TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 165, March 1972

RELATIVE IMAGINARY QUADRATIC FIELDS OF CLASS NUMBER 1 OR 2

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Abstract. Let K be a normal totally real algebraic number field. It is shown how to effectively classify all totally imaginary quadratic extensions of class number 1. Let K be a real quadratic field of class number 1, whose fundamental unit has norm -1. Then it is shown how to effectively classify all totally imaginary quadratic extensions of class number 2.

1. **Introduction.** Let K be a totally real field, and let L be a totally imaginary quadratic extension of K. Holding K fixed, there are only a finite number of fields L having a given class number h_L [3]. For $K=\boldsymbol{Q}$, the problem of effectively determining all L having a given class number has its origins in Gauss' *Disquisitiones.* Although extensive work has been done over a period of many years, the first notable successes have come only fairly recently. In 1966, Stark [6] settled the problem $K=\boldsymbol{Q}$, $h_L=1$. In 1970, the author [4] reduced the problem $K=\boldsymbol{Q}$, $h_L=2$ to a conjecture on linear forms in the logarithms of algebraic numbers. A weak form of this conjecture was proven by Baker [1] and Stark [6]. Their form of the conjecture sufficed to settle the class number 2 problem. In the present work we will consider the problems $h_L=1$ and $h_L=2$ for algebraic number fields other than $\boldsymbol{Q}$. The basic idea is to reduce the analytic difficulties back to the case $K=\boldsymbol{Q}$, where the situation has been carefully studied. Therefore, the methods of this paper have a distinctly algebraic character and are somewhat more elementary than was required in the cited references.

The starting point of this work is the result of Heilbronn and Linfoot [5] who proved that all imaginary quadratic fields of class number 1 have discriminants $\geqq -200$, with one possible exception. It is an immediate consequence of this result that there are at least 9 and at most 10 imaginary quadratic fields of class number 1. It is interesting to note, however, that all known proofs that there are exactly 9 imaginary quadratic fields of class number 1 make no use of the result of Heilbronn and Linfoot, but rather classify the relevant fields directly.

The work of Heilbronn and Linfoot was generalized by T. Tatuzawa [9], who proved that for $h\geqq 1$, $d=$the discriminant of an imaginary quadratic field of class

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Received by the editors March 31, 1971.

AMS 1969 *subject classifications.* Primary 1065; Secondary 1068.

Key words and phrases. Number field, class number, quadratic field.

[1] Research supported by the National Science Foundation Grant GP-20538.

number $\leqq h$, then

$$|d| \leqq 2100h^2 \log^2 (13h),$$

with one possible exceptional d.

Recently, J. Sunley [8] has generalized Tatuzawa's result as follows:

THEOREM [S]. *Let $h \geqq 1$ and let K be a fixed totally real algebraic number field. Further, let L be a totally imaginary quadratic extension of K, having class number h_L and discriminant d_L. Then there exists an effectively computable constant $c = c(K, h)$ such that if $h \leqq h_L$, then $|d_L| \leqq c$, with the possible exception of one field L.*

In this paper, we will prove

THEOREM 1. *Let all notations be as in* [S], *and assume that K is normal. Then the exceptional field (if it exists) must be normal over $\boldsymbol{Q}$.*

THEOREM 2. *Let all notations be as in* [S], *and assume that K is normal. Assume that $L/\boldsymbol{Q}$ is normal.*

(1) *If $h_L = 1$, then either $L = K((-p)^{1/2})$, where $\boldsymbol{Q}((-p)^{1/2})$ is an imaginary quadratic field of class number* 1, *or L belongs to a finite, effectively determined collection of fields.*

(2) *Let K be a real quadratic field of class number* 1 *and fundamental unit of norm* $= -1$. *If $h_L = 2$, then either $L = K((-pq)^{1/2})$, where $\boldsymbol{Q}((-pq)^{1/2})$, is an imaginary quadratic field of class number* 2, *or L belongs to a finite, effectively determined collection of fields.*

As immediate consequences of [S], Theorems 1 and 2, and the effective classification of imaginary quadratic fields of class number 1 and 2, we get

THEOREM 3. *Assume that K is normal. There exists an effectively determined constant $c = c(K, h)$ such that if L has class number h and $L/\boldsymbol{Q}$ is nonnormal, then $|d_L| \leqq c$.*

THEOREM 4. *Assume that K is normal. It is possible to effectively determine all those L for which $h_L = 1$.*

THEOREM 5. *Let K be a real quadratic field of class number* 1 *whose fundamental unit has norm* -1. *Then it is possible to effectively determine all those L for which $h_L = 2$.*

It may be possible to generalize Theorem 5 to the case of arbitrary totally real K, but we do not see how to accomplish this at the present time.

The author wishes to acknowledge, with sincere thanks, many helpful conversations with J. Sunley, R. Greenberg, and Y. Furuta, and a number of valuable suggestions of the referee.

2. **Proof of Theorem 1.** Throughout this paper, let K be a normal totally real algebraic number field of degree n and class number h_K. Let $N = N_{K/\boldsymbol{Q}}$ and let

$x \to x^{(i)}$ $(1 \leqq i \leqq n)$ denote the distinct conjugation maps of K. Further, let $L = K(\delta^{1/2})$ be a totally imaginary quadratic extension of K and let h_L denote the class number of L.

