TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 164, February 1972

CONNECTIONS ON SEMISIMPLE LIE GROUPS

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Abstract. The plus and minus connections of Cartan and Schouten, which exist on any Lie group, have the following three properties: (1) the connection is left invariant, (2) the curvature of the connection is zero, (3) the set of maximal geodesics through the identity of the Lie group is equal to the set of one-parameter subgroups of the Lie group. It is shown that the plus and minus connections are the only ones with these properties on a real simple Lie group. On a real semisimple Lie group the connections with these properties are in one-to-one correspondence with the ways of choosing an ideal of the Lie algebra and then choosing a complementary subspace to it.

1. **Introduction.** To find all the connections on a real semisimple Lie group with the properties listed in the abstract, we restate the properties in terms of representations of Lie algebras and classify the representations which correspond to connections. Specifically, these properties induce skew-symmetric representations of the corresponding Lie algebra. By using computational methods of representation theory, the skew-symmetric representations of a complex simple Lie algebra can be determined. With standard techniques of representation theory these results are extended to complex semisimple Lie algebras in §3 and then to real semisimple Lie algebras in §4.

Let G be a real Lie group with $T_e(G)$, the tangent space at the identity, considered as its Lie algebra g. Let ∇ be a left invariant affine connection on G. Nomizu [4] has shown that ∇ arises from a bilinear map $\alpha\colon g\times g\to g$. When the curvature of ∇ is zero, it is easily shown that α is a Lie algebra representation of g on itself, considering $\alpha\colon g\to \mathrm{gl}\,(g)$. Helgason [2] shows that the geodesics of ∇ have property (3) if α is skew-symmetric. Consequently, the desired connections arise from representations (ρ, g) of g on itself which are *skew-symmetric* ($\rho(x)x=0$ for all $x\in g$).

Let g be a complex semisimple Lie algebra and let h be a Cartan subalgebra of g. Let Δ be the set of roots of g with respect to h and let $\pi=\{\alpha_1,\dots,\alpha_n\}$ be a set of simple roots of Δ. Let $\{H_1,\dots,H_n\}$ be a basis of h chosen so that $\alpha_i(H_j)\in \boldsymbol{Q}$, the field of rational numbers, for $i,j=1,\dots,n$. In this paper, we will call an element $H\in h$ *generic* if $H=\sum_{i=1}^n a_iH_i$ where the complex coefficients $a_1,\dots,a_n$ are linearly independent over $\boldsymbol{Q}$. It can be shown that generic elements have the properties listed in the following lemma.

Presented to the Society, January 29, 1969 under the title *On counting connections on Lie groups* and May 12, 1969; received by the editors June 24, 1970.

AMS 1970 *subject classifications*. Primary 22E15, 17B10.

Key words and phrases. Real semisimple Lie group, affine connection, real semisimple Lie algebra, skew-symmetric representation.

LEMMA 2. (a) $\alpha(H)\neq 0$ *for every root* α *of* g *and every generic element of* h.

(b) *If* ω, ω' *are weights of some arbitrary representation* (σ, V) *of* g, *and if* $\omega(H)=\omega'(H)$ *for some generic element of* h, *then* $\omega=\omega'$.

(c) *Let* (σ, V) *be a representation of* g. *Assume that for a generic element,* H, *of* h, *there is a nonzero* $v \in V$ *and a* $\omega \in h^*$ *such that* $\sigma(H)v=\omega(H)v$. *Then* ω *is a weight of* (σ, V).

2. **Skew-symmetric representations.** Throughout this section let (ρ, g) be a skew-symmetric representation of g. Let $V_0=\{x \in g \mid \rho(g)x=0\}$. Using the definition of skew-symmetry it is easy to show that

$$(\rho(x))^2 = \rho(x)\,\mathrm{ad}\,x \quad \text{for all } x \in g \tag{1}$$

and that $\ker \rho = V_0$. Hence, V_0 is an ideal in g.

