t \pmod 4$. Since $(d^t, {}_1d^z)=0$, we may assume $d_5^t=-2$, ${}_1d_5^z=2$. Then clearly $d_j^t={}_1d_j^z$ for $j>5$. For $i>1$, $({}_id^z, {}_1d^z-d^t)=12$ yields ${}_id_5^z=2$. It now follows easily that we have one of the cases (A)–(E) of Table I, with Z_1's.

(b) Suppose ${}_1A_2=-1$. Then ${}_1A_3=0$ implies ${}_1A_4=-1$. It follows that $\sum_{j=0}^{4} {}_0A_j^2=7$; since $({}_0A, {}_0A)=8$, we may assume ${}_0A_5=1$, ${}_0A_j=0$ for $j>5$. The conditions on $({}_0A, {}_iA)$ imply ${}_1A_5={}_2A_5=-{}_3A_5=-{}_4A_5=-1$. By Galois symmetry there exist at least four $j>4$ with ${}_1A_j\neq 0$, contradicting $({}_1A, {}_1A)=8$.

Case 3. ${}_0A_1=-1$. Suppose first that ${}_2A_1$ or ${}_4A_1$ is ± 2. As $({}_1A, {}_1A)=8$, it must be -2. If ${}_4A_1=-2$ we replace the first row by the fourth with a sign change and so may assume ${}_2A_1=-2$; then ${}_3A_1={}_4A_1=0$. As in Case 2(a), we easily conclude that we may assume $d_1^t=-1$ and $v_1=(1, 0, 2, 0, 0)$.

As in Case 2(a) we may assume ${}_jd_5^z=-d_5^t=2$, and we get (A)–(E) in Table I, with Z_2's.

Now suppose $|{}_iA_1|\leqq 1$, $2\leqq i\leqq 4$. It follows that ${}_2A_1={}_4A_1=-1$. As $\chi_1(z)\neq 0$, we may assume by (3.3), (3.4), that $d_1^t=1$ and $v_1=(1, 2, 0, 1, 1)$ or $(1, 0, 2, 1, 1)$.

The arguments in both these cases are the same so we consider only the first. We have $\sum_{j=1}^{4} {}_1A_j\,{}_3A_j=2$. Since $({}_1A, {}_3A)=0$ and $|{}_jA_k|\leqq 1$ for $j>0$, $k>4$, we may assume ${}_1A_5=-{}_3A_5=1$. By Galois symmetry, χ_5 has at least four algebraic conjugates under σ, and it follows easily from $({}_iA, {}_iA)=8$ that ${}_2A_5={}_4A_5=0$. We may assume $\chi_5^{\sigma^i}=\chi_{5+i}$, $0\leqq i\leqq 3$. As $({}_0A, {}_0A)=8$, ${}_0A_5=0$ or -1. By replacing the fifth row with the seventh with a sign change if necessary, we may assume ${}_0A_5=0$. Then we may assume $d_5^t=-1$, and $v_5=(1, 1, 0, 0, 1)$. As in 2(a) we may assume $d_9^t=-{}_1d_9^z=-2$; $({}_jd^z, {}_1d^z-d^t)=12$ yields ${}_jd_9^z=2$ for $2\leqq j\leqq 5$. We then clearly get (F) or (G) in Table I.

Case 4. ${}_0A_1=-2$. If ${}_2A_1=-2$, then ${}_0A_2=0$ and ${}_1A_2=-2$, and Case 3 applies. Similarly, if ${}_4A_1=-2$, then ${}_0A_4=0$ and ${}_1A_4=-2$ and Case 3 applies again. As $\sum_{i=1}^{4} {}_iA_1=0$, we may assume ${}_2A_1={}_4A_1=-1$; an argument like that in Case 2(b) gives a contradiction.

Now we may assume that $|{}_iA_j| \leqq 1$ for $i \geqq 1$ and all j.

Case 5. ${}_1A_1 = {}_2A_1 = -{}_3A_1 = -{}_4A_1 = 1$. By Galois symmetry, we may assume ${}_1A_2 = {}_4A_2 = {}_3A_3 = {}_4A_3 = {}_2A_4 = {}_3A_4 = 1$ and other ${}_iA_j$, $2 \leqq i \leqq 4$, $1 \leqq j \leqq 4$, are -1. The conditions on the inner products $({}_iA, {}_jA)$ imply that we may assume ${}_1A_5 = {}_2A_6 = {}_1A_7 = {}_2A_8 = {}_3A_5 = {}_4A_6 = {}_3A_7 = {}_4A_8 = 1$ and other ${}_iA_j$, $5 \leqq j \leqq 8$, $1 \leqq i \leqq 4$, are -1. From $({}_0A, {}_0A) = 8$ we may assume ${}_0A_1 = -{}_0A_3 = {}_0A_5 = {}_0A_7 = -1$, ${}_0A_2 = {}_0A_4 = {}_0A_6 = {}_0A_8 = 0$. We then have $d_j^t \equiv {}_1d_j^z \pmod 4$ for $j > 8$. (3.3) and $\chi_1(z) \neq 0$ imply that we may assume $d_1^t = 1$, $v_1 = (1, 2, 1, 0, 1)$, by replacing the first row with the third with a sign change, if necessary. By Galois symmetry, $({}_2d^z, {}_2d^z) = 16$, and (3.3), we have $v_5 = (1, 2, 1, 2, 1)$, $(0, 1, 0, 1, 0)$, $(1, 0, 1, 0, 1)$ or $(2, 1, 2, 1, 2)$, and similarly for v_7.

(a) If $v_5 = (2, 1, 2, 1, 2)$, then $d_5^t = 0$; (3.2) implies $v_7 = (1, 0, 1, 0, 1)$ and $d_7^t = -1$. Then $(d^t, {}_1d^z) = 0$ implies we may assume $d_9^t = -3$, ${}_1d_9^z = 1$. $({}_jd^z, d^t) = 0$ yields ${}_jd_9^z = 1$ for $2 \leqq j \leqq 5$ and we have (H) in Table I.

