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A METHOD FOR SHRINKING DECOMPOSITIONS OF CERTAIN MANIFOLDS

BY
ROBERT D. EDWARDS(1) AND LESLIE C. GLASER(2),(3)

Abstract. A general problem in the theory of decompositions of topological manifolds is to find sufficient conditions for the associated decomposition space to be a manifold. In this paper we examine a certain class of decompositions and show that the nondegenerate elements in any one of these decompositions can be shrunk to points via a pseudo-isotopy. It follows then that the decomposition space is a manifold homeomorphic to the original one. As corollaries we obtain some results about suspensions of homotopy cells and spheres, including a new proof that the double suspension of a Poincaré 3-sphere is a real topological 5-sphere.

1. **Introduction.** In this paper we examine a special case of the following question: if $\mathscr{G}$ is a decomposition of a space X, where $X \times E^k$ is a manifold for some k, then is the decomposition space $X/\mathscr{G}$ crossed with E^k a manifold, and if so, is it homeomorphic to $X \times E^k$? This question assumes special significance when one knows that X is a manifold, but $X/\mathscr{G}$ is not. The first important result in this context was Bing's paper [2] in which he showed that his dogbone space (which is noneuclidean) crossed with E^1 is E^4; indeed, some of the techniques introduced there are used in this paper.

We are interested in the special case given by the following.

Hypothesis. (X^n, S, E^k): X is a compact metric space with closed subset S such that (i) X is contractible, (ii) S is collared in X and is simply connected, and (iii) $(X-S) \times E^k$ is an open $(n+k)$-manifold.

For example, X may be a fake n-cell (for $n=3$ or 4), that is, a contractible combinatorial n-mainfold with $(n-1)$-sphere boundary S. Given the *Hypothesis*, we present an elementary proof, using a version of radial engulfing, that if $n+k \geqq 5$, then for any topological k-manifold M^k without boundary, $X \times M$ is homeomorphic to $(v * S) \times M$ by a homeomorphism which is bounded as small as desired in the M coordinate and is the identity on $S \times M$. A pleasant corollary is a simple proof of the known fact that the double suspension of a fake 3-cell and the single

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suspension of a fake 4-cell are each topologically homeomorphic to I^5 (see below and Corollary 7).

If in the *Hypothesis* S is assumed to be an $(n-1)$-sphere, then the results of this paper follow from work of Siebenmann [21]. However, the generality of the *Hypothesis* is necessitated in part by the requirements of the authors of [11], where X is merely assumed to be a contractible polyhedron.

Concerning fake cells and the fact that their suspensions are real topological cells, the original result in this direction appeared in [8]. There it was shown that the double suspension of a homotopy 3-sphere which bounds a contractible 4-manifold is topologically S^5. The first proof of this fact without the assumption about bounding was given by Siebenmann in [20], and since then several other proofs of varying degrees of simplicity have been discovered. For example, Glaser [9], following a suggestion of Kirby, gave a proof using the topological h-cobordism theorem and the local contractibility of the homeomorphism group of a manifold. Kirby and Siebenmann [13] announced a proof using the topological h-cobordism theorem and an infinite meshing technique of Černavskiĭ. Glaser [10] generalized his original theorem mentioned above to get the full result. Mindful of possible unnecessary repetition, we present our proof in the belief that it represents a useful and elementary alternative.

It should be noted that in [18], Rosen considers a situation resembling that of the *Hypothesis*. Namely, he assumes $k=1$ and S is a suspension n-sphere (that is, $\Sigma S \approx S^n$), and using an elementary decomposition argument, he shows that $\Sigma X \approx I^{n+1}$.

2. **Preliminary results.** If Y is a metric space with metric d and $\varepsilon > 0$, then we denote the open ε-neighborhood of a subset D of Y by $N_\varepsilon(D)$. The metric on a product $X \times Y$ of metric spaces will always be taken to be the usual cartesian product metric $d_{X\times Y} = (d_X^2 + d_Y^2)^{1/2}$. The following theorem is the cornerstone of the paper.

THEOREM 1. *Suppose we have Hypothesis* (X^n, S, E^k) *and* $g\colon S\times[0,1) \to X$ *is a collar for* S *in* X. *If* $n+k \geqq 5$, *then given any* $\varepsilon > 0$ *and* $A \subset E^k$, *A compact, there exists an isotopy* f_t, $t \in [0,1]$, *of* $X \times E^k$ *onto itself such that*

(1) $f_0 = \mathit{identity}$,

(2) $f_t = \mathit{identity}$ *on* $X\times E^k - N_\varepsilon(D\times A)$ *for each* t, *where* $D = X - g(S\times[0,1))$,

(3) *if* $w \in E^k$, *then* $f_t(X\times w) \subset X \times N_\varepsilon(w)$ *for each* t, *and*

(4) *if* $w \in A$, *then* $\operatorname{diam} f_1(D\times w) < \varepsilon$.