LEMMA 1. *If* ρ *is a faithful skew-symmetric representation of a semisimple Lie algebra, then every invariant subspace is an ideal.*

Proof. Let V_1 be an invariant subspace of g under ρ. Since any representation of a semisimple Lie algebra is completely reducible [3, p. 79], we may write $g=V_1 \oplus V_2$ where V_2 is an invariant subspace of g under ρ. Now $\rho(g)[x, V]\subseteq V_1$ where $x \in g$ and $v \in V_1$. Writing $[x, v]=[x, v]_1+[x, v]_2$ where $[x, v]_i \in V_i$, $i=1, 2$, we have that

$$V_1 \supseteq \rho(g)[x, v] = \rho(g)([x, v]_1)+\rho(g)([x, v]_2).$$

From the invariance of V_i, $i=1, 2$, we conclude that $[x, v]_2=0$ and that V_1 is an ideal.

We are now able to characterize the weights of a skew-symmetric representation of a complex semisimple Lie algebra.

THEOREM 3. *Let* g *be a complex semisimple Lie algebra. Every nonzero weight of* ρ *is a root of* g. *Furthermore,* 0 *is a weight of* ρ.

Proof. With respect to ad we may write

$$g = g(0)+\sum_{\alpha} g(\alpha), \quad \text{where } \alpha \text{ is a root of } g.$$

With respect to (ρ, g) we may write

$$g = V(0)+\sum_{\omega} V(\omega), \quad \text{where } \omega \text{ is a weight of } (\rho, g).$$

We first show that $g(0)$, which is a Cartan subalgebra, h, of g is a subspace of $V(0)$. Let $Y \in h$. By equation (1) for all $H \in h$ we have $(\rho(H))^2 Y=\rho(H)[H, Y]=0$. It can be shown that for all $H \in h$, $\rho(H)$ is semisimple. Thus, $(\rho(H))^2 Y=0$ implies that $\rho(H)Y=0$ and $Y \in V(0)$. Hence, 0 is a weight of ρ.

Now let $H \in h$ be generic. Then $\alpha(H)\neq 0$ for every root α of g. Let $g(\alpha)=CE_\alpha$ and consider $\rho(H)E_\alpha$.

$$\rho(H)(\rho(H)E_\alpha) = \rho(H)[H, E_\alpha] = \alpha(H)\rho(H)E_\alpha. \tag{2}$$

There are two cases:

(a) $\rho(H)E_\alpha=0$ which, by Lemma 2, choosing $\omega=0$, implies that $E_\alpha \in V(0)$.

(b) $\rho(H)E_\alpha \neq 0$ which, by Lemma 2 and equation (2), implies that α is a weight and $\rho(H)E_\alpha \in V(\alpha)$.

Label the roots so that (a) is true for $\alpha^{(1)}, \ldots, \alpha^{(k)}$ and (b) is true for $\alpha^{(k+1)}, \ldots, \alpha^{(r)}$. Then

$$V(0) \supseteq h+g(\alpha^{(1)})+\cdots+g(\alpha^{(k)}), \tag{3}$$

$$V(\alpha^{(j)}) \supseteq \rho(H)g(\alpha^{(j)}), \qquad j = k+1, \ldots, r. \tag{4}$$

By a dimension argument, we have equality in (3) and (4). Therefore,

$$g = V(0)+V(\alpha^{(k+1)})+\cdots+V(\alpha^{(r)})$$

is a weight space decomposition of g and $\alpha^{(k+1)}, \ldots, \alpha^{(r)}$ are the only nonzero weights of (ρ, V).

Knowing about the weights of (ρ, g) where g is complex semisimple, we can prove this theorem about equivalent representations.

THEOREM 4. *Let g be complex semisimple. If ρ is equivalent to* ad, *then* $\rho=\mathrm{ad}$.

Proof. There is an element $B \in \mathrm{GL}(g)$ such that, for all $X \in g$, $\mathrm{ad}\, X=B^{-1}\rho(X)B$. Since ρ skew-symmetric, we have $B\,\mathrm{ad}\, XB^{-1}=0$ for all $X \in g$, and using the nonsingularity of B, $[X, B^{-1}X]=0$. By polarization, $[X, B^{-1}Y]=[B^{-1}X, Y]$, or $[BX, Y]=[X, BY]$ for all $X, Y \in g$. Since ρ is equivalent to ad, $h \subseteq V(0)$ implies that $h=V(0)$.