(b) If $v_5 = (0, 1, 0, 1, 0)$, then $d_5^t = \pm 2$. If $d_5^t = 2$, then the Schwarz inequality on the columns $(d_j^t)_{j>6}$ and $({}_2d_j^z)_{j>6}$ yields $6 \leqq 27^{1/2}$, a contradiction. So $d_5^t = -2$. Now $(d^t, d^t) = 16$ implies $v_7 = (1, 0, 1, 0, 1)$ or $(1, 2, 1, 2, 1)$; $(d^t, {}_1d^z) = 0$ implies we may assume $d_9^t = -1$, ${}_1d_9^z = 3$, and so $({}_1d^z, {}_2d^z) \not\equiv 0 \pmod 3$, a contradiction.

(c) We may now assume that v_5 and v_7 are either $(1, 0, 1, 0, 1)$ or $(1, 2, 1, 2, 1)$. If both are $(1, 2, 1, 2, 1)$, then $\sum_{j=0}^{8} ({}_2d_j^z)^2 = 16$ and $\sum_{j=0}^{8} {}_1d_j^z\, {}_2d_j^z = 10$, a contradiction. Hence we may assume $v_5 = (1, 0, 1, 0, 1)$, $d_5^t = -1$. As $(d^t, {}_1d^z) = 0$, we may assume $d_9^t = -{}_1d_9^z = -2$, $d_j^t = {}_1d_j^z = 1$, $10 \leqq j \leqq 12$; we easily get (J), (K), or (L) in Table I. This disposes of Case 5.

Since $\sum_{i=1}^{4} {}_iA_j = 0$ for all j, we may assume that for each j, $({}_iA_j)_{i=1}^4$ is some cyclic permutation of $(1, 0, -1, 0)$; $(1, -1, 1, -1)$; $(1, -1, 0, 0)$; or $(0, 0, 0, 0)$, possibly with a sign change. (3.2), (3.3), (3.4), Galois symmetry, $\chi_j(z) \neq 0$, and the conditions on $({}_iA, {}_jA)$ yield the possibilities for v_j shown in Table II.

Case 6. No $({}_iA_j)_{i=1}^4$ is $(1, 0, -1, 0)$. Then since $({}_1A, {}_3A) = 0$, no $({}_iA_j)_{i=1}^4$ is $(1, -1, 1, -1)$. Hence we may assume $({}_iA_{4n+1})_{i=1}^4 = (1, -1, 0, 0)$ and $\chi_{4n+k} = \chi_{4n+1}^{\sigma^{k-1}}$, $0 \leqq n \leqq 3$, $1 \leqq k \leqq 4$. If ${}_0A_1 = -1$, we get ${}_0A_3 = {}_0A_4 = -1$. Since ${}_0A_0 = 1$ and $({}_0A, {}_0A) = 8$, we may assume ${}_0A_1 = {}_0A_5 = {}_0A_9 = 0$. We have $d_j^t \equiv {}_1d_j^z \pmod 4$ for $j > 16$.

(a) ${}_0A_{13} = 0$. Then v_1, v_5, v_9, and v_{13} are each either $(0, 0, -1, 0, 0)$ or $(1, 1, 0, 1, 1)$. Correspondingly, d_1^t, d_5^t, d_9^t, and d_{13}^t are either -1 or 0. Since $({}_1d^z, {}_1d^z) = (d^t, d^t) = 16$, we cannot have $v_1 = v_5 = v_9 = v_{13}$. Depending on whether one, two, or three of v_1, v_5, v_9, and v_{13} are $(1, 1, 0, 1, 1)$, we get (by permuting rows and changing signs) cases (M); (N) or (P); (Q) or (R) in Table I.

(b) ${}_0A_{13} = -1$. If $v_{13} = (1, 2, 1, 2, 2)$, then, by $({}_2d^z, {}_2d^z) = 16$, $v_1 = v_5 = v_9 = (0, 0, -1, 0, 0)$, against $({}_1d^z, {}_2d^z) = 12$. So $v_{13} = (0, 1, 0, 1, 1)$. Thus $d_{13}^t = -1$, and as $(d^t, d^t) = 16$, we may assume $v_9 = (1, 1, 0, 1, 1)$. Suppose that k of v_5 and v_7 are $(0, 0, -1, 0, 0)$. By $(d^t, {}_1d^z) = 0$, we may assume ${}_1d_{17}^z = k + 1$ and $d_{17}^t = k - 3$ $(k = 0, 1,$ or $2)$. From $({}_2d^z, {}_1d^z - d^t) = 12$, we get ${}_2d_{17}^z = k$. The Schwarz inequality on $({}_1d_j^z)_{j \geqq 17}$ and $({}_2d_j^z)_{j \geqq 17}$ yields $(3 + 2k - k^2)^2 \leqq (4 + 2k - k^2)(2 + 2k - k^2)$ which is impossible for $0 \leqq k \leqq 2$.

TABLE II

Possible v_j for given $({}_iA_j)_{i=1}^4$

$({}_iA_j)_{i=1}^4$	${}_0A_j$	Possible v_j (up to sign change and Galois conjugacy)
$(1, 0, -1, 0)$	0	$(1, 1, 0, 0, 1)$
		$(1, 1, 2, 2, 1)$
$(1, -1, 1, -1)$	0	$(2, 2, 1, 2, 1)$
		$(1, 1, 0, 1, 0)$
		$(0, 0, 1, 0, 1)$
		$(1, 1, 2, 1, 2)$
$(1, -1, 1, -1)$	1	$(2, 1, 0, 1, 0)$
		$(1, 0, -1, 0, -1)$
		$(0, 1, 2, 1, 2)$
$(1, -1, 0, 0)$	0	$(1, 1, 0, 1, 1)$
		$(0, 0, -1, 0, 0)$
$(1, -1, 0, 0)$	-1	$(1, 2, 1, 2, 2)$
		$(0, 1, 0, 1, 1)$

Case 7. $({}_iA_j)_{i=1}^4=(1, 0, -1, 0)$ for two distinct values of j, say $j=1$ and $j=5$. Then ${}_1d_j^z=-d_j^t=\pm 1$ for $1\leqq j\leqq 8$, by Table II. As $({}_1d^z, d^t)=0$, we get ${}_1d_j^z=d_j^t$ for $j>8$. Since $({}_1A, {}_3A)=0$, we may assume $({}_iA_9)_{i=1}^4=(1, -1, 1, -1)$; then Table II and (3.4) yield ${}_1d_9^z\neq d_9^t$, a contradiction.