In order to prove Theorem 1, let us assume that $L/\boldsymbol{Q}$ is not normal. Since K is normal, the conjugate fields of L are all of the form $L_i = K((\delta^{(i)})^{1/2})$ $(1 \leqq i \leqq n)$, and since L is not normal, there exists an i $(1 \leqq i \leqq n)$ such that

$$K(\delta^{1/2}) \neq K((\delta^{(i)})^{1/2}).$$

Since L is isomorphic to L_i over $\boldsymbol{Q}$, L and L_i have the same discriminant d. Thus, if $|d| > c$, where c is as in [S], then we get an immediate contradiction to [S]. Therefore, $|d| \leqq c$ and L cannot be the exception field. This completes the proof of Theorem 1.

3. **Preliminaries to the Proof of Theorem 2.**

LEMMA 3-1. *If L has class number* 1, *then K has class number* 1. *If L has class number* 2, *then K has class number* 1 *or* 2.

Proof. Let K^* (resp. L^*) denote the Hilbert class field of K (resp. L). Then K^*L is an unramified, abelian extension of L, so that $K^*L \subseteq L^*$. But since K^*/K is unramified, K^* is totally real and thus K^* is linearly disjoint from L over K. Therefore, $\deg(K^*L/L) = \deg(K^*/K) = h_K$. But if L has class number 1, $L^* = L$, so that $\deg(K^*L/L) = 1 \Rightarrow h_K = 1$. On the other hand, if L has class number 2, then $\deg(L^*/L) = 2$ and therefore $h_K = 1$ or 2.

For the sake of simplicity, let us assume throughout the remainder of this paper that *if L has class number* 2, *then K has class number* 1.

By Lemma 3-1 and the assumption, the ring of integers $\mathfrak{O}_K$ of K is a unique factorization domain and therefore we may write $L = K(\mu^{1/2})$ where μ is a square-free integer of K. Let

$$2\mathfrak{O}_K = \mathfrak{p}_1^{\kappa_1} \cdots \mathfrak{p}_t^{\kappa_t}$$

where $\mathfrak{p}_i$ is a K-prime. Suppose that

$$\mathfrak{p}_i \nmid \mu\mathfrak{O}_K \quad (1 \leqq i \leqq s), \qquad \mathfrak{p}_i | \mu\mathfrak{O}_K \quad (s+1 \leqq i \leqq t).$$

Let r_i $(1 \leqq i \leqq s)$ denote the largest nonnegative integer $\leqq \kappa_i$ such that there exists $u_i \in K$ such that $\mu \equiv u_i^2 \pmod{\mathfrak{p}_i^{2r_i}}$. Then, by a classical result of Kummer theory,

$$d_{L/K} = \prod_{i=1}^{s} \mathfrak{p}_i^{2(\kappa_i - r_i)} \cdot \prod_{1=s+1}^{t} \mathfrak{p}_i^{2\kappa_i} \cdot \mu\mathfrak{O}_K, \tag{1}$$

where $d_{L/K}$ denotes the relative discriminant of the extension L/K. Since the $\mathfrak{p}_i$ are principal, we can choose an integer $\delta \in K$ such that

$$L = K(\delta^{1/2}), \qquad d_{L/K} = \delta\mathfrak{O}_K. \tag{2}$$

Note that δ is determined up to multiplication by the square of a unit of K.

Choose $b \in \mathfrak{O}_K$ so that

$$b \equiv u_i \pmod{\mathfrak{p}_i^{r_i}} \qquad (1 \leqq i \leqq s),$$

let $\mathfrak{p}_i = \pi_i \mathfrak{O}_K$ $(1 \leqq i \leqq t)$, and let $\lambda = \prod_{i=1}^{s} \pi_i^{r_i}$. Then $1, (b-\mu^{1/2})/\lambda$ is an integral basis of L. In particular, every integer of L can be written in the form

$$(3) \qquad (x+y\mu^{1/2})/\lambda \qquad (x, y \in \mathfrak{O}_K).$$

LEMMA 3-2. *Let U_K (resp. U_L) denote the group of units of $\mathfrak{O}_K$ (resp. $\mathfrak{O}_L$). If $|N(\mu)| > 4^n$, then $U_K = U_L$.*

Proof. Let $u = (x+y(\mu^{1/2}))/\lambda \in U_L$. Then $|N_{L/Q}(u)| = 1$, so that

$$(4) \qquad |N(x^2-\mu y^2)| = |N_{L/Q}(\lambda)| = |N(\lambda)|^2 \leqq 4^n.$$

However, if $y \neq 0$,

$$|N(x^2-\mu y^2)| \geqq |N(\mu y^2)| > 4^n |N(y)|^2 \geqq 4^n,$$

which is a contradiction to (4). Therefore, $y=0$ and $u \in U_K$. Thus, $U_L \subseteq U_K$. The converse inclusion is obvious.