Let H be a generic element of h. We have $\rho(H)BE_\alpha=B\,\mathrm{ad}\, HE_\alpha=\alpha(H)BE_\alpha$ for every root, α, of g. Since H is generic, $BE_\alpha \in V(\alpha)$ and, from the proof of Theorem 3, $\rho(H)E_\alpha \in V(\alpha)$. Therefore, there is a nonzero constant depending on α and H which will be denoted by $C(\alpha, H)$ such that $BE_\alpha=C(\alpha, H)\rho(H)E_\alpha$.

For any root α and any $H \in h$, equation (1) implies that

$$(\rho(H))^2E_\alpha-\rho(H)[H, E_\alpha] = \rho(H)(\rho(H)E_\alpha-\alpha(H)E_\alpha) = 0.$$

Therefore, if H is a generic, $\rho(H)E_\alpha-\alpha(H)E_\alpha=H^{(\alpha)} \in h$. Let β be any root of g. Then for generic H,

$$[BE_\alpha, E_\beta] = [C(\alpha, H)\rho(H)E_\alpha, E_\beta] = C(\alpha, H)\alpha(H)[E_\alpha, E_\beta]+C(\alpha, H)\beta(H^{(\alpha)})E_\beta$$

and

$$[E_\alpha, BE_\beta] = [E_\alpha, C(\beta, H)\rho(H)E_\beta] = C(\beta, H)\beta(H)[E_\alpha, E_\beta]-C(\beta, H)\alpha(H^{(\beta)})E_\alpha.$$

But $[BE_\alpha, E_\beta]=[E_\alpha, BE_\beta]$ which gives us on equating terms

$$(C(\alpha, H)\alpha(H)-C(\beta, H)\beta(H))[E_\alpha, E_\beta] = 0 \tag{5}$$

and

$$C(\alpha, H)\beta(H^{(\alpha)})E_\beta+C(\beta, H)\alpha(H^{(\beta)})E_\alpha = 0. \tag{6}$$

If $\alpha=\beta$, we have $\alpha(H^{(\alpha)})=0$. If $\alpha\neq\beta$, we have $\beta(H^{(\alpha)})=0$ from (6).

Therefore, for any root α, $H^{(\alpha)}=0$, and $\rho(H)E_\alpha=\text{ad } HE_\alpha$ for all generic $H\in h$. Consequently $BE_\alpha=C(\alpha, H)\alpha(H)E_\alpha$ for all roots α. Since $\rho(H)H'=0=\text{ad } HH'$ for all $H, H'\in h$ and by skew-symmetry $\rho(E_\alpha)H=\text{ad } E_\alpha H$ for all roots α and all generic elements of h, to show $\rho(X)=\text{ad } X$ for all X we must only show $\rho(E_\alpha)E_\beta=\text{ad } E_\alpha E_\beta$ for $\alpha\neq\beta$. Now

$$\rho(E_\alpha)E_\beta = B\,\text{ad } E_\alpha B^{-1}E_\beta = B[E_\alpha, E_\beta]/C(\beta, H)\beta(H).$$

If $\alpha+\beta$ is not a root, then $\rho(E_\alpha)E_\beta=0=\text{ad } E_\alpha E_\beta$. Otherwise $[E_\alpha, E_\beta]=kE_{\alpha+\beta}$ where k is a constant and

$$\rho(E_\alpha)E_\beta = (C(\alpha+\beta, H)(\alpha+\beta)(H)/C(\beta,H)\beta(H))\,\text{ad } E_\alpha E_\beta.$$

But from (5), $C(\beta, H)\beta(H)=C(-\beta, H)(-\beta(H))$ since $[E_\alpha, E_\beta]\neq 0$ and $C(-\beta, H)\times(-\beta(H))=C(\alpha+\beta, H)(\alpha+\beta)(H)$ since $[E_{\alpha+\beta}, E_{-\beta}]\neq 0$. Therefore, $\rho(E_\alpha)E_\beta=\text{ad } E_\alpha E_\beta$ and, by linearity, $\rho(X)=\text{ad } X$ for all $X\in g$.