Case 8. We may now assume $({}_iA_1)_{i=1}^4=(1, 0, -1, 0)$, $({}_iA_5)_{i=1}^4=(1, -1, 1, -1)$, $({}_iA_7)_{i=1}^4=({}_iA_{11})_{i=1}^4=(1, -1, 0, 0)$. Since $({}_0A, {}_0A)=8$, ${}_0A_1={}_0A_5={}_0A_7=0$; ${}_0A_{11}=-1$ or 0. From Table II, $\sum_{j=7}^{10}(d_j^t-{}_2d_j^z)^2=3$, $\sum_{j=11}^{15}(d_j^t-{}_2d_j^z)^2\geqq 3$. If $v_1=(1, 1, 2, 2, 1)$, then we get $(d^t-{}_2d^z, d^t-{}_2d^z)>32$, a contradiction. Therefore, $v_1=(1, 1, 0, 0, 1)$, and $d_1^t=-1$. Suppose ${}_0A_{11}=-1$. Then ${}_0A_j=0$ for $j>14$; $({}_2d^z, {}_2d^z)=16$ implies $v_{11}=(0, 1, 0, 1, 1)$, so $d_{11}^t=-1$. Now $(d^t-{}_1d^z, d^t-{}_1d^z)=32$ implies $d_j^t={}_1d_j^z$ for $j>14$. Hence $4+\sum_{j=5}^{10}(d_j^t)^2=\sum_{j=5}^{10}({}_1d_j^z)^2=\sum_{j=5}^{10}({}_2d_j^z)^2$, as $(d^t, d^t)=({}_1d^z, {}_1d^z)=({}_2d^z, {}_2d^z)$. None of the possibilities for v_j, $5\leqq j\leqq 10$, listed in Table II satisfy these equations.

Therefore ${}_0A_{11}=0$. As above, ${}_1d_j^z=d_j^t$ for $j>14$. If $v_7=v_{11}=(0, 0, -1, 0, 0)$, then $(d^t, d^t)=({}_1d^z, {}_1d^z)=16$ implies $v_5=(2, 1, 2, 1, 2)$, so $({}_2d^z, {}_1d^z-d^t)=8$, a contradiction. Therefore we may assume $v_{11}=(1, 1, 0, 1, 1)$. If $v_7=(1, 1, 0, 1, 1)$, then $(d^t, d^t)=({}_1d^z, {}_1d^z)$ and $({}_2d^z, {}_1d^z-d^t)=12$ imply $v_5=(0, 1, 0, 1, 0)$ and $d_5^t=-2$. This yields (S) in Table I. Finally, if $v_7=(0, 0, -1, 0, 0)$, then $({}_2d^z, {}_1d^z-d^t)=12$ implies $v_7=(1, 2, 1, 2, 1)$, and we easily get (T), (U), or (V).

4. **Character degrees.** We show in this section that either case (U) or (V) in Table I holds, that G has a rational character of degree 12, and

(4.1) $$|G| = 195|C_G(z)|^3/|C_G(Z(T))|^2.$$

Let d be one of the columns d^t or ${}_id^z$; let $x=t$ or z, respectively. Let $\tilde{G}=C_G^*(x)$ and let $\tilde{d}$ be the corresponding column of generalized decomposition numbers for $B_0(\tilde{G})$ (with respect to the basic set $\{1\}$ if $x=t$, $\{1, \mu, \mu^2, \mu^4, \mu^3\}$ if $x=z$). Let $\tilde{\chi}_0, \tilde{\chi}_1, \ldots, \tilde{\chi}_n$ be the ordinary characters in $B_0(\tilde{G})$ and define $h_j=\sum_\alpha \tilde{\chi}_j(z_\alpha)/|C_{\tilde{G}}(z_\alpha)|$, where z_α runs over $\tilde{G}$-conjugacy classes of involutions. Then by a result of Brauer [4],

$$|G| \sum_{j=0}^{m} \chi_j(z)^2 d_j/\chi_j(1) = |\tilde{G}|\, |C_G(z)|^2 \sum_{j=0}^{n} h_j^2 \tilde{d}_j/\tilde{\chi}_j(1).$$

We denote the left and right sides of this equation by $\mathrm{L}(d)$ and $\mathrm{R}(d)$, respectively. If A is any column indexed by $B_0(G)$ which is a linear combination of d^t and the ${}_id^z$, $\mathrm{L}(A)$ is defined as the corresponding linear combination of $\mathrm{L}(d^t)$ and the $\mathrm{L}({}_id^z)$.

LEMMA 4. (a) $\mathrm{L}(d^t)=0$.

(b) $\mathrm{L}({}_1d^z-{}_2d^z)=0$.

(c) $\mathrm{L}({}_1d^z)=128|C_G(z)|^3/|C_G(Z(T))|^2$.

Proof. Lemma 1(b) and a result of Brauer [4] imply (a). By Lemma 1, $|C_T(x)|=2^4$ for all elements x of T of order 4, so T has 16 linear characters and three irreducible characters ψ, ψ^β, and ψ^{β^2} of degree 4 vanishing off $Z(T)$. Choose notation so that $\ker \psi=\langle z\rangle$.

Let $\bar{C}=C_G(z)/O_{2'}(C_G(z))$. As argued in Lemma 2, all characters of $\bar{C}$ lie in $B_0(\bar{C})$. Let $\bar{T}$ be the image of T in $\bar{C}$. Thus $\bar{T}\cong T$.