LEMMA 3-3. *Let L^* = the Hilbert class field of L and assume that $h_L = 2$. Then there exists an integer $\alpha \in K$ such that*

(i) $L^* = K(\alpha^{1/2}, \delta^{1/2})$,

(ii) $d_{K(\alpha^{1/2})/K} = \alpha \mathfrak{O}_K$.

Proof. Since $\deg (L^*/L) = 2$, there exist $\alpha, \beta \in K$ such that

$$L^* = L((\alpha+\beta\delta^{1/2})^{1/2}).$$

If $\beta \neq 0$, then $L^{**} = L((\alpha+\beta\delta^{1/2})^{1/2}, (\alpha-\beta\delta^{1/2})^{1/2})$ is an unramified extension of L of degree 4 (since $L((\alpha+\beta\delta^{1/2})^{1/2}) = L((\alpha-\beta\delta^{1/2})^{1/2})$ implies that $(\alpha+\beta\delta^{1/2})^{1/2} = a(\alpha-\beta\delta^{1/2})^{1/2}+b$ $(a, b \in L)$) which is impossible since L has class number 2. Therefore, $\beta = 0$. Using the same reasoning as led to (1), we may choose α to be an integer in K and alter it by the square of K to ensure that (ii) holds.

The next result requires Furuta's formula [2] for the genus number of an abelian extension; so let us briefly review Furuta's theory. Let M and N be number fields, M/N abelian. Let M^* = the maximal unramified extension of M which is normal over N. Then M^* is called the *genus field* of the extension M/N, and the degree of M^* over M is called the *genus number* of the extension M/N. Furuta has found the following beautiful formula for the genus number:

$$(5) \qquad \deg (M^*/M) = \frac{h_N \prod_{\mathfrak{p}} e'_{\mathfrak{p}}}{\deg (M_0/N)[U_N : U_M^0]},$$

where

h_N = the class number of N,

$\mathfrak{p}$ runs over all N-primes,

M_0 = the maximum abelian extension of N contained in M,

$e'_{\mathfrak{p}}$ = the ramification index of the maximum abelian extension of $N_{\mathfrak{p}}$ contained in $M_{\wp}$, where $\wp$ is any M-prime dividing $\mathfrak{p}$, and $N_{\mathfrak{p}}$ and $M_{\wp}$ are the respective completions of N and M at $\mathfrak{p}$ and $\wp$,

U_N = the group of units of $\mathfrak{O}_N$,

U_M^0 = the group of units of $\mathfrak{O}_N$ which are locally everywhere norms of units in M.

PROPOSITION 3-4. *Assume that* $|N(\mu)| > 4^n$.

(a) *If* $h_L = 1$, *then* $d_{L/K}$ *is divisible by exactly one prime.*

(b) *If* $h_L = 2$, *then* $d_{L/K}$ *is divisible by exactly two primes.*

Proof. As above, let L^* denote the Hilbert class field of L. In case (a), $L^* = L$, so that L^* is abelian over K. In case (b), Lemma 3-3 asserts that L^* is abelian over K. In either case, the genus field of the extension L/K is L^* and the genus number is h_L. Let us use (5) to compute the genus number in a different way. We set $M = L$, $N = K$, $M_0 = L$. Suppose that $d_{L/K}$ is divisible by r distinct finite K-primes. Since K is totally real and L is totally imaginary, every infinite K-prime ramifies in L. Therefore,

$$\prod_{\mathfrak{p}} e'_{\mathfrak{p}} = 2^{r+n}.$$

Moreover, by Lemma 3-1, and our restrictive assumption in case $h_L = 2$, we have $h_K = 1$. Further, by Hasse's theorem, an element of $K^{\times}$ which is everywhere a local norm from L is a global norm from L. Therefore, by Lemma 3-2,

$$U_L^0 = N_{L/K} U_K \supseteq U_K^2.$$

But by Dirichlet's theorem, $U_K \approx \{\pm 1\} \times \boldsymbol{Z}^{n-1}$, so that $[U_K : U_L^0] \geqq 2^n$. Assembling all the data into Furuta's formula, we get $h_L \geqq 2^{r-1}$, from which (a) and (b) follow immediately, since $h_L = 2$ implies that K is a real quadratic field with fundamental unit of norm -1.

In the remainder of this section, we will study the case $h_L = 2$ more thoroughly. We found that in this case $L^* = L(\alpha^{1/2}, \delta^{1/2})$. Let $K' = K(\alpha^{1/2})$. Then by the transitivity formula for the discriminant, we see that

$$d_{L^*/K} = d_{L/K}^2 N_{L/K} d_{L^*/L} = \delta^2 \mathfrak{O}_K$$

since L^*/L is unramified. On the other hand,

$$d_{L^*/K} = d_{K'/K}^2 \cdot N_{K'/K} d_{L^*/K'} = \alpha^2 N_{K'/K} d_{L^*/K'}.$$