Before proving the theorem, we describe a refined version of an engulfing theorem that was presented by Bing in [4] and subsequently sharpened to handle the codimension 3 case by Wright [24]. Let M^n be a piecewise linear manifold without boundary, and let $V \subset U$ be two open subsets of M and $\{X_\alpha\}$ a collection of subsets of M. Then (generalizing Bing's definition slightly) the statement that *finite r-complexes in M can be pulled into U rel V along* $\{X_\alpha\}$ means the following: suppose P is a closed polyhedron in M with closed subpolyhedron Q in V, and suppose that $R=\mathrm{cl}\,(P-Q)$ is compact. Then, if $\dim R \leqq r$, there is a homotopy $H_t: P \to M$, $t \in [0, 1]$, such that H_0=identity, H_t=identity on Q, $H_1(P) \subset U$ and for each point $x \in P$, the path $H_t(x)$, $t \in [0, 1]$, lies in some element of $\{X_\alpha\}$.

ENGULFING LEMMA (CF. [4] AND [24] WHICH THIS GENERALIZES SLIGHTLY). *Suppose* M^n *is a piecewise linear manifold without boundary and suppose* $U_0 \supset U_1 \supset \cdots \supset U_{r+1}$ *is a collection of open subsets of* M *and* $\{X_\alpha\}$ *is a collection of subsets of* M *such that for each* i, $0 \leqq i < r+1$, *finite i-complexes in* M *can be pulled into* U_i *rel* U_{i+1} *along* $\{X_\alpha\}$. *Suppose* P^p *is a closed polyhedron in* M *with closed subpolyhedron* Q *in* U_{r+1} *such that* $R=\mathrm{cl}\,(P-Q)$ *is compact. Suppose* $p \leqq n-3$ *and* $\dim R \leqq r$. *Then, for each* $\varepsilon > 0$, *there is an engulfing isotopy* $h_t: M \to M$, $t \in [0, 1]$, *such that* h_0=*identity*, h_t=*identity on* $Q \cup (M-C)$, *where* C *is some compact subset of* M, $h_1(U_0) \supset P$ *and for each* $x \in M$ *there are* $r+1$ *elements of* $\{X_\alpha\}$ $[r+2$ *if* $r=n-3]$ *such that the path* $h_t(x)$, $t \in [0, 1]$, *lies in the* ε*-neighborhood of the union of these* $r+1$ [*or* $r+2$] *elements.*

The proof, which proceeds by induction on r, is essentially given in the references already mentioned and requires only trivial modifications to handle our slightly more general hypothesis.

Proof of Theorem 1. The proof uses the above Engulfing Lemma and a dual skeleton argument due to Stallings [22]. First, note that it follows as a corollary to Connell's argument in [6] that $(X-S) \times E^k$ is euclidean $(n+k)$-space, since it is a contractible open topological manifold which has a 1-connected open collar neighborhood of ∞. Alternatively, this also follows from the more general result given in [19].

Given $\lambda > 0$, let $M' = X - g(S \times [0, 1-\lambda])$, $V' = g(S \times (1-\lambda, 1))$ and let $U_0' \supset U_1' \supset \cdots \supset U_{n+k-2}'$ be a collection of open subsets of M' such that $\mathrm{diam}\, U_0' < \varepsilon/2$ and, for each i, $0 \leqq i < n+k-2$, M' can be deformed (that is, homotoped) into U_i' keeping U_{i+1}' fixed (in addition to the contractibility of M', this also uses the fact that M' is an ANR, which follows since M' is a retract of the manifold $M' \times E^k$; see [12]). Let $M = M' \times N_\lambda(A)$, $V = V' \times N_\lambda(A)$ and $U_i = U_i' \times N_\lambda(A)$ for each i. Then $M \subset (X-S) \times E^k$ is a piecewise linear manifold and if λ is sufficiently small, then $M \subset N_\varepsilon(D \times A)$. Let $\{X_\alpha\}$ be the collection of subsets of M given by

$$\{M' \times w \mid w \in N_\lambda(A)\}.$$

Then finite 2-complexes in M can be pulled into V rel V along $\{X_\alpha\}$, since S is 1-connected and hence (M, V) is 2-connected. Also, for each i, $0 \leqq i < n+k-2$, finite $(n+k)$-complexes in M can be pulled into U_i rel U_{i+1} along $\{X_\alpha\}$.

Let P^{n+k} be a closed subpolyhedron of M such that $P-V$ is compact and $P \supset M' \times N_\eta(A)$ for some small $\eta > 0$. Let $\delta > 0$, and let T be a triangulation of M

of mesh $<\delta$ such that P is a subcomplex of T. Let $T_{(2)}$ denote the dual 2-skeleton of T, that is, the subcomplex of the barycentric first derived subdivision T' of T given by $T_{(2)}=\{\hat{\alpha}\cdots\hat{\gamma} \mid \alpha<\cdots<\gamma\in T \text{ and } \dim \alpha \geqq n+k-2\}$.