The characters $1_{\bar{T}}$, ψ, ψ^β, ψ^{β^2} are all invariant in $\bar{C}$ and hence extend in five ways each to $\bar{C}$. Since $\exp \bar{C}=20$ and ψ is rational, it is easily seen that at least one extension $\tilde{\psi}$ of ψ is rational, whence $\tilde{\psi}(zf)=-1$ for all $f\in \bar{C}$ of order 5. The extensions of ψ are then $\tilde{\psi}\lambda^i$, $0\leqq i\leqq 4$. We have $\tilde{\psi}\lambda^i(zf)=\sum_{j\neq i}\lambda^j(f)$ for all $f\in\bar{C}$ of odd order. Hence the generalized decomposition numbers at z for the characters $\tilde{\psi}\lambda^i$ are the cyclic permutations of (0, 1, 1, 1, 1). Similarly, those for $\tilde{\psi}^\beta\lambda^i$ and $\tilde{\psi}^{\beta^2}\lambda^i$ are the cyclic permutations of $(0, -1, -1, -1, -1)$. Finally, the fifteen linear nonprincipal characters of $\bar{T}$ form three orbits under the action of $\bar{C}$ and so by induction to $\bar{C}$ yield three irreducible characters of $\bar{C}$ of degree 5 vanishing off $\bar{T}$ and with $\bar{z}$ in their kernels. Thus the generalized decomposition numbers for each of these characters at $\bar{z}$ are (1, 1, 1, 1, 1). Now expand $\mathrm{R}({}_1d^z-{}_2d^z)$ from its definition. Apart from a constant factor, there is a sum of terms indexed by $B_0(\bar{C})$. It is clear that the only nonzero terms arise from 1 and λ; $\tilde{\psi}$ and $\tilde{\psi}\lambda$; $\tilde{\psi}^\beta$ and $\tilde{\psi}^\beta\lambda$; $\tilde{\psi}^{\beta^2}$ and $\tilde{\psi}^{\beta^2}\lambda$; and these cancel in pairs, proving (b). Put $c(z)=|C_G(z)|$, $c(Z(T))=|C_G(Z(T))|$. The $\bar{C}$-classes of involutions are represented by $\bar{z}$, $\bar{y}$, and $\bar{y}\bar{z}$ where $Z(T)=\langle y, z\rangle$. We find

$$\begin{aligned}\mathrm{R}({}_1d^z) &= c(z)^3\Big[\Big(\frac{1}{c(z)}+\frac{2}{c(Z(T))}\Big)^2+\Big(\frac{4}{4}\Big)\Big(\frac{4}{c(z)}-\frac{8}{c(Z(T))}\Big)^2 \\ &\qquad -\Big(\frac{4}{4}\Big)\Big(-\frac{4}{c(z)}\Big)^2-\Big(\frac{4}{4}\Big)\Big(-\frac{4}{c(z)}\Big)^2+\Big(\frac{3}{5}\Big)\Big(\frac{5}{c(z)}+\frac{10}{c(Z(T))}\Big)^2\Big] \\ &= 128c(z)^3/c(Z(T))^2,\end{aligned}$$

proving (c).

LEMMA 5. (a) $\chi_j(1) \geqq 12$ *if* $0 < j \leqq m$.

(b) $\chi_j(1) + 3\sum_{i=1}^{5} {}_i d_j^z + 60 d_j^t$ *is a nonnegative integral multiple of* 64.

(c) $\sum_{j=0}^{m} \chi_j(1) d_j^t = \sum_{j=0}^{m} \chi_j(1)_i d_j^z = 0$ *for each i.*

Proof. (a) Since $\chi_j|T$ is faithful and $(\chi_j|T, \psi) = (\chi_j^{\beta}|T, \psi^{\beta}) = (\chi_j|T, \psi^{\beta}) = (\chi_j|T, \psi^{\beta^2})$, $\chi_j|T$ must contain $\psi + \psi^{\beta} + \psi^{\beta^2}$. (b) simply restates that $(\chi|T, 1_T)$ is a nonnegative integer; (c) is due to Brauer [3].

We shall use also the following consequence of a theorem of Schur [13]:

(*) If $\chi_j(1) = e > 5$ and $Q(\chi_j) \subseteq Q(\lambda)$, then no prime divisor of $|G|$ exceeds $e+1$.

The jth row of the column $\pm\chi(1)$ in Table I is defined as $\pm\chi_j(1)$, according as the jth row of generalized decomposition numbers for G is $\pm$ the jth row in Table I. We now eliminate (A)–(T) case by case.

(A) $L({}_1d^z - {}_2d^z) = (\pm\chi(1), {}_1d^z - {}_2d^z) = 0$ yield

(A1) $1 + (18/x_1) + (1/y_1) + (1/y_2) - (81/y_3) = 0$,

(A2) $1 + 2x_1 + y_1 + y_2 - y_3 = 0$.

By Lemma 5(b), $x_1 \equiv 51$, $y_1 \equiv 1$, $y_2 \equiv 1$, $y_3 \equiv 41$ (mod 64). If $x_1 > 0$ or $x_1 < -77$, (A1) implies $y_3 = 41$, whence $(18/51) + (1/65) + (1/65) \geqq (18/x_1) + (1/y_1) + (1/y_2) = 40/41$, a contradiction. Therefore $-77 \leqq x_1 < 0$. If $x_1 = -13$, (A1) implies $y_3 < 0$; it is clear from Table I that $Q(\chi_1) \subseteq Q(\lambda)$, so (*) implies $y_3 < -343$, so $(2/65) + (81/343) \geqq (1/y_1) + (1/y_2) - (81/y_3) = 5/13$, a contradiction. Hence $x_1 = -77$. $|(1/y_1) + (1/y_2)| \leqq 2/63$ implies $|(59/77) - (81/y_3)| \leqq 2/63$, so $y_3 = 105$. Then $(1/y_1) + (1/y_2) = 2/385$, and (A2) gives $y_1 + y_2 = 258$. These equations have no solution, so (A) is impossible. (D) and (G) yield the same equations as (A) and so are also impossible.