3. **Skew-symmetric representations of complex semisimple Lie algebras.** In this section we find all the skew-symmetric representations of a complex semisimple Lie algebra. The technique used is to first study skew-symmetric representations of complex simple Lie algebras and then extend these results to the semisimple case. Throughout this section g will be complex semisimple and (ρ, g) will be skew-symmetric.

Initially, our goal is to show that the only nonzero skew-symmetric representation of a complex simple Lie algebra is the adjoint representation. Since V_0 is an ideal, when g is simple a skew-symmetric representation of g will be either 0 or irreducible and faithful. If $\rho\neq 0$, in light of Theorem 4, we need only prove that ρ is equivalent to ad, or equivalently the highest weight of ρ is equal to highest root of g. We then will be able to conclude

THEOREM 5. *If g is simple, $\rho=0$ or $\rho=$* ad.

LEMMA 6. *If g is simple and $\rho\neq 0$, then the highest weight of ρ is equal to the highest root of g.*

Proof. We first consider the case that all the roots of g have the same length, i.e. Δ is simply laced ($g=A_n, D_n, E_n$). Let $\Lambda=$set of weights of ρ. Let $\omega\in\Lambda$ and $\alpha\in\Delta$. By Theorem 3, $\omega\in\Delta$. Hence, there is an element s of the Weyl group of g such that $s(\omega)=\alpha$. But $s(\omega)\in\Lambda$ since the Weyl group preserves weights. Therefore $\Lambda=\Delta$, and the highest weight equals the highest root.

Now if the roots of g have two different lengths, i.e. Δ is doubly laced, consider the case where the length of the highest weight$=$the length of the highest root. By the same argument as above, one can show that the highest root is a weight, and, therefore, must be the highest weight.

Finally, if the length of the highest weight$\neq$the length of the highest root, the same argument shows that the highest weight$=$the highest root with length not equal to the length of the highest root of g. In fact, the highest root of a doubly laced root system always has the longer length. The highest root of short length is the highest weight of an irreducible representation of g, but in each case $\dim(\rho) < \dim(g)$ which is contrary to hypothesis. Consequently, this situation is not possible, and the other two cases give the desired result.

By examining the possible dimensions for irreducible representations, the author has also proved Lemma 6 in the following form.

LEMMA 6'. *Let (σ, V) be an irreducible representation of a complex simple Lie algebra g such that* $\dim(\sigma)=\dim(g)$. *Then* $\sigma\sim\mathrm{ad}$.

Turning to the semisimple case we first consider representations for which

$$V_0 = \{x \mid \rho(g)X = 0\}$$

is zero. It has been shown that with this hypothesis ρ is faithful and every invariant subspace is an ideal. The converse is also true.

LEMMA 7. *Let g be semisimple and (ρ, g) be skew-symmetric. If $V_0=0$, every ideal of g is an invariant subspace of ρ.*

Proof. Let $g=V_1\oplus\cdots\oplus V_s$ where V_j, $j=1,\ldots,s$, are irreducible invariant subspaces of g. Then V_j is an ideal for each j. If each V_j is a simple ideal, we are finished, so suppose that V_k is not simple. Then ρ restricted to V_k is irreducible, faithful, and skew-symmetric. Let $V_k=g_1\oplus\cdots\oplus g_r$ where the g_i are simple ideals of g. Let $\rho_i=\rho\,|\,g_i$. Using the same method as in the proof of Theorem 3, one may show that every nonzero weight of ρ_i is a root of g_i. Since ρ is irreducible it may be written as the tensor sum of the representations σ_i where $\sigma_i\colon g_i\to\mathrm{gl}(W_i)$ and W_i is a minimal ρ_i invariant subspace of g. That is, for ρ restricted to V_k