Let $g_t\colon M\to M$, $t\in[0,1]$, be an engulfing isotopy such that $g_0=\text{identity}$, g_t is the identity off of some compact subset of M, $g_1(V)\supset P\cap T_{(2)}$ and for each $x\in M$, there are 4 (or fewer) elements of $\{X_\alpha\}$ such that the path $g_t(x)$, $t\in[0,1]$, lies in the δ-neighborhood of the union of these elements. If δ is sufficiently small, then $g_t(D\times A)$ remains in $M'\times N_\eta(A)$, and therefore $g_1(D\times A)\cap T_{(2)}=\varnothing$. Let R be a finite subcomplex of T such that $g_1(D\times A)\subset R$. By a second application of the engulfing lemma, there is an engulfing isotopy h_t of M, $t\in[0,1]$, such that $h_0=\text{identity}$, h_t is the identity off of some compact subset of M, $h_1(U_0)\supset R^{(n+k-3)}$ ($=$the $(n+k-3)$-skeleton of R) and for each $x\in M$, there are $n+k-1$ elements of $\{X_\alpha\}$ such that the path $h_t(x)$, $t\in[0,1]$, lies in the δ-neighborhood of the union of these elements. Let $\theta_t\colon M\to M$, $t\in[0,1]$ be an isotopy with compact support such that $\theta_0=\text{identity}$, $\theta_1(h_1(U_0))\supset g_1(D\times A)$ and each θ_t moves points less than δ. To get the desired isotopy of the theorem, let f_t be the isotopy of M defined by taking the isotopy g_t on the first third of the interval $[0,1]$, and then $\theta_t^{-1}g_1$ on the second third of the interval, and finally $h_t^{-1}\theta_1^{-1}g_1$ on the final third. Note that f_t is the identity off a compact subset of M, and so can be extended via the idęntity to an isotopy of $X\times E^k$. It is a trivial matter to verify that if δ is chosen small enough, then the isotopy f_t satisfies the conditions of the theorem, completing the proof.

The following theorem generalizes Theorem 1 in two useful directions, replacing the E^k factor by an arbitrary topological manifold M^k without boundary (where *manifold* means a separable metric space which is locally euclidean) and replacing the compact subset A by a closed subset. The constant ε of Theorem 1 is replaced by a map $\varepsilon\colon M\to(0,\infty)$.

If Y is a metric space and $\varepsilon\colon Y\to(0,\infty)$ is a map, then the ε-neighborhood of a subset D of Y is the set $N_\varepsilon(D)=\bigcup_{x\in D} N_{\varepsilon(x)}(x)$.

THEOREM 2. *Suppose we have Hypothesis* (X^n, S, E^k), *with* $n+k\geqq 5$, *and* $g\colon S\times[0,1)\to X$ *is a collar for* S *in* X, *and suppose that* M^k *is a topological manifold without boundary.*

If A *is a closed subset of* M, *then given any map* $\varepsilon\colon M\to(0,\infty)$, *there exists an isotopy* f_t, $t\in[0,1]$, *of* $X\times M$ *onto itself such that*

(1) $f_0=\textit{identity}$,

(2) $f_t=\textit{identity}$ *on* $X\times M-N_\varepsilon(D\times A)$ *for each* t, *where* $D=X-g(S\times[0,1))$ *and* $\varepsilon(x,w)=\varepsilon(w)$,

(3) *if* $w\in M$, *then* $f_t(X\times w)\subset X\times N_\varepsilon(w)$ *for each* t, *and*

(4) *if* $w\in A$, *then* $\operatorname{diam} f_1(D\times w)<\varepsilon(w)$.

Proof. The proof is broken up into two cases.

Case (1). *A compact*. For this case we may as well assume that ε is constant. Suppose $A \subset \bigcup_{i=1}^{s} h_i(E^k)$, where $h_i\colon E^k \to M$, $1 \leqq i \leqq s$, are coordinate homeomorphisms of M, and suppose the theorem is true for compact subsets of M which lie in the union of $s-1$ or fewer coordinate neighborhoods. Express A as a union of two compact subsets, $A = A_1 \cup A_2$, such that $A_1 \subset \bigcup_{i=1}^{s-1} h_i(E^k)$ and $A_2 \subset h_s(E^k)$. By the induction hypothesis, there is an isotopy g_t of $X \times M$ such that conditions (1) through (4) of the theorem hold with A replaced by A_1 and ε by $\varepsilon/3$. Now g_t has compact support (assuming ε is sufficiently small) and is therefore uniformly continuous, so there is a δ, $0 < \delta < \varepsilon/3$, such that

(*) If B is a subset of $X \times M$ with diam $B < 2\delta$, then diam $g_t(B) < \varepsilon/3$ for each t.