Therefore, $\alpha | \delta$. Let $\beta = \delta/\alpha$. Then $\beta \in \mathfrak{O}_K$ and $L^* = K(\alpha^{1/2}, \beta^{1/2})$, $\alpha\beta = \delta$, $d_{L/K} = \delta \mathfrak{O}_K$. Set $L' = K(\alpha^{1/2})$ and $L'' = K(\beta^{1/2})$. We have already normalized α so that $d_{L'/K} = \alpha \mathfrak{O}_K$. We claim that α and β are each divisible by one prime. This will follow from

THEOREM 3-5. *One of the fields* L', L'' *is totally real and the other is totally imaginary.*

Proof. Let J_K denote the idele group of K. If F/K is an abelian extension, let $H(F)$ denote the admissible subgroup of J_K corresponding to F. Let $\mathfrak{p}_{\infty,0}, \ldots, \mathfrak{p}_{\infty,r}$ denote the infinite primes of K, and let

$$\delta_{\infty,i} = (\alpha_{\mathfrak{p},i}) \in J_K \qquad (0 \leqq i \leqq r)$$

be the idele defined by

$$\begin{aligned} \alpha_{\mathfrak{p},i} &= 1 \qquad (\mathfrak{p} \neq \mathfrak{p}_{\infty,i}) \\ &= -1 \qquad (\mathfrak{p} = \mathfrak{p}_{\infty,i}). \end{aligned}$$

Since L is a quadratic extension of K, $[J_K : H(L)] = 2$. Moreover, since K is totally real and L is totally imaginary, $\delta_{\infty,0} \notin H(L)$. Therefore, we have the coset decomposition

$$J_K = H(L) \cup \delta_{\infty,0} H(L). \tag{6}$$

Let S denote the set of all K-primes $\mathfrak{p}$ such that $f_{\mathfrak{p}}(L/K) = 1, f_{\mathfrak{p}}(L^*/K) = 2$, where $f_{\mathfrak{p}}$ denotes the residue class degree. Since $L^* = K(\alpha^{1/2}, \beta^{1/2})$, L^*/K is an abelian extension with the Klein 4-group as Galois group. Therefore, by Tchebotarev's density theorem, S has Dirichlet density $\frac{1}{4}$ and, in particular, is infinite. Therefore, let us choose $\mathfrak{p} \in S$ such that (i) $\mathfrak{p}$ is finite and (ii) $\mathfrak{p}$ does not ramify in L^*. Let π be a local uniformizing parameter at $\mathfrak{p}$. Then π is a local norm from $L_{\mathfrak{p}}$, but is not a local norm from $L^*_{\mathfrak{p}}$. Therefore, the idele

$$b = (1, \ldots, 1, \underset{\substack{\uparrow \\ \mathfrak{p}}}{\pi}, 1, \ldots, 1) \in J_K$$

is in $H(L) - H(L^*)$. However, by class field theory, $H(L^*) \subseteq H(L)$ and $[H(L) : H(L^*)] = 2$. Therefore,

$$H(L) = H(L^*) \cup bH(L^*).$$

Thus, by (6), we see that

$$J_K = H(L^*) \cup bH(L^*) \cup \delta_{\infty,0} H(L^*) \cup \delta_{\infty,0} bH(L^*). \tag{7}$$

Furthermore, since L^*/K is abelian and L^* is the Hilbert class field of L, L^* is the maximal unramified extension of L which is abelian over K. Thus, L^* is the genus field of L over K. Therefore, by [2, Proposition 1], we know that

$$H(L^*) = K^* N_{L/K} U_L, \tag{8}$$

where $N_{L/K}$ denote the idele norm from L to K and U_L is the group of unit ideles of L. By class field theory, we have

$$H(L') = H(L^*) \cup \delta_{\infty,0} H(L^*), \tag{9}$$

$$H(L'') = H(L^*) \cup \delta_{\infty,0} bH(L^*). \tag{10}$$

Let us show that L' is totally real and that L'' is totally imaginary. It clearly suffices to prove the former since $\alpha\beta$ is totally negative. But

$$(11) \qquad L' \text{ is totally real} \Leftrightarrow \delta_{\infty,i} \in H(L') \qquad (1 \leqq i \leqq r).$$

Let us assume that L' is not totally real. Then for some i $(1 \leqq i \leqq r)$, we have $\delta_{\infty,i} \notin H(L')$. By (7) and (9), we must have either (a) $\delta_{\infty,i} \in bH(L^*)$ or (b) $\delta_{\infty,i} \in \delta_{\infty,0}bH(L^*)$. On account of (8), there exist $k \in K^\times$, $u \in N_{L/K}U_L$, such that either

(a) $\delta_{\infty,i}bu = k$ or

(b) $\delta_{\infty,i}\delta_{\infty,0}bu = k$.