$$\rho(X_1+\cdots+X_r)(w_1\otimes\cdots\otimes w_r) = \sum_{i=1}^{r} w_1\otimes\cdots\otimes\rho(X_i)w_i\otimes\cdots\otimes w_r$$

for $X_i\in g_i$, and $w_i\in W_i$. Every weight of ρ is the sum of weights of σ_i. But weights of σ_i are also weights of ρ_i and hence roots of g_i. Therefore, we have on one hand every weight of ρ is a root of g, and on the other hand every weight of ρ is the sum of roots g_i and hence not a root of g. This contradicts the assumption that V_k was not simple.

THEOREM 8. *If (ρ, g) is a skew-symmetric faithful representation of a complex semisimple Lie algebra g, then* $\rho=\mathrm{ad}$.

Proof. Let $g=g_1\oplus\cdots\oplus g_k$ where g_j is a simple ideal. Write $\rho=\rho_1+\cdots+\rho_k$ where ρ_j is an irreducible representation of g on g_j. Let σ_j be equal to ρ_j restricted to g_j. Then σ_j is skew-symmetric and faithful. From Theorem 5, $\sigma_j=\mathrm{ad}_{g_j}$ and hence $\rho=\mathrm{ad}_g$.

We now consider the case where $V_0 \neq 0$. We have seen that V_0 is an invariant subspace under ρ. Since g is semisimple we may write $g = V_0 \oplus V_1$ as vector spaces where V_1 is a complementary invariant subspace to V_0 and also $g = g_1 \oplus g_2$ where $g_1 = V_0$ and g_2 is an ideal. Then as vector spaces $V_1 \cong g_2$. Let $\pi_i: g \to g_i$, $i=1, 2$, be projection maps. Then π_2 is the isomorphism between V_1 and g_2. Also if we let $\phi = \pi_1 \circ \pi_2^{-1}$, then $\phi: g_2 \to g_1$ and

$$V_1 = \{x + \phi(x) \mid x \in g_2\}.$$

We see that ϕ is a vector space homomorphism, and it is a Lie algebra homomorphism if and only if V_1 is a subalgebra of g.

Now let ρ_1 be the representation of g_2 on V_1 induced by restricting ρ to g_2. Then ρ_1 is faithful. For if $\rho_1(x) = 0$ for some $x \in g_2$, we have $\rho_1(x)v_1 = \rho(x)v_1 = \rho(v_1)x = 0$ for all $v_1 \in V_1$. Furthermore, $\rho(x)v_0 = \rho(v_0)x = 0$ for all $v_0 \in V_0$. Therefore, $\rho(v)x = 0$ for all $v = v_0 + v_1 \in g$ which implies that $x \in g_1$ and consequently that $x = 0$.

Let $\rho_2(x) = \pi_2 \circ \rho_1(x) \circ \pi_2^{-1}$ where $x \in g_2$. Then ρ_2 is a faithful representation of g_2 on itself. Also we have

$$\rho_2(x)x = \pi_2(\rho_1(x)(x+\phi(x))) = \pi_2(\rho(x)(x) + \rho(x)\phi(x)) = 0.$$

From Theorem 5 we conclude that $\rho_2(x)y = [x, y]$ and that

$$\rho(x+\phi(x))(y+\phi(y)) = [x, y] + \phi([x, y])$$

where $x, y \in g_2$. We have shown

THEOREM 9. *Let (ρ, g) be a skew-symmetric representation of a complex semisimple Lie algebra of whose zero space is g_1. Let V_1 be a complementary subspace to g_1 defined by $\phi: g_2 \to g_1$ where g_2 is a complementary ideal to g_1. Then for all $x, y \in g_2$, we have $\rho(x+\phi(x))(y+\phi(y)) = [x, y] + \phi([x, y])$ and $\rho|_{V_1} = \mathrm{ad}|_{V_1}$ if V_1 is a subalgebra of g.*

4. **Skew-symmetric representations of real semisimple Lie algebras.** In this section we extend the results of the preceding section to real semisimple Lie groups. We use the standard techniques of complexifying real Lie algebras and their representations which are described below.