This implies that g_t satisfies the following conditions:

(3′) If $w \in M$, then $g_t(X \times N_\delta(w)) \subset X \times N_\varepsilon(w)$ for each t, and

(4′) if $w \in A_1$, then diam $g_1(N_{2\delta}(D \times w)) < \varepsilon$.

By Theorem 1, there is an isotopy h_t, $t \in [0, 1]$, of $X \times M$ such that conditions (1) through (4) hold with f_t replaced by h_t, A replaced by A_2 and ε by δ. Define an isotopy f_t, $t \in [0, 1]$, of $X \times M$ by

$$\begin{aligned} f_t &= h_{2t}, & t &\in [0, \tfrac{1}{2}], \\ &= g_{2t-1}h_1, & t &\in [\tfrac{1}{2}, 1]. \end{aligned}$$

Case (2). *The General Case*. Since M is a locally compact separable metric space, there are two collections $\mathscr{C}$ and $\mathscr{D}$ of compact subsets of M such that $M = \bigcup(\mathscr{C} \cup \mathscr{D})$, and each collection is countable and discrete, where *discrete* means that each point of M has a neighborhood which intersects at most one member of the collection (for a short proof of this fact, see the paragraph on pp. 165, 166 of [17]). Let $C_1, C_2, \ldots$ and $D_1, D_2, \ldots$ be the intersections of the members of these collections with A. Then by infinitely many simultaneous applications of Case (1), the theorem holds for the closed set $A_1 = \bigcup_{i=1}^{\infty} C_i$, and likewise for $A_2 = \bigcup_{i=1}^{\infty} D_i$. From this point on the argument of Case (1) applies, since A can be written as the union of two closed subsets, $A = A_1 \cup A_2$, each of which satisfies the theorem. The ε and δ are now assumed to be maps, and (*) should be replaced by

(*′) If B is a subset of $X \times N_{2\delta}(w)$ with diam $B < 2\delta(w)$, then diam $g_t(B) < \varepsilon(w)/3$, for each t.

3. **Main results.** A *pseudo-isotopy* of a space Y is a homotopy $h_t\colon Y \to Y$, $t \in [0, 1]$, such that for each $t \in [0, 1)$, h_t is a homeomorphism. A pseudo-isotopy is a desirable means of shrinking an upper semicontinuous decomposition of a manifold, for if the limiting map h_1 is a closed surjection, then the decomposition space is a manifold homeomorphic to the original one (see Corollary 6). In this section, we construct a pseudo-isotopy by generalizing a theorem of Bing [2, Theorem 3] to our situation.

First we establish some notation. Suppose we have *Hypothesis* (X^n, S, E^k) and $g: S\times[0, 1)\to X$ is a fixed collar for S in X. Let $D=X-g(S\times[0, 1))$. If M is a topological manifold and A is a closed subset of M, let $\mathcal{G}(X\times M, D\times A)$ denote the decomposition of $X\times M$ having nondegenerate elements only of the form $D\times w$, $w\in A$; that is,

$$\mathcal{G}(X\times M, D\times A)$$
$$= \{G \subset X\times M \mid G = D\times w, w\in A \text{ or } G = (x, w)\in X\times M-D\times A\}.$$

Let $X\times M/\{D\times w \mid w\in A\}$ denote the decomposition space given by

$$X\times M/\mathcal{G}(X\times M, D\times A).$$

THEOREM 3. *Suppose we have Hypothesis (X^n, S, E^k), with $n+k\geqq 5$, and $g: S\times[0, 1)\to X$ is a collar for S in X, and suppose that M^k is a topological manifold without boundary.*

If A is a closed subset of M, then given any map $\varepsilon: M\to(0, \infty)$, there exists a pseudo-isotopy h_t of $X\times M$ such that

(1) *$h_0=$ identity,*

(2) *$h_t=$ identity on $X\times M-N_\varepsilon(D\times A)$ for each t,*

(3) *if $w\in M$, then $h_t(X\times w)\subset X\times N_\varepsilon(w)$ for each t, and*

(4) *h_1 is a closed map taking $X\times M$ onto itself and each element of $\mathcal{G}(X\times M, D\times A)$ onto a distinct element of $X\times M$.*

COROLLARY 4. *Under the assumptions of Theorem 3, there exists a homeomorphism $f: X\times M\to X\times M/\{D\times w \mid w\in A\}$, with $f=$ identity on $S\times M$, such that*

$$d(w, p_2 f(x, w)) < \varepsilon(w)$$

for each $(x, w)\in X\times M$, where $p_2: X\times M/\{D\times w \mid w\in A\}\to M$ is the natural projection and d is the metric on M.

