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REGULAR FUNCTIONS $f(z)$ FOR WHICH $zf'(z)$ IS α-SPIRAL

BY

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Abstract. A function $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ regular in the open unit disk $\Delta=\{z : |z|<1\}$ is a (univalent) α-spiral function for real α, $|\alpha|<\pi/2$, if $\operatorname{Re}\{e^{i\alpha}zf'(z)/f(z)\}>0$ for z in Δ; in this case we write $f(z)\in\mathscr{F}_\alpha$. A fundamental result of this paper shows that the transformation

$$f_*(z) = \frac{azf((z+a)/(1+\bar{a}z))}{f(a)(z+a)(1+\bar{a}z)^{e^{-2i\alpha}}}$$

defines a function in $\mathscr{F}_\alpha$ whenever $f(z)$ is in $\mathscr{F}_\alpha$ and a is in Δ.

If $g(z)$ is regular in Δ, $g(0)=0$ and $g'(0)=1$, then $g(z)$ is in $\mathscr{G}_\alpha$ if and only if $zg'(z)$ is in $\mathscr{F}_\alpha$. The main result of the paper is the derivation of the sharp radius of close-to-convexity for each class $\mathscr{G}_\alpha$; it is given as the solution of an equation in r which is dependent only on α. (Approximate solutions of this equation were made by computer and these suggest that the radius of close-to-convexity of the class $\mathscr{G}=\bigcup_\alpha \mathscr{G}_\alpha$ is approximately $.99097^+$.) Additional results are also obtained such as the radius of convexity of $\mathscr{G}_\alpha$, a range of α for which $g(z)$ in $\mathscr{G}_\alpha$ is always univalent is given, etc. These conclusions all depend heavily on the transformation cited above and its analogue for $\mathscr{G}_\alpha$.

1. **Introduction.** A function $f(z)=z+a_2z^2+\cdots$ regular in the open unit disk $\Delta=\{z : |z|<1\}$ and satisfying the condition

$$\operatorname{Re}\{e^{i\alpha}zf'(z)/f(z)\} > 0, \qquad z \text{ in } \Delta, \tag{1.1}$$

for some real α, $|\alpha|<\pi/2$, was shown to be univalent in Δ by Špaček [13]. Because of their mapping properties, later writers (see [6]) called such functions "spirallike". If $f(z)$ satisfies the above conditions for a given α, then $f(z)$ is called an α-spiral function [6] and we write $f(z)\in\mathscr{F}_\alpha$; the class of spirallike functions is $\mathscr{F}=\bigcup_\alpha \mathscr{F}_\alpha$.

For $\alpha=0$, (1.1) defines a starlike function [7], i.e. $\mathscr{F}_0$ is the class of regular functions starlike with respect to the origin, ordinarily denoted by $\mathscr{S}^*$. A brief calculation shows that $f(z)\in\mathscr{F}_\alpha$ if and only if there is a function $s(z)$ in $\mathscr{S}^*$ such that

$$f(z) = z[s(z)/z]^{\cos\alpha\, e^{-i\alpha}}, \qquad z \text{ in } \Delta, \tag{1.2}$$

where $[s(z)/z]^{\cos\alpha\, e^{-i\alpha}}$ has the value 1 at the origin.

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Recently Robertson [10] introduced and studied some mapping properties of functions $f(z)$ regular in Δ and satisfying the condition that $zf'(z)$ is in $\mathscr{F}_\alpha$. We say that $f(z)$, regular in Δ, with the usual normalization $f(0)=0$ and $f'(0)=1$, is in $\mathscr{G}_\alpha$ when and only when $zf'(z)$ is in $\mathscr{F}_\alpha$. As above, we let $\mathscr{G}=\bigcup_\alpha \mathscr{G}_\alpha$. $\mathscr{G}_0$ is the class of convex functions frequently represented by $\mathscr{K}$; and, as is well known [7], $f(z)$ is in $\mathscr{S}^*$ if and only if $zf'(z)$ is in $\mathscr{K}$. Now using this last relation together with (1.2) we may conclude that $f(z)$ is in $\mathscr{G}_\alpha$ whenever

$$f(z) = \int_0^z [g'(\zeta)]^{\cos\alpha\, e^{-i\alpha}}\, d\zeta, \qquad z \text{ in } \Delta, \tag{1.3}$$

for $g(z)$ in $\mathscr{G}_0$ and conversely. Integrals of the type (1.3) have been studied by several authors ([9], [11]) and have recently been found to be useful in the study of a Banach space of analytic functions ([1], [2]). Robertson [10] has shown that functions satisfying (1.3) need not be univalent.