Since L^*/L is unramified, $N_{L/K}U_L = N_{L^*/K}U_{L^*} \subseteq N_{L'/K}U_{L'}$. Therefore, $u \in N_{L'/K}U_L$. Let $[\ ,L'/K]\colon J_K \to \{\pm 1\}$ denote the global norm residue symbol for L'/K. Since $H(L')$ is the kernel of $[\ ,L'/K]$ and since $\delta_{\infty,0} \in H(L')$, $u \in H(L')$, $k \in H(L')$, we see that $[b, L'/K] = -1 \Rightarrow \mathfrak{p}$ is inert in $L'/K \Rightarrow \mathfrak{p}$ is decomposed in L''/K (since $f_{\mathfrak{p}}(L/K) = 1$, $f_{\mathfrak{p}}(L^*/K) = 2$). Thus, with finitely many exceptions, every prime of S is decomposed in L''/K. Therefore, with finitely many exceptions, every K-prime $\mathfrak{p}$ which decomposes in L decomposes in L'. Therefore, by Bauer's theorem[2] $L \subseteq L'' \Rightarrow L = L''$, which gives a contradiction, since then L'' is totally imaginary and L' is totally real.

Let us assume that α is totally positive and β is totally negative. Recall that we normalized α so that $d_{L'/K} = \alpha\mathfrak{O}_K$. Claim that

$$(12) \qquad d_{L''/K} = \beta\mathfrak{O}_K.$$

We have already shown that

$$(13) \qquad d_{L'L''/K} = d_{L^*/K} = \delta^2\mathfrak{O}_K.$$

But by Theorem 3-5, the fields L' and L'' are linearly disjoint over K. Therefore,

$$\mathfrak{D}_{L'L''/K} = \mathfrak{D}_{L'/K} \cdot \mathfrak{D}_{L''/K},$$

where $\mathfrak{D}_{M/N}$ denotes the difference of the extension M/N. Thus, $d_{L'L''/K} = d^2_{L'/K} \cdot d^2_{L''/K}$, and therefore $d_{L''/K} = \beta\mathfrak{O}_K$.

COROLLARY 3-6. *Assume that $h_L = 2$ and $|N(\mu)| > 4^n$. Then α and β are each divisible by exactly one K-prime.*

Proof. Since K has class number 1, α and β are both divisible by at least one K-prime, since otherwise L' or L'' would be an unramified abelian extension of K. However, $\alpha\beta = \delta$ and, by Proposition 3-4(b), δ is divisible by exactly 2 K-primes. Therefore, it suffices to show that α and β cannot both be divisible by the same K-prime $\mathfrak{p}$. But if $\mathfrak{p} | \alpha$ and $\mathfrak{p} | \beta$, then $\mathfrak{p}$ is totally ramified in $L'L'' = L^*$. But since L^*/L

[2] See M. Bauer, *Zur Theorie der Algebraischen Zahlkörper*, Math. Ann. **77** (1916), or H. Hasse, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*, II, Jber. Deutsch. Math. Verein. **35** (1926), §25.

is unramified, no K-prime can be totally ramified in L^*. Thus, a contradiction is reached and the corollary is established.

4. **Proof of Theorem 2 in Case $h_L=1$.** We assume throughout this section that $L=K(\delta^{1/2})$ is of class number 1 and normal over $\boldsymbol{Q}$. If $x \in K^\times$, we will denote by (x) the principal K-ideal generated by x. Let $\varepsilon_1, \ldots, \varepsilon_r$ $(r=n-1)$ be a basis for the free part of the group of K-units.

Reduction 1. We may assume that $|(N(\mu)|>4^n$.

For there are only finitely many integral K-ideals $\mathfrak{A}$ such that $N\mathfrak{A} \leqq 4^n$. Since K has class number 1, every such ideal is principal. Let η run through some fixed set of generators for these ideals. Then, if $|N(\delta)| \leqq 4^n$, L is one of the finite collection of fields $K((\varepsilon\eta)^{1/2})$, where

$$\varepsilon = \prod_{j=1}^{t} \varepsilon_{i_j} \qquad (1 \leqq i_1 < i_i < \cdots < i_t \leqq r). \tag{14}$$

These fields may be individually tested to determine which has class number 1 (using the Minkowski bound, say). Thus, we may restrict ourselves to L such that $|N(\delta)|>4^n$.

By Proposition 3-4(a) and Reduction 1, we know that δ is divisible by exactly one K-prime $\mathfrak{p}$.

Reduction 2. We may assume that $\mathfrak{p}$ is not a divisor of 2.

For if $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the K-primes dividing 2 and if π_i is a generator of $\mathfrak{p}_i$, then the fields L for which δ is divisible by some $\mathfrak{p}_i$ are of the form $K((\varepsilon\pi_i)^{1/2})$ or $K(\varepsilon^{1/2})$, where ε is given by (14). Once again these fields may be individually studied.

Reduction 3. We may assume that $\mathfrak{p} \nmid d_K$.

For if $\mathfrak{p} | d_K$ and if the primes dividing d_K are $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, then L is of the form $K((\varepsilon\lambda_i)^{1/2})$, where λ_i is a generator of $\mathfrak{q}_i$ and ε is of the form (14).