Theorem 3 easily generalizes to the broader context of upper semicontinuous decomposition theory, and it seems to be more transparent and useful in this setting. Thus we present Theorem 5 and Corollary 6 below, which include Theorem 3 and Corollary 4 as special cases (upon application of Theorem 2 and Remark 1 below). Theorem 5 presents a useful general statement of a result that has been applied many times in more specific situations (e.g. [1], [5], [14], and [25]).

Let M be a metric space and let $\mathcal{G}=\{G_\alpha\}$ be an upper semicontinuous decomposition of M into compact subsets, that is, a collection of disjoint compact subsets of M, whose union is all of M, such that the quotient map $\rho: M\to M/\mathcal{G}$ of M onto the decomposition space is closed. Following McAuley [15] we say that such a $\mathcal{G}$ is *shrinkable* if given any map $\varepsilon: M\to(0, \infty)$ and any saturated open cover $\mathcal{U}$ of M (where *saturated* means that for any $U\in\mathcal{U}$ and $G_\alpha\in\mathcal{G}$, either $G_\alpha\cap U=\varnothing$

or $G_\alpha \subset U$), there is an isotopy $f_t \colon M \to M$, $t \in [0, 1]$, such that $f_0 = \text{identity}$ and, for each $G_\alpha \in \mathscr{G}$,

(i) there is a $U \in \mathscr{U}$ such that $U \supset G_\alpha \cup f_t(G_\alpha)$ for all $t \in [0, 1]$, and

(ii) $\operatorname{diam} f_1(G_\alpha) < \inf \varepsilon(G_\alpha)$.

The significance of this notation is demonstrated by the following result.

THEOREM 5 (CF. [2, THEOREM 3] AND [15, THEOREM 2]). *Suppose that $\mathscr{G}$ is an upper semicontinuous decomposition into compact subsets of the metric space M, and suppose that $\mathscr{G}$ is shrinkable and M is complete. Then given any saturated open cover $\mathscr{U}$ of M, there is a pseudo-isotopy $h_t \colon M \to M$, $t \in [0, 1]$, such that*

(1) *for each $G_\alpha \in \mathscr{G}$, there is a $U \in \mathscr{U}$ such that $U \supset G_\alpha \cup h_t(G_\alpha)$ for all $t \in [0, 1]$, and*

(2) *h_1 is a closed map taking M onto itself and each element of $\mathscr{G}$ onto a distinct element of M.*

COROLLARY 6. *Given the hypotheses of Theorem 5, then the decomposition space $M/\mathscr{G}$ is homeomorphic to M.*

Proof of Corollary. The relation $f = h_1 \rho^{-1}$ gives a well-defined homeomorphism from $M/\mathscr{G}$ to M.

REMARK 1. If in addition in the definition of shrinkable we assume that each isotopy f_t can be chosen to be the identity on some given subset D of M (for example, the complement of the ε-neighborhood of the union of all nondegenerate elements of $\mathscr{G}$), then we can further assume in Theorem 5 that the pseudo-isotopy h_t is also the identity on D for each t. We shall not dwell here on this modification (which of course we need for Theorem 3), inasmuch as it is trivial.

REMARK 2. It is useful to note that the notion of shrinkability is independent of the metric chosen for M. This can be seen by showing that an equivalent definition is the following: given any two open covers $\mathscr{U}$, $\mathscr{V}$ of M, with $\mathscr{U}$ saturated, there is an isotopy $f_t \colon M \to M$, $t \in [0, 1]$, such that for each $G_\alpha \in \mathscr{G}$,

(i) there is a $U \in \mathscr{U}$ such that $U \supset G_\alpha \cup f_t(G_\alpha)$ for all t, and

(ii) there is a $V \in \mathscr{V}$ such that $f_1(G_\alpha) \subset V$.

Thus, in the proof that follows, we are justified in assuming that the given metric d on M is complete, thereby simplifying things considerably.

REMARK 3. Theorem 5 shows that the shrinking criterion is a sufficient condition for $M/\mathscr{G}$ to be homeomorphic to M. It is by no means a necessary condition; see Bing's well-known figure eight example in [3, p. 7]. However, Siebenmann [21], generalizing some results of Armentrout for three dimensions, has shown that if M is a manifold without boundary ($\dim M > 4$) and the elements of $\mathscr{G}$ are cell-like (that is, cellular as subsets of some euclidean space), then if $M/\mathscr{G}$ is locally euclidean, then $M/\mathscr{G} \approx M$ and the decomposition $\mathscr{G}$ is shrinkable.

Question. Is there an explicit expression or formulation for a metric on $M/\mathscr{G}$, given a metric on M? If $\mathscr{G}$ has only one nondegenerate element G, then such a metric is $d^*(x, y) = \min\{d(x, y), d(x, G) + d(G, y)\}$ for $x, y \in \mathcal{G}$, where d is the given metric on M. That $M/\mathcal{G}$ is in fact a metric space is proved in [23, Theorem 1], see also [7, p. 235].