The purpose of this paper is to continue the study of functions satisfying (1.3), that is, of functions in $\mathscr{G}_\alpha$. In particular, a range of α for which $f(z)$ in $\mathscr{G}_\alpha$ is always univalent in Δ is given; this is a slight improvement of a result of Robertson [10] and is obtained by an entirely different method. The main result is the derivation of the sharp radius of close-to-convexity for each class $\mathscr{G}_\alpha$; it is given as the solution of an equation in r which is dependent only on α. The key to the solution of these problems is a transformation, Lemma 2 below, which preserves membership in $\mathscr{G}_\alpha$.

2. **Preliminary ideas.** This section contains several fundamental lemmas which are essential for proof of the principal results; the first two are new [14], whereas the remaining ones are consequences of known results.

LEMMA 1. *If $f(z)$ is in $\mathscr{F}_\alpha$ and a is in Δ, then*

$$f_*(z) = \frac{azf((z+a)/(1+\bar{a}z))}{f(a)(z+a)(1+\bar{a}z)^{e^{-2i\alpha}}}, \qquad z \text{ in } \Delta, \tag{2.1}$$

is likewise in $\mathscr{F}_\alpha$.

Proof. For ρ real, $0<\rho<1$, let

$$f_\rho(z) = \frac{azf(\rho(z+a)/(1+\bar{a}z))}{f(\rho a)(z+a)(1+\bar{a}z)^{e^{-2i\alpha}}}, \qquad z \text{ in } \Delta, \tag{2.2}$$

then

$$\begin{aligned}\frac{zf_\rho'(z)}{f_\rho(z)} = 1 &+ \frac{\rho((z+a)/(1+\bar{a}z))f'(\rho(z+a)/(1+\bar{a}z))}{f(\rho(z+a)/(1+\bar{a}z))}\cdot\frac{z(1-|a|^2)}{(z+a)(1+\bar{a}z)} \\ &- \frac{z[1+\bar{a}z+\bar{a}ze^{-2i\alpha}+|a|^2e^{-2i\alpha}]}{(z+a)(1+\bar{a}z)}.\end{aligned} \tag{2.3}$$

Letting $z=e^{i\theta}$ and $\omega=\rho(e^{i\theta}+a)/(1+\bar{a}e^{i\theta})$ and multiplying (2.3) by $e^{i\alpha}$ gives

$$e^{i\alpha}\frac{zf_\rho'(z)}{f_\rho(z)} = e^{i\alpha}\frac{\omega f'(\omega)}{f(\omega)}\cdot\frac{1-|a|^2}{|1+ae^{-i\theta}|^2} + e^{i\alpha}\left[1-\frac{1+\bar{a}e^{i\theta}+ae^{i\theta}e^{-2i\alpha}+|a|^2e^{-2i\alpha}}{|1+ae^{-i\theta}|^2}\right]. \tag{2.4}$$

Further calculation shows that the second term on the right of (2.4) is imaginary, therefore, for $|z|=1$,

$$\operatorname{Re}\left\{e^{i\alpha}\frac{zf_\rho'(z)}{f_\rho(z)}\right\} = \frac{1-|a|^2}{|1+ae^{-i\theta}|^2}\operatorname{Re}\left\{e^{i\alpha}\frac{\omega f'(\omega)}{f(\omega)}\right\} \geqq 0, \tag{2.5}$$

and we conclude that $f_\rho(z)$ is in $\mathscr{F}_\alpha$ for every admissible ρ. From the compactness of $\mathscr{F}_\alpha$ and (2.5) we infer that $f_*(z)=\lim_{\rho\to 1}f_\rho(z)$ is in $\mathscr{F}_\alpha$.