If V is a vector space over $\boldsymbol{R}$, the field of real numbers, denote by V^c the complex vector space obtained from V by extending the ground field. That is,

$$V^c = V \otimes_{\boldsymbol{R}} \boldsymbol{C} \cong V \oplus iV \quad \text{and} \quad \dim_{\boldsymbol{R}} V = \dim_{\boldsymbol{C}} V^c.$$

If g is a real Lie algebra, then g^c is a complex Lie algebra, called the complexification of g, and the bracket product in g^c is defined by

$$[X+iY, Z+iW] = ([X, Z] - [Y, W]) + i([Y, Z] + [X, W]).$$

It can be shown that g is semisimple if and only if g^c is semisimple.

If (ρ, V) is a real representation of a real Lie algebra g, denote by (ρ^c, V^c) the complex representation of g induced by ρ where

$$\rho^c(X)(v+iw) = \rho(X)v + i\rho(X)w$$

for any $X \in g$ and $v+iw \in V^c$. (ρ^c, V^c) can be extended to a representation of g^c, denoted by (P, V^c) and defined by

$$P(X+iY) = \rho^c(X) + i\rho^c(Y)$$

for all $X+iY \in g^c$.

The properties of faithfulness, equivalence, skew-symmetry, and adjoint all lift from a real representation of a real Lie algebra to the induced representation of the complexification of the algebra, and conversely they are induced on a real representation of a real Lie algebra by its lift to the complexification of the algebra. Consequently, we may extend Theorem 8 to read

THEOREM 10. *If (ρ, g) is a skew-symmetric faithful representation of a real semisimple Lie algebra, g, then $\rho = \mathrm{ad}$.*

Using Theorem 10 in place of Theorem 8 we may extend Theorem 9 to read

THEOREM 11. *Let (ρ, g) be a skew-symmetric representation of a real semisimple Lie algebra g whose zero-space is g_1. Let V_1 be a complementary subspace to g_1 defined by $\phi: g_2 \to g_1$ where g_2 is a complementary ideal to g_1. Then for all $x, y \in g_2$, we have*

$$\rho(x+\phi(x))(y+\phi(y)) = [x, y] + \phi([x, y]).$$

Converting the Lie algebra results to Lie group ones as described in §1, we have shown

THEOREM 12. *For a real semisimple Lie group G, the left invariant linear connections on G which have*

(a) *zero curvature, and*

(b) *set of maximal geodesics through the identity = set of one-parameter subgroups, are in one-to-one correspondence with the decompositions of the Lie algebra g of G into the vector space sum of an ideal g_1 and a subspace V_1 where the representation defining the connection is zero on g_1 and as in Theorem* 11 *on V_1.*

However, if we assume that the connection is also right invariant, we can show

THEOREM 13. *The bi-invariant connections on a real semisimple Lie group G which have properties* (a) *and* (b) *of Theorem* 12 *are in one-to-one correspondence with the decompositions of the Lie algebra g of G into the direct sum of two ideals, $g = g_1 \oplus g_2$, where the representation defining the connection is zero on g_1 and equal to the adjoint representation on g_2.*

Proof. Using Nomizu's results [4], we have that a bi-invariant connection satisfying (a), and (b) is defined by a skew-symmetric representation (ρ, g) of g such that

$$\text{ad } X\rho(Y)Z = \rho(\text{ad } X\cdot Y)Z+\rho(Y)(\text{ad } X)Z.$$

Applying this identity to the result of Theorem 11 yields $[X, \phi([Y, Z])]=0$ for all $X\in g_1$ and $Y, Z\in g_2$. Therefore, $\phi([Y, Z])=0$. But since g_2 is semisimple $[g_2, g_2]=g_2$ and we have $\phi=0$. Therefore, given g_1, the only choice for V_1 is g_2.

The author would like to thank Professor M. Goto, his dissertation advisor, for his helpful suggestions on the material of this paper which is part of the author's Ph.D. dissertation written at the University of Pennsylvania, Spring 1969.

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