In what follows, we make extensive use of the fact that $M/\mathcal{G}$ is paracompact. For this and other useful properties of paracompact spaces, we refer the reader to [7].

Proof of Theorem 5. Let $\varepsilon_1, \varepsilon_2, \ldots$ be any sequence of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. Inductively we construct a sequence $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ of successively finer saturated open covers of M such that $\{\overline{U} \mid U \in \mathcal{U}_1\}$ refines $\mathcal{U}$ and $\mathcal{U}_i$ refines $\{N_{\varepsilon_i}(G_\alpha) \mid G_\alpha \in \mathcal{G}\}$ for each $i > 1$, and a sequence of isotopies of M,

$$h(x, t),\ t \in [0, \tfrac{1}{2}]; \quad h(x, t),\ t \in [\tfrac{1}{2}, \tfrac{2}{3}]; \ \ldots$$

such that $h(x, 0) = x$, any two adjacent $h(x, t)$'s agree on their common end, and for each $i \geqq 1$,

(a) for each $G_\alpha \in \mathcal{G}$, there is a $U_i \in \mathcal{U}_i$ and a $\lambda > 0$ such that $h(U_i, (i-1)/i) \supset h(N_\lambda(U_{i+1}), t)$ for any $U_{i+1} \in \mathcal{U}_{i+1}$ containing G_α and any $t \in [(i-1)/i, i/(i+1)]$, and

(b) for each $U_{i+1} \in \mathcal{U}_{i+1}$, $\operatorname{diam} h(U_{i+1}, i/(i+1)) < 2\varepsilon_i$.

From these conditions it follows that for each $i \geqq 1$,

(c) each point $h(x, t) \in M$ moves less than $2\varepsilon_{i-1}$ during $t \in [(i-1)/i, i/(i+1)]$, and

(d) for each $G_\alpha \in \mathcal{G}$, there is a $U_i \in \mathcal{U}_i$ such that $h(U_i, (i-1)/i) \supset h(G_\alpha, t)$ for all $t \in [(i-1)/i, 1)$.

Taking the limit of such a sequence of isotopies provides the desired pseudo-isotopy (the only nontrivial verification is that the limit map is surjective).

To start the construction of the h's and the $\mathcal{U}$'s, let $h(x, 0) = \text{identity}$ and let $\mathcal{U}_1$ be any saturated open cover such that $\{\overline{U} \mid U \in \mathcal{U}_1\}$ refines $\mathcal{U}$. In general, for $i \geqq 1$, given $h(x, (i-1)/i)$ and $\mathcal{U}_i$, first choose a map $\delta\colon M \to (0, \infty)$ so small that

(δ) for each $G_\alpha \in \mathcal{G}$, if S is a subset of $N_\delta(G_\alpha)$ of diameter $< \sup \delta(N_\delta(G_\alpha))$, then $\operatorname{diam} h(S, (i-1)/i) < \varepsilon_i$.

The construction of δ uses the paracompactness of $M/\mathcal{G}$. Next, let $\mathcal{W}$ be a saturated open cover of M which refines $\{N_\delta(G_\alpha) \mid G_\alpha \in \mathcal{G}\}$, and let $f(x, t)$, $t \in [(i-1)/i, i/(i+1)]$, be an isotopy of M given by the assumption of shrinkability, reparametrized as indicated, using the cover $\{W \cap U_i \mid W \in \mathcal{W}, U_i \in \mathcal{U}_i\}$ and the map δ. To extend the sequence of $h(x, t)$'s one more step, define $h(x, t) = h(f(x, t), (i-1)/i)$ for $t \in [(i-1)/i, i/(i+1)]$. Then for each $G_\alpha \in \mathcal{G}$,

(a′) there is a $U_i \in \mathcal{U}_i$ such that for all $t \in [(i-1)/i, i/(i+1)]$, $h(U_i, (i-1)/i) \supset h(G_\alpha, t)$ and

(b′) $\operatorname{diam} h(G_\alpha, i/(i+1)) < \varepsilon_i$.

From these conditions it follows that there exists a saturated open cover $\mathcal{V}$ of M, refining $\{N_{\varepsilon_{i+1}}(G_\alpha) \mid G_\alpha \in \mathcal{G}\}$, such that for each $V \in \mathcal{V}$, there exists a $U_i \in \mathcal{U}_i$ and there exists a $\lambda > 0$ such that for all $t \in [(i-1)/i, i/(i+1)]$, $h(U_i, (i-1)/i) \supset h(N_\lambda(V), t)$, and also $\operatorname{diam} h(V, i/(i+1)) < 2\varepsilon_i$. Let $\mathcal{U}_{i+1}$ be a saturated open cover of M which is a barycentric refinement of $\mathscr{V}$, that is, the set $\bigcup\{U \mid G_\alpha \subset U \in \mathscr{U}_{i+1}\}$, for each $G_\alpha \in \mathscr{G}$, is contained in some element of $\mathscr{V}$ [7, p. 168]. Then $\mathscr{U}_{i+1}$ is the desired cover. This completes the proof.