Henceforth, we will assume that Reductions 1–3 have been carried out. By Reduction 2 and equation (1), $\delta=\mu$ is irreducible. Let p be the rational prime which $\mathfrak{p}$ divides. Let us renumber the conjugates $\delta^{(i)}$ so that the distinct K-primes dividing p are $(\delta^{(1)}), \ldots, (\delta^{(r)})$. Then, by Reduction 3, $(p)=(\delta^{(1)})\cdots(\delta^{(r)})$. Thus, $\delta^{(1)}\cdots\delta^{(r)}=\zeta p$ for some K-unit ζ. If r is even, then since δ is totally negative, we have ζp is totally positive. Since L is normal over $\boldsymbol{Q}$, $(\delta^{(i)})^{1/2} \in L$ $(1 \leqq i \leqq r) \Rightarrow (\zeta p)^{1/2} \in L \Rightarrow (\zeta p)^{1/2} \in K$ since ζp is totally positive. But then, p ramifies in K, which contradicts Reduction 3. Thus r is odd and ζ is totally negative. Assume that $\zeta \neq -x^2$ for some $x \in K$. Then, if $p \equiv 1 \pmod 4$, $L=K((\zeta p)^{1/2})$ and $L(p^{1/2})=K(\zeta^{1/2}, p^{1/2})$ is an unramified extension of L of degree 2. But this contradicts the fact that L has class number 1. On the other hand, if $p \equiv 3 \pmod 4$, then $L=K((\zeta p)^{1/2})$ and $L((-p)^{1/2})=K((-\zeta)^{1/2}, (-p)^{1/2})$ is an unramified quadratic extension of L, which is again a contradiction to the fact that L has class number 1. Thus, $\zeta=-x^2$ for some $x \in K$ and $(-p)^{1/2} \in L \Rightarrow L = K((-p)^{1/2})$.

Claim. $\boldsymbol{Q}((-p)^{1/2})$ has class number 1.

Let H^* denote the Hilbert class field of $\mathbf{Q}((-p)^{1/2})$. Then $K\cdot H^*$ is an unramified, abelian extension of $K((-p)^{1/2})$. However, since $K((-p)^{1/2})=L$ has class number 1, we see that

$$K\cdot H^* = K((-p)^{1/2}) \Rightarrow H^* \subseteq K((-p)^{1/2}). \tag{15}$$

Moreover, it is clear that

$$(-p)^{1/2} \in H^*. \tag{16}$$

Let H_0^* denote the maximal real subfield of H^*. Then $H^*=H_0^*\mathbf{Q}((-p)^{1/2})$, $H_0^*\subseteq K$, by (15) and (16). Assume that $\deg(H_0^*/\mathbf{Q})>1$. Then there exists a $\mathbf{Q}$-prime $q\neq p$ such that q ramifies in H_0^*. For by Minkowski's theorem, there exists a $\mathbf{Q}$-prime q which ramifies in H_0^* and $q\neq p$ by Reduction 3. Moreover, if 2 does not ramify in $\mathbf{Q}((-p)^{1/2})/\mathbf{Q}$, we see that q ramifies in H_0^*, but does not ramify in $\mathbf{Q}((-p)^{1/2})/\mathbf{Q}$. But this is a contradiction to the fact that $H^*/\mathbf{Q}((-p)^{1/2})$ is unramified. Thus, $\deg(H_0^*/\mathbf{Q})=1 \Rightarrow H_0^*=\mathbf{Q} \Rightarrow H^*=\mathbf{Q}((-p)^{1/2}) \Rightarrow \mathbf{Q}((-p)^{1/2})$ has class number 1. If 2 ramifies in $\mathbf{Q}((-p)^{1/2})/\mathbf{Q}$, then $p\equiv 1 \pmod 4$ and $p^{1/2}\in H_0^*$. Thus p ramifies in K, contrary to Reduction 3.

From the claim and Stark's theorem, we know that the only possibilities for p are $p=3, 7, 11, 19, 43, 67, 163$. Thus, except for the fields set aside in making Reductions 1–3, the only possibilities for L are

$$L = K((-3)^{1/2}), K((-7)^{1/2}), K((-11)^{1/2}), K((-19)^{1/2}), K((-43)^{1/2}), \\ K((-67)^{1/2}), K((-163)^{1/2}).$$

This completes the proof of Theorem 2 in case $h_L=1$.

5. **Proof of Theorem 2 in Case $h_L=2$.** Throughout this section, we will assume that K is a real quadratic field of discriminant d, whose fundamental unit ε has norm -1. Further, we will assume that L is a totally imaginary quadratic extension of K of class number 2 such that $L/\mathbf{Q}$ is normal.

From the results of §3, we know that $L^*=K((\alpha^{1/2}, \beta^{1/2})$, where $\alpha, \beta\in K$, and α is totally negative and β totally positive. Moreover, since $L/\mathbf{Q}$ is normal and L has class number 2, L^* is normal over $\mathbf{Q}$. As in §4, it suffices to consider the case where $|N(\delta)|>4^n$. We will assume throughout that we are in this case. Then, from the results of §3, we know that α and β are both divisible by exactly one K-prime. Suppose that α is divisible by $\mathfrak{p}$ and β by $\mathfrak{q}$. Let p and q be, respectively, the $\mathbf{Q}$-primes dividing $\mathfrak{p}$ and $\mathfrak{q}$. Also note that, since K has class number 1, $d=8$ or d is an odd prime discriminant (that is, $d=8$ or d is a prime and $d\equiv 1 \pmod 4$)). Note that the situation we are considering actually occurs. In fact, there are 11 values of d less than 100 for which our assumptions are satisfied:

$$d = 5, 8, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97.$$

Let us consider four separate cases corresponding to p and q both odd; $p=2$ and q odd; p odd and $q=2$; $p=q=2$. First, however, let us collect some observations which are common to the four cases.