(B) From $(\pm\chi(1), {}_1d^z - d^t) = 0$ we get $x_2 = -2x_1$. Then $L({}_1d^z - d^t) > 0$ implies $x_1 < 0$. $L({}_1d^z - {}_2d^z) = L({}_1d^z - {}_2d^z + d^t) = (\pm\chi(1), {}_1d^z - {}_2d^z) = (\pm\chi(1), d^t) = 0$ yield

(B1) $1 + (18/x_1) - (81/y_3) + (36/y_4) - (16/y_5) = 0$,

(B2) $2 + (82/x_1) + (25/y_1) + (25/y_2) + (108/y_4) - (16/y_5) = 0$,

(B3) $1 + 2x_1 - y_3 + y_4 - y_5 = 0$,

(B4) $1 + y_1 + y_2 + y_3 + 2y_4 = 0$.

From Lemma 5(b), $x_1 \equiv 51$, $y_1 \equiv 53$, $y_2 \equiv 53$, $y_3 \equiv 41$, $y_4 \equiv 54$, $y_5 \equiv 52$ (mod 64); $x_2 \geqq 90$, and if $y_4 < 0$, then $y_4 \leqq -138$. Since $x_2 = -2x_1$, we get $x_1 \leqq -77$. Adding (B3) and (B4), we find that we cannot have $y_1, y_2, y_4 < 0$, $y_5 > 0$ at the same time. Suppose $x_1 < -77$. Then by (B2), we get $y_4 = -138$, $x_1 = -141$, and we may assume $y_1 = -75$. If $y_2 < 0$, then (B4) implies $y_3 \geqq 425$, and subtracting (B1) from (B2) yields $81/425 > (64/141) + (25/75) + (72/138) - 1$, a contradiction. So $y_2 > 0$; by (B2), $y_5 = 52$; (B3) yields $y_3 = 471 = 3 \cdot 157$, violating (*) applied to a character of degree 141. Therefore, $x_1 = -77$. If $y_4 > 0$, (B2) implies $y_1 = y_2 = -75$, $y_5 = 52$; (B3) and (B4) give $y_4 = 118$, violating (*) as $y_5 = 52$. Therefore $y_4 < 0$, and (B3) implies either y_3 or $y_5 < 0$. If $y_3 < 0$, (B1) implies $1 < (18/77) + (36/138) + (16/52)$, a contradiction. Therefore $y_5 < 0$, and (B1) implies $y_3 = 41$ or 105. If $y_3 = 41$, (B1) implies $y_5 = -12$, violating (*); therefore $y_3 = 105$. Subtracting (B1) from (B2) yields $(25/y_1) + (25/y_2)$

$< -(81/105)-(13/77)+(72/138)$. By (B4) we may assume $y_1>0$, so $25/y_2<-1/3$, $-75<y_2<0$, which is impossible.

(C) $L({}_1d^z-{}_2d^z)=0$ yields $1+(18/x_1)+(1/y_1)-(16/y_5)-(16/y_6)=0$. From Lemma 5(b), $x_1\equiv 51$, $y_1\equiv 1$, $y_5\equiv 52$, $y_6\equiv 52 \pmod{64}$. As in (B) we get $x_1<0$. If $x_1<-13$, then the above equation gives $(1/63)+(16/52)\cdot 2\geqq 59/77$, a contradiction. Thus $x_1=-13$. If $y_5<-12$, (*) implies $y_5<-140$; similarly for y_6. If both are <-12, we get $-1/y_1>(59/77)-(32/140)$, against $|y_1|\geqq 63$. Thus we may assume $y_5=-12$; then $(1/y_1)-(16/y_6)=37/39$, violating $y_6\equiv 52 \pmod{64}$. Cases (E) and (F) yield similar contradictions.

(H) $L(d^t)=(\pm\chi(1), d^t)=0$ yield

(H1) $1+(100/x_1)-(18/y_2)-(75/y_3)=0$,

(H2) $1+4x_1-2y_2-3y_3=0$.

We have $x_1\equiv 53$, $y_2\equiv 51$, $y_3\equiv 37 \pmod{64}$, and if $y_3>0$, then $y_3\geqq 165$. It follows easily from (H1) that $x_1<0$. By (H2) either $y_2<0$ or $y_3<0$. Therefore $100/x_1 > 1-(75/165)$, and $x_1=-75$ or -139. In either case (H1) and (H2) yield a quadratic equation for y_3 which has no integral solutions, a contradiction.

(J) $L({}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-{}_2d^z)=(\pm\chi(1), d^t-{}_1d^z)=0$ yield

(J1) $1+(9/y_1)-(49/y_2)+(36/y_3)+(1/y_6)-(16/y_7)=0$,

(J2) $1+y_1-y_2+y_3+y_6-y_7=0$,

(J3) $y_1+y_2+y_3=0$.

$y_1\equiv 51$, $y_2\equiv 39$, $y_3\equiv 38$, $y_6\equiv 1$, $y_7\equiv 52 \pmod{64}$. By Lemma 4,

$$L(-3{}_1d^z+4{}_2d^z-d^t) > 0,$$

and this easily yields $y_2=39$ or 103. However, if $y_2=103$, then (*) implies $|y_j|\geqq 102$, $j=1, 3, 6$, and 7; the congruences and (J1) yield a contradiction. Therefore $y_2=39$. Suppose $y_1\geqq -13$. By (J3), $(9/y_1)+(36/y_3)<0$, and (J1) implies $y_7=-12$, $y_1=-13$, $y_3=-26$, $|y_6|\leqq 2$, a contradiction. Therefore $y_1<-13$, $y_3>0$, and $(36/y_3)+(9/y_1)>0$. From (J1), $y_7\neq -12$, otherwise $|y_6|<1$. Now if $y_1=-77$, then (J1) yields $0<y_7<40$, which is impossible. Applying (*) to a character of degree 39, we find $y_1\leqq -333$. The function $(9/y_1)+(36/-y_1-39)$ is increasing for $y_1<0$, so (J1) implies $y_7>-112$, $y_7<0$. Therefore $y_7=-76$. Now (J2) and (J3) imply $y_6=1$, a contradiction.