4. Corollaries and applications.

Recall that $D = X - g(S \times [0, 1))$.

COROLLARY 7. *Suppose we have Hypothesis* (X^n, S, E^k), *with* $n+k \geqq 5$. *Then*

(1) *there exists a homeomorphism* $f\colon X \times E^k \to (v * S) \times E^k$, *bounded as small as desired on the* E^k *factor, such that* $f \mid S \times E^k = \text{identity}$, *and*

(2) *there exists a homeomorphism* $f\colon X * S^{k-1} \to (v * S) * S^{k-1}$ *with* $f \mid S * S^{k-1} = \text{identity}$, *and*

(3) *if* M^k *is a topological manifold with boundary, then there is a homeomorphism* $f\colon X \times M/\{D \times w \mid w \in \partial M\} \to (v * S) \times M$ *which is the "identity" on* $(v * S) \times \partial M \cup S \times M$.

REMARK 1. Statement (2) in particular says that $\Sigma^2 F^3 \approx I^5$ and $\Sigma F^4 \approx I^5$, and therefore $\Sigma^2 H^3 \approx S^5$ and $\Sigma H^4 \approx S^5$, where F and H are a homotopy cell and a homotopy sphere, respectively, and $\approx$ means "is homeomorphic to." Statement (3) provides an interesting fact for homotopy 3-cells, saying that

$$F^3 \times I^2/\{D \times w \mid w \in \partial I^2\} \approx I^5$$

by a homeomorphism which is the "identity" on the boundary.

REMARK 2. An important problem in relation to simplicial triangulations of topological manifolds is whether there exists some nonsimply-connected homology n-sphere H whose kth suspension, for some k, is homeomorphic to S^{n+k}. Since $n+k$ is necessarily $\geqq 5$, it follows by part (2), using $((\Sigma H) - \operatorname{int} B, \partial B, E^{k-1})$ for (X^n, S, E^k), where B is a locally flat $(n+1)$-cell in ΣH missing H and the suspension points, that $\Sigma^k H \approx S^{n+k}$ if and only if $v * H \times E^{k-1}$ is an $(n+k)$-manifold with boundary.

REMARK 3. An application of Corollary 7, where the S in *Hypothesis* (X^n, S, E^k) may not be a sphere, is obtained as follows: Suppose $S \subset H$ is a locally flat embedding of a homotopy 3-sphere S in a homotopy 4-sphere H. Then $(\Sigma^2 H, \Sigma^2 S) \approx (\Sigma S^5, S^5)$. This is because S separates H into two contractible manifolds X_1 and X_2, each containing S as a collared subset. Thus by part (1) of the corollary, $X_i \times E^1 \approx (v_i * S) \times E^1$ and, by part (2), $\Sigma X_i \approx \Sigma(v_i * S)$, $i = 1, 2$. Hence $(\Sigma H, \Sigma S)$ $(\Sigma(v_1 * S * v_2), \Sigma S)$. But then

$$\begin{aligned}(\Sigma^2 H, \Sigma^2 S) &\approx (\Sigma^2(v_1 * S * v_2), \Sigma^2 S) = (v_1 * (\Sigma^2 S) * v_2, \Sigma^2 S) \\ &\approx \text{[by Remark 1 above]}\ (v_1 * S^5 * v_2, S^5) = (\Sigma S^5, S^5).\end{aligned}$$

Proof of Corollary 7. *Part* (1). This is immediate from Corollary 4.

Part (2). The homeomorphism f is induced by the homeomorphism given in part (1) (call it f_1) in a natural manner. Simply let $\lambda_1\colon X \times E^k \to X * S^{k-1} - S^{k-1}$ and $\lambda_2\colon (v * S) \times E^k \to (v * S) * S^{k-1} - S^{k-1}$ be the homeomorphisms which are naturally induced by a homeomorphism $\lambda\colon E^k \to (v * S^{k-1}) - S^{k-1}$ which preserves radial lines. Then f is defined as indicated by the diagram.

$$\begin{tikzcd}
X * S^{k-1} \arrow[rr, dashed, "f", "f|S^{k-1} = \text{identity}"'] & & (v * S) * S^{k-1} \\
X \times E^k \arrow[u, "\lambda_1"] \arrow[rr, "f_1"'] & & (v * S) \times E^k \arrow[u, "\lambda_2"']
\end{tikzcd}$$