Let $L_0^* = \boldsymbol{Q}(d^{1/2}, \beta^{1/2})$ denote the maximal real subfield of L^*.

A. *There exist no fields L for which p = the unique prime dividing d.*

For suppose that p = the unique prime dividing d. If d is odd, then p is odd and α is an irreducible element of $\mathfrak{O}_K$. Therefore, $\alpha = -\varepsilon^a d^{1/2}$ for some $a \in \boldsymbol{Z}$. But since α is totally negative, and $N(\varepsilon) = -1$, we see that a is even and, without loss of generality, we may assume that $\alpha = (-\sqrt{d})^{1/2}$. Since $L^*/\boldsymbol{Q}$ is normal, $d^{1/4}, (-1)^{1/2} \in L^* \Rightarrow L^* = \boldsymbol{Q}(d^{1/4}, (-1)^{1/2})$ since $\deg(L^*/\boldsymbol{Q}) = 8$. Therefore,

$$L = K((-4\sqrt{d})^{1/2}).$$

But for this field, $p = 2$. Thus, we contradict the fact that p is odd. If $d = 8$, then $p = 2$ and $\alpha = \pm \varepsilon^a (2^{1/2})^b$ $(a, b \in \boldsymbol{Z})$. Since α is totally negative and $N(\varepsilon) = -1$, we may assume that $\alpha = -(2^{1/2})^b$. Since $L^*/\boldsymbol{Q}$ is normal, $((\sqrt{2})^b)^{1/2} \in L^*$ and thus $L^* = \boldsymbol{Q}(2^{1/2}, (-1)^{1/2}, ((\sqrt{2})^b)^{1/2}, \beta^{1/2})$. If b is odd, then

$$L^* = \boldsymbol{Q}(2^{1/4}, (-1)^{1/2}, \beta^{1/2}) = \boldsymbol{Q}(2^{1/4}, (-1)^{1/2})$$

since $\deg(L^*, \boldsymbol{Q}) = 8$. However, only one K-prime then ramifies in L^*, which contradicts Proposition 3-4(b). Thus b is even and $L^* = \boldsymbol{Q}(2^{1/2}, (-1)^{1/2}, \beta^{1/2})$. However, since $\deg(L^*/\boldsymbol{Q}) = 8$, $\beta \in \boldsymbol{Q}$ and thus since β is totally positive,

$$L^* = \boldsymbol{Q}(2^{1/2}, (-1)^{1/2}, q^{1/2}).$$

Therefore, $L = K((-q)^{1/2}) = K((-8q)^{1/2})$.

B. *$p = q = 2$ occurs for only a finite number of fields and these can be effectively determined.*

This is clear.

C. *There are no fields L for which $p = q$ and both are odd.*

For if $p = q$, then either $\beta = -\varepsilon^a \alpha$ or $= -\varepsilon^a \alpha'$, where α' denotes the conjugate of α over K. Since β is totally positive, and $N(\varepsilon) = -1$, we see that a is even. And since $L^*/\boldsymbol{Q}$ is normal, $(\pm\alpha)^{1/2}, (\pm\alpha')^{1/2} \in L^* \Rightarrow (-1)^{1/2} \in L^* \Rightarrow 2$ ramifies in L^*. This contradicts the fact that d, p, q are all odd.

D. *There exists at most one field L for which $q = d$, namely*

$$L^* = \boldsymbol{Q}(d^{1/4}, (-1)^{1/2}), \qquad L = K((-4\sqrt{d})^{1/2}).$$

This follows from the proof of A.

Case I. *p, q odd.* In this case, both α and β are irreducible. Upon examining the factorization of p in K, we see that $(p) = (\alpha)^{e_1}(\alpha')^{e_2}$, for some integers $e_i = 0, 1, 2$, where (x) denotes the K-ideal generated by x. Since $L^*/\boldsymbol{Q}$ is normal, $(\alpha')^{1/2} \in L^*$ $\Rightarrow$ there exists a K-unit ζ of constant signature such that $(\zeta p)^{1/2} \in L^*$. If ζ is totally positive, p ramifies in L_0^*, so that $p = d$ or $p = q$. This is a contradiction by A and C. Thus ζ is totally negative and of the form $-\varepsilon^a$, a even. Thus, $(-p)^{1/2} \in L^*$. Applying to q the same reasoning as we applied to p, we see that $(\eta q)^{1/2} \in L^*$ for some K-unit η of constant signature. If η is totally negative, then

$$\begin{aligned}(-\eta pq)^{1/2} \in L^* &\Rightarrow p \text{ ramifies in } L_0^* \quad (\text{since } -\eta pq \text{ is totally positive})\\ &\Rightarrow p = d \quad \text{or} \quad p = q,\end{aligned}$$