(K) $L({}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-{}_2d^z)=0$ yield

(K1) $1+(9/y_1)+(9/y_2)-(81/y_4)+(1/y_5)+(1/y_6)=0$,

(K2) $1+y_1+y_2-y_4+y_5+y_6=0$.

$y_1\equiv 51$, $y_2\equiv 51$, $y_4\equiv 41$, $y_5\equiv 1$, $y_6\equiv 1 \pmod{64}$. If $y_1=y_2$, we can argue as in (A) to a contradiction. So we may assume $y_1\neq y_2$. If neither is -13, (K1) implies $y_4<105$, $y_4>0$, so $y_4=41$; thus from (K1), $(40/41)+(9/y_1)+(9/y_2)+(1/y_5)+(1/y_6)=0$, which is impossible. We may thus assume $y_1=-13$. As $y_2\neq y_1$, $Q(\chi)\subseteq Q(\lambda)$ where χ is a character of degree 13, and (*) applies. Now (K1) yields $y_4<540$, $y_4>0$. If $y_4\leqq 169$, then (K1) implies $(-9/y_2)+(1/y_5)+(1/y_6)\geqq 29/169$, so $y_2=51$, violating (*). It follows from (*) that $y_4=297$. Suppose $y_5<-63$. Then (*) implies $y_5<-500$; similarly for y_6. It follows easily from (K1) that we may assume $y_5=-63$. (K1) and (K2) yield $(9/y_2)+(1/y_6)=-172/9009$, $y_2+y_6=372$. Therefore $y_6>0$, $y_2<0$; (*) implies $y_2=-77$, and so $1/y_6=(9/77)-(172/9009)>1/63$, a contradiction.

(L) $L({}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-{}_2d^z)=(\pm\chi(1), {}_1d^z-d^t)=0$, $L({}_1d^z-d^t)>0$ yield

(L1) $1+(9/y_1)+(9/y_2)+(1/y_6)-(16/y_7)-(16/y_8)=0$,

(L2) $1+y_1+y_2+y_6-y_7-y_8=0$,

(L3) $y_1+y_2+y_3=0$,

(L4) $(36/y_1)+(36/y_2)+(400/y_3)>0$.

$y_1\equiv 51$, $y_2\equiv 51$, $y_6\equiv 1$, $y_7\equiv 52$, $y_8\equiv 52 \pmod{64}$. By (L1), either y_1 or $y_2=-13$. So we may assume $y_1=-13$. Then (L4) implies $0<y_3<200$. If $y_3=154$, then (L3) gives $y_2=-141$, violating (*). Hence $y_3=90$, $y_2=-77$. If y_7 and y_8 both exceed 52, then (L1) and (L3) give $y_7=y_8=180$, $y_6=-63$, against (L2). So we may assume $y_7=52$. Then (L1) and (L2) imply $(1/y_6)-(16/y_8)=9/77$, $y_6-y_8=141$. Thus $-160<y_8<0$, so by (*) $y_8=-140$. Thus $y_6=1$, a contradiction.

(M) $L({}_1d^z-{}_2d^z)=0$ yields

$$1-(1/x_1)-(1/x_2)-(1/x_3)+(16/x_4)+(1/y_2)+(1/y_3)-(16/y_4)\ =\ 0.$$

Also, $x_i\equiv 1$ $(1\leqq i\leqq 3)$, $x_4, y_4\equiv 52$, $y_2, y_3\equiv 1 \pmod{64}$. It follows easily that $x_4=-12$. Then $|(16/y_4)+(1/3)|\leqq 5/63$, so $-140<y_4<-12$; by (*) applied to a character of degree 12, $y_4\neq -76$, a contradiction.

Remark. These are the generalized decomposition numbers for $B_0(T\langle\beta\rangle)$.

(N) $L(d^t)=L({}_1d^z-{}_2d^z+d^t)=0$ yield

(N1) $1+(4/x_1)+(4/x_2)-(200/x_5)+(1/x_6)+(81/x_7)+(1/x_8)=0$,

(N2) $2+(3/x_1)+(3/x_2)+(16/x_3)+(16/x_4)-(200/x_5)+(2/x_6)+(2/x_8)=0$.

$x_1, x_2, x_6, x_8\equiv 1$; $x_3, x_4\equiv 52$; $x_5\equiv 26$, $x_7\equiv 41 \pmod{64}$; if $x_5>0$, then $x_5\leqq 90$, and if $x_7<0$, then $x_7\leqq -87$. First suppose $x_5=90$. (N1) implies $x_7=41$, which is impossible as $|x_i|\geqq 63$, $i=1, 2, 6, 8$. So $x_5\neq 90$. Then (N2) implies that we may assume $x_3=-12$. If also $x_4=-12$, then subtracting (N1) from (N2) we find $-81<x_7<0$, which is impossible. So $x_4\neq -12$. Now we can apply (*) to a character of degree 12. Thus $x_4\neq -76$. Suppose $x_5\neq 154$. By (*), $x_5>600$ if $x_5>0$. This contradicts (N2), so $x_5=154$. By (N2), $0<x_3<40$, a contradiction.

(P) $L(d^t)=L({}_1d^z-{}_2d^z)=0$ and $L({}_1d^z-d^t)>0$ yield

(P1) $1+(4/x_1)+(4/x_2)-(200/x_5)+(1/x_6)+(25/x_7)+(25/x_8)=0$,

(P2) $1-(1/x_1)-(1/x_2)+(16/x_3)+(16/x_4)+(1/x_6)-(16/x_9)-(16/x_{10})=0$,

(P3) $(-4/x_1)-(4/x_2)+(64/x_3)+(64/x_4)+(400/x_5)>0$.