Part (3). Consider the subsets $X \times \operatorname{int} M \subset X \times M/\{D \times w \mid w \in \partial M\}$ and $(v * S) \times \operatorname{int} M \subset (v * S) \times M$. By Corollary 4, given any $\varepsilon\colon \operatorname{int} M \to (0, \infty)$, there exists a homeomorphism $f\colon X \times \operatorname{int} M \to (v * S) \times \operatorname{int} M$, $f = \text{identity}$ on $S \times M$, such that $d(w, p_2 f(x, w)) < \varepsilon(w)$ for all $w \in \operatorname{int} M$. We can further assume (Theorem 3) that f is the identity off of the $\varepsilon(x, w) = \varepsilon(w)$ neighborhood of $D \times \operatorname{int} M$ in $X \times \operatorname{int} M$. Then if ε is sufficiently small, f entends via the identity to the desired homeomorphism of the corollary.

REMARK. Part (1) (and therefore part (2)) of Corollary 7 holds under the weaker hypothesis that $S \times E^k$ merely be collared in $X \times E^k$. This result follows by attaching a collar to X to get $X_+ = X \cup_{S = S \times 0} S \times [0, 1]$ and applying part (1) above to get $X_+ \times E^k \approx (v * S) \times E^k$, and then observing that $X_+ \times E^k \approx X \times E^k$ using the collar of $S \times E^k$ in $X \times E^k$.

This remark provides for a proof of the following fact, originally proved by Siebenmann in [20] under the added assumption that ∂M is a manifold when $m = 5$.

COROLLARY 8. *Suppose M^m is a simplicial homotopy m-manifold, $m \neq 4$. Then $M - \partial M$ is a topological m-manifold without boundary and ∂M is collared in M. In particular, then, if $m \neq 4$, 5, or if $m = 5$ and ∂M is a topological manifold, then M is a topological manifold.*

Proof. Assume M is triangulated with ∂M as a full subcomplex. Then by definition for each k-simplex $\sigma^k \in M$, $lk(\sigma, M)$ is homotopically equivalent to either an $(m-k-1)$-sphere or ball according as $\sigma \in M - \partial M$ or $\sigma \in \partial M$, and if $\sigma \in \partial M$, then $lk(\sigma, \partial M)$ is homotopically equivalent to an $(m-k-2)$-sphere.

First, consider $\operatorname{int} M = |M| - |\partial M|$. Suppose inductively that $\operatorname{int} M - M^{(k)}$ ($M^{(k)} = k$-skeleton of M) is an open m-manifold and suppose σ is a k-simplex in $M - \partial M$. The induction assumption implies that $lk(\sigma, M) \times E^{k+1}$ is an open m-manifold, since $lk(\sigma, M) \times E^{k+1} \approx \text{open star}(\sigma, M) - \mathring{\sigma} \subset \operatorname{int} M - M^{(k)}$, where $\text{open star}(\sigma, M) = \bigcap \{\text{open star}(v, M) \mid v \text{ vertex of } \sigma\}$. We will show that $lk(\sigma, M) * S^k \approx S^m$, which then implies that open star (σ, M) is locally euclidean, being homeomorphic to an open subset of $lk(\sigma, M) * S^k$.

Without loss assume $k \leqq m-4$ (for $k \geqq m-3$, $lk(\sigma, M)$ is a real PL sphere). Let B be a small collared $(m-k-1)$-cell in $lk(\sigma, M)$ and let

$$(X, S) = (lk(\sigma, M) - \operatorname{int} B, \partial B).$$

X is acyclic by duality in $X \times S^q$, large q. So by Corollary 7, part (2) we have $(X, S) * S^k \approx (B^{m-k-1}, \partial B^{m-k-1}) * S^k$ and therefore sewing back in $B * S^k$ along $\partial B * S^k$ we get that $lk(\sigma, M) * S^k \approx S^m$.

Now consider ∂M. Inductively assume that $\partial M - M^{(k)}$ is collared in $M - (\partial M \cap M^{(k)})$ and suppose $\sigma = \sigma^k \in \partial M$. Again without loss $k \leqq m-4$. Let

$$(X, S) = (lk(\sigma, M), lk(\sigma, \partial M)).$$

The induction hypothesis implies that $S \times E^{k+1}$ is collared in $X \times E^{k+1}$, and the first part of the corollary implies that $(X-S) \times E^{k+1}$ is an open m-manifold. Now applying the preceding remark to the pair (X, S), we have

$$(X, S) * S^k \approx (v * S, S) * S^k = (v * (S * S^k), S * S^k),$$

and therefore $S * \mathring{\sigma}$ is collared in $X * \mathring{\sigma}$. Thus the induction can proceed.

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Department of Mathematics, Princeton University, Princeton, New Jersey 08540

Department of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Department of Mathematics, University of California, Los Angeles, California 90024*

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112**

* Current address of Robert D. Edwards.

**Current address of Leslie C. Glaser.