which is a contradiction by A and C. Thus, η is totally positive. But then since $N(\varepsilon)=-1$, we have $\eta=\varepsilon^a$ for a even which implies $q^{1/2}\in L^*$. Thus, we have shown that

$$\boldsymbol{Q}(d^{1/2},(-p)^{1/2},q^{1/2})\subseteq L^*,\qquad \boldsymbol{Q}(d^{1/2},(-p)^{1/2},q^{1/2}) = L^*,$$

since $\deg(L^*/\boldsymbol{Q})=8$ and d,p,q are distinct by A–D. Finally, we have $L=K((-pq)^{1/2})$. Moreover, since $\mathfrak{p}$ and $\mathfrak{q}$ are the only K-primes which ramify in L, we see that p and q are inert in $\boldsymbol{Q}(d^{1/2})$ and $-p$ and q are prime discriminants.

Case II. *p odd*, $q=2$. As in Case I, we can prove that $(-p)^{1/2}\in L^*$. The difficulty in this case is that β may no longer be irreducible. Let π be a K-integer such that $\mathfrak{q}=(\pi)$. Then $\beta=\varepsilon^a\pi^b$ for some $a,b\in\boldsymbol{Z}$. If b is even, then a is even since β is totally positive and $N(\varepsilon)=-1$. But then $\beta^{1/2}\in K$, which is a contradiction to the fact that $\deg(L^*/K)=4$, $L^*=K(\alpha^{1/2},\beta^{1/2})$. Thus, b is odd and $(\varepsilon^a\pi)^{1/2}\in L^*$. If 2 is inert in K, we may choose $\pi=2$. Then, since $N(\varepsilon)=-1$ and $\varepsilon^a\pi$ is totally positive, we see that a is even and $2^{1/2}\in L^*$. If 2 decomposes in K, $(\varepsilon^a\pi)\cdot(\varepsilon^a\pi)'=2$, and thus since $L^*/\boldsymbol{Q}$ is normal, $2^{1/2}\in L^*$. Finally, reasoning as in Case I, we see that

$$L^* = \boldsymbol{Q}(d^{1/2},(-p)^{1/2},8^{1/2}),\qquad L = K((-8p)^{1/2}),$$

where $-p$ is an odd prime discriminant.

Case III. $p=2$, *q odd.* As in Case I, $q^{1/2}\in L^*$. Let $(\pi)=\mathfrak{p}$, $\alpha=-\varepsilon^a\pi^b$ where $a,b\in\boldsymbol{Z}$. If b is even, then a is even since $N(\varepsilon)=-1$ and $(-1)^{1/2}\in L^*$. On the other hand, if b is odd, then by reasoning as in Case II, we see that $(-2)^{1/2}\in L^*$. Thus, by reasoning as in Case I, we get

$$L^* = \boldsymbol{Q}(d^{1/2},(-4)^{1/2},q^{1/2})\quad\text{or}\quad L^* = \boldsymbol{Q}(d^{1/2},(-8)^{1/2},q^{1/2})$$

and

$$L = K((-4q)^{1/2})\quad\text{or}\quad L = K((-8q)^{1/2}),$$

where q is an odd prime discriminant.

Case IV. $p=q=2$. By B, we may neglect this case from consideration.

We will now prove

THEOREM 5-1. *Let d_1 and d_2 be distinct prime discriminants, neither equal to d, such that $d_1d_2<0$ and $K((d_1d_2)^{1/2})$ has class number* 2. *Then $\boldsymbol{Q}((d_1d_2)^{1/2})$ has class number* 2.

Proof. Let $H=$the Hilbert class field of $\boldsymbol{Q}((d_1d_2)^{1/2})$, $H_0=$the maximal real subfield of H. The only primes which can ramify in H_0 are the primes dividing d_1 and d_2. Since $K((d_1d_2)^{1/2})$ has class number 2, its Hilbert class field is $K((d_1)^{1/2},(d_2)^{1/2})$. Since $H\cdot K$ is an unramified abelian extension of $K((d_1d_2)^{1/2})$, we see that $H\cdot K\subseteq K((d_1)^{1/2},(d_2)^{1/2})$. Let $d_1<0$, $d_2>0$. Then $(d_1)^{1/2}\in H$ and $H=H_0\cdot\boldsymbol{Q}((d_1)^{1/2})$, $H_0\subseteq K((d_2)^{1/2})$. Moreover, d ramifies in K, but since $d\neq d_i$ $(i=1,2)$, d does not ramify in H_0. Therefore, $H_0 \neq K((d_2)^{1/2})$, so that

$$\begin{aligned} H_0 = Q((d_2)^{1/2}) \Rightarrow H &= Q((d_1)^{1/2}, (d_2)^{1/2}) \\ &\Rightarrow \deg (H/Q((d_1 d_2)^{1/2})) = 2 \\ &\Rightarrow Q((d_1 d_2)^{1/2}) \text{ has class number 2.} \end{aligned}$$

By Cases I–IV and Theorem 5-1, it suffices to effectively determine all imaginary quadratic fields of class number 2. But this is possible by the theorem of Baker and Stark. This completes the proof of Theorem 2 in case $h_L = 2$.

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