$x_1, x_2, x_6\equiv 1$, $x_3, x_4, x_9, x_{10}\equiv 52$, $x_5\equiv 26$, $x_7\equiv 53$, $x_8\equiv 53 \pmod{64}$; if $x_5>0$, then $x_5\geqq 90$. From (P1), we easily get $x_5>0$. If $x_5=90$, (P1) implies either x_7 or x_8 is >0 and <50, a contradiction. Hence $x_5\geqq 154$. If $x_4=-12$, then (P3) implies $0<x_3<32$, a contradiction. Therefore x_4, and similarly x_3, is $\neq -12$. Since $|x_i|\geqq 63$, $i=1, 2, 6$, it follows easily from (P2) that $x_3=x_4=-52$, $x_9=x_{10}=76$; thus $(1/x_1)+(1/x_2)-(1/x_6)=-9/247$. We may then assume that $x_1=-63$, and either $x_2=-63$ or $x_6=65$. In either case the third x_i turns out not to be an integer, a contradiction.

(Q) $L(d^t+{}_1d^z-{}_2d^z)=0$, $L({}_1d^z-d^t)>0$ yield

(Q1) $2+(3/x_1)+(16/x_2)+(16/x_3)+(16/x_4)-(75/x_5)+(2/x_7)-(16/x_8)=0$,

(Q2) $(-4/x_1)+(64/x_2)+(64/x_3)+(64/x_4)+(100/x_5)>0$.

$x_1, x_7\equiv 1$, $x_2, x_3, x_4, x_8\equiv 52$, $x_5\equiv 37 \pmod{64}$; if $x_5>0$, then $x_5\geqq 165$. Suppose $x_2=-12$. Then (Q2) implies either x_3 or x_4 is positive and <32, a contradiction. Hence $x_2\neq -12$, and similarly for x_3 and x_4. Now (Q1) implies $75/x_5>\frac{1}{2}$, so $0<x_5<150$, a contradiction.

(R) $L(d^t)=(\pm\chi(1), d^t)=0$ yield

(R1) $1+(4/x_1)-(75/x_5)+(25/x_6)+(25/x_7)=0$,

(R2) $1+4x_1-3x_5+x_6+x_7=0$.

$x_1\equiv 1$, $x_5\equiv 37$, $x_6\equiv 53$, $x_7\equiv 53 \pmod{64}$; if $x_5>0$, then $x_5\geqq 165$. From (R2), it is impossible that $x_1, x_6, x_7<0$ and $x_5>0$ at the same time. Then (R1) easily yields $0<x_5<225$, so $x_5=165$. Also from (R1), we may assume that $x_6=-75$. Then we obtain $(4/x_1)+(25/x_7)=-7/33$, $4x_1+x_7=569$. Therefore $x_7\equiv 53 \pmod{256}$. From the first equation, $-165<x_7<0$, a contradiction.

(S) From Lemma 5(b), $x_1\equiv 51$, $x_4\equiv 50$, $x_6, x_7\equiv 53$, $x_8\equiv 52 \pmod{64}$; if $x_4>0$, then $x_4\geqq 114$. Now $0>L(5d^t+3{}_1d^z-4{}_2d^z)\geqq 8-(144/51)-(96/114)-(100/75)-(100/75)-(64/52)$, a contradiction.

(T) $L(d^t+{}_1d^z-{}_2d^z)=L(-d^t+2{}_1d^z-2{}_2d^z)=(\pm\chi(1), d^t+3{}_1d^z-4{}_2d^z)=0$, $L({}_1d^z)>0$ yield

(T1) $2-(18/x_1)+(16/x_2)+(3/x_3)-(147/x_4)+(108/x_6)=0$,

(T2) $1+(72/x_1)+(32/x_2)-(6/x_3)-(243/x_5)=0$,

(T3) $1-2x_4-x_5+x_6=0$,

(T4) $1+(36/x_1)+(64/x_2)+(98/x_4)+(81/x_5)+(72/x_6)>0$.

$x_1\equiv 51$, $x_2\equiv 52$, $x_3\equiv 1$, $x_4\equiv 39$, $x_6\equiv 54 \pmod{64}$; if $x_5<0$, $x_5\leqq -87$; if $x_6<0$, then $x_6\leqq -138$. We show first that $x_2\neq -12$, $x_1\neq -13$. If $x_2=-12$, (T2) implies $x_5<0$, and (T4) implies $98/x_4>2$, so $x_4=39$. As $x_5<-87$, (T3) gives $x_6<0$. Then (T4) yields $98/x_4>3$, which is impossible. If $x_1=-13$, (T2) implies $243/x_5<-3$, so $-81<x_5<0$, a contradiction. Now since $x_2\neq -12$, (T1) gives $0<x_4<295$. If $x_4=39$, (T1) implies $x_6<108$ and $x_6>0$, so $x_6=54$. Then (T3) implies $x_5=-23$, a contradiction. If $x_4=167$, then (*) implies $|x_i|>165$, $1\leqq i\leqq 6$, and (T1) cannot hold, a contradiction. If $x_4=231$, (T1) implies $108/x_6<-\frac{2}{3}$ so $x_6=-138$; (T3) gives $x_5=-599$, a prime, violating (*). Therefore, $x_4=103$. If $x_6>0$, (*) implies $|x_i|\geqq 102$, $1\leqq i\leqq 3$, and (T1) cannot hold, a contradiction. Therefore $x_6>0$, and from (T3), $x_5<0$. By (*), $x_2\leqq -140$ if $x_2<0$. Then (T2) implies $72/x_1<-2/3$, so $-108<x_1<0$, violating (*).

(U) We show in this case and (V) also that G has a rational character of degree 12, and prove (4.1). From Lemma 4,

(U1) $1-(36/x_1)+(4/x_3)-(98/x_4)+(200/x_5)+(1/x_6)=0$,

(U2) $1+(18/x_1)+(16/x_2)-(1/x_3)-(49/x_4)+(1/x_6)-(16/x_7)=0$, (U3) $1+(36/x_1)+(64/x_2)+(98/x_4)+(200/x_5)+(1/x_6)>0$.

Also, $x_1\equiv 51$, $x_2\equiv 52$, $x_3\equiv 1$, $x_4\equiv 39$, $x_5\equiv 42$, $x_6\equiv 1$, $x_7\equiv 52 \pmod{64}$; if $x_5<0$, then $x_5\leqq -150$. Suppose first that $x_2=-12$. Adding (U1) and (U3) yields $400/x_5>3$, so $x_5=42$ or 106. If $x_5=42$, then (U1) implies $(36/x_1)+(98/x_4)>5$, which is impossible. So $x_5=106$, violating (*). Thus, $x_2\neq -12$. Suppose $x_1\neq -13$. (U2) implies $0<x_4<228$. If $x_4=103$ or 167, then (U2) cannot hold without a violation of (*). Thus $x_4=39$. From (U1) we get $200/x_5>12/13$; thus $x_5=42$, 106, or 170. By (*) applied to a character of degree 39, $x_5\neq 106$. If $x_5=42$, then (U1) clearly cannot hold. So $x_5=170$. Now (U1) yields $-140<x_1<0$, so $x_1=-77$; again by (U1), $(4/x_3)+(1/x_6)<-1/10$, which is impossible. We have proved that $x_1=-13$. It now follows easily from (U1) that $x_4=39$, $x_5=-150$; then $(4/x_3)+(1/x_6)=1/13$, so $x_3=x_6=65$. From (U3), $64/x_2>7/12$, so $x_2=52$. (U2) yields $x_7=-12$. The character with degree $-x_7$ is clearly rational. By Lemma 4(c),

$$\frac{128|C_G(z)|^3}{|C_G(z,y)|^2} = |G|\left\{1-\frac{36}{13}+\frac{64}{52}+\frac{98}{39}-\frac{200}{150}+\frac{1}{65}\right\},$$

proving (4.1).

(V) We get the same equations as in (U), with $\sum_{i=1}^4 25/x_5^{(i)}$ substituted for $200/x_5$, and now $x_5^{(i)}\equiv 53 \pmod{64}$, $1\leqq i\leqq 4$. Suppose $x_1\neq -13$. If $x_2=-12$, then adding (U1) and (U3) we get $\sum_{i=1}^4 50/x_5^{(i)}>42/12$, so some $x_5^{(i)}=53$, violating (*). Thus if $x_1\neq -13$, then $x_2\neq -12$. As in (U), we conclude that $x_4=39$. Now (U1) implies that some $x_5^{(i)}$ is 53, again contradicting (*). Therefore $x_1=-13$. As in (U) we find $x_4=39$. Now (U1) gives $\sum_{i=1}^4 25/x_5^{(i)}<-964/819$. (*) applied to a character of degree 13 implies that $x_5^{(i)}<-200$ if $x_5^{(i)}<-75$. It follows that each $x_5^{(i)}=-75$. We can now argue as in (U).

5. **Completion of the proof.** Since G has a rational character of degree 12, a theorem of Schur [13] implies $|G| \mid 2^6\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13$. We show $C_G(z)=C_G(Z(T))$. Let p be a prime divisor of $|O_{2'}(C_G(z))|$, and let P_0 and P be T-invariant Sylow p-subgroups of $O_{2'}(C_G(z))\cap C_G(Z(T))$ and $O_{2'}(C_G(z))$, respectively, with $P_0\leqq P$. Suppose $P_0<P$. Then from the character theory of T we conclude $p^4 \mid |P:P_0|$, so $p^4 \mid |C_G(z)|/|C_G(Z(T))|$. By (4.1), we get $p^{12} \mid |G|$, a contradiction. Therefore $P_0=P$ and, as p was arbitrary, $O_{2'}(C_G(z))\leqq C_G(Z(T))$. The structure of $C_G(z)$ modulo core yields $C_G(Z(T))=C_G(z)$. Let $N=N_G(Z(T))$. Thus N is strongly embedded in G. By a theorem of Bender [2], $G\cong Sz(8)$, $U_3(4)$, or $L_2(64)$, since $|T|=2^6$. As T has exactly 3 involutions, $G\cong U_3(4)$, completing the proof of Theorem 2.

We turn to the corollary to Theorem 1. Let N be a minimal normal subgroup of G. If $T\leqq N$, then $N=G$ and since T is indecomposable, G is simple; thus $G\cong U_3(4)$ by Theorem 1. So assume $T\nleqq N$.

If N is nonsolvable, then by the Z^*-theorem, N is simple and $N\geqq Z(T)$, since T contains only 3 involutions. As argued in the proof of Lemma 1, N contains an element α normalizing $N\cap T$ and cycling $Z(T)^{\#}$. Therefore $|N\cap T|\equiv 1 \pmod 3$. If $|N\cap T|=16$, then the existence of α implies that $N\cap T\cong Z_4\times Z_4$, contradicting the main theorem of [14]. So $N \cap T = Z(T)$, and $N \cong L_2(q)$ for some $q \equiv \pm 3 \pmod 8$, by [10]. But then $2^4 \nmid |\mathrm{Aut}\, N|$ so $C_G(N)$ contains an involution; this implies $C_G(N) \cap N \neq 1$, which is impossible.

Therefore N is solvable, so $N \leqq Z(T)$. If $|N| = 2$, then since $Z(T)$ is weakly closed in T, we get $Z(T)/N \lhd G/N$. Hence $Z(T) \lhd G$ in any case. Since $G = O^{2'}(G)$, $Z(T) \leqq Z(G)$. Denote residues modulo $Z(T)$ by bars. The proof of Lemma 1(c) implies that T has no automorphism of order 3 or 7 acting trivially on $Z(T)$. Hence 3 and 7 do not divide $|N_{\bar{G}}(\bar{T})/C_{\bar{G}}(\bar{T})|$. Clearly $\bar{G}$ is core free. By the main theorem of [14], a minimal normal subgroup of $\bar{G}$ is solvable, and it follows easily that $\bar{T} \lhd \bar{G}$. Therefore $T = G$, as required.

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