The univalence preserving transformation

$$[F((z+\zeta)/(1+\bar{\zeta}z))-F(\zeta)]/(F'(\zeta)(1-|\zeta|^2)), \qquad \zeta\in\Delta,$$

is used to obtain the radius of convexity for the class $\mathscr{S}$ of normalized univalent functions in Δ (see [7, pp. 215–216]). In an analogous fashion, the transformation of Lemma 1 gives the radius of starlikeness of $\mathscr{F}_\alpha$ and other results which follow.

COROLLARY 1. *The sharp radius of starlikeness of $\mathscr{F}_\alpha$ is*

$$[|\sin\alpha|+\cos\alpha]^{-1}. \tag{2.6}$$

Proof. If $f(z)$ is in $\mathscr{F}_\alpha$ and $f_*(z)=z+b_2^*z^2+\cdots$ is given by (2.1), then

$$b_2^* = f_*''(0)/2 = (1-|a|^2)f'(a)/f(a)-(1+e^{-2i\alpha}|a|^2)/a. \tag{2.7}$$

Replacing a by z and letting $|z|=r$ in (2.7) and making use of the bound $|b_2^*| \leqq 2\cos\alpha$, given in [6], we may write

$$\left|\frac{zf'(z)}{f(z)}-\frac{1+e^{-2i\alpha}r^2}{1-r^2}\right| \leqq \frac{2r\cos\alpha}{1-r^2}, \qquad z\in\Delta, \tag{2.8}$$

which implies that

$$\operatorname{Re}\{zf'(z)/f(z)\} \geqq (1-2r\cos\alpha+r^2\cos 2\alpha)/(1-r^2). \tag{2.9}$$

From (2.9) we conclude that the radius of starlikeness of $\mathscr{F}_\alpha$ is the smallest positive zero of

$$1-2r\cos\alpha+r^2\cos 2\alpha, \tag{2.10}$$

which is (2.6). To show this result is sharp we let $f(z)=z/(1-z)^{2\cos\alpha e^{-i\alpha}}$ and $\zeta=r(r-e^{i\alpha})/(1-re^{i\alpha})$ and obtain

$$\zeta f'(\zeta)/f(\zeta) = (1-2\cos\alpha\, r+e^{-2i\alpha}r^2)/(1-r^2)$$

which has real part of value zero at $r=[|\sin\alpha|+\cos\alpha]^{-1}$.

This result was obtained independently and using different methods by both Robertson [9] and Libera [6]. Minimizing either (2.6) or (2.10) as a function of α we obtain the following which appears also in [9].

COROLLARY 2. *The radius of starlikeness of* $\mathscr{F}$ *is* $2^{1/2}/2$.

Making use of the relationship between $\mathscr{F}_\alpha$ and $\mathscr{G}_\alpha$ we may write the following as a consequence of both Corollaries 1 and 2.

COROLLARY 3. *The radius of convexity of* $\mathscr{G}_\alpha$ *is* $[|\sin\alpha|+\cos\alpha]^{-1}$ *and the radius of convexity of* $\mathscr{G}$ *is* $2^{1/2}/2$. *These results are sharp.*

LEMMA 2. *If* $g(z)$ *is in* $\mathscr{G}_\alpha$, $a\in\Delta$, *and* $g_*(z)$ *is defined by*

$$g_*'(z)=\frac{g'((z+a)/(1+\bar{a}z))}{g'(a)(1+\bar{a}z)^{e^{-2i\alpha}+1}},\quad z\in\Delta,\quad \textit{and}\quad g_*(0)=0, \tag{2.11}$$

then $g_*(z)$ *is in* $\mathscr{G}_\alpha$.

Proof. Let $f(z)$ be chosen in $\mathscr{F}_\alpha$ so that $zg'(z)=f(z)$ and let $f_*(z)$ be defined by (2.1). Defining $g_*(z)$ by the relation $zg_*'(z)=f_*(z)$ gives (2.11).

Corollary 3 can be obtained directly from Lemma 2 using the methods of Corollary 1.

The remaining two lemmas have not appeared as stated, but are consequences of known results and are given here for use later in this paper.

LEMMA 3. *If* $g(z)=z+\sum_{k=2}^\infty a_kz^k$ *is in* $\mathscr{G}_\alpha$ *and* μ *is any complex number, then*

$$|a_3-\mu a_2^2|\leqq \tfrac{1}{3}\cos\alpha\max\{1,|1-\cos\alpha\, e^{-i\alpha}(3\mu-2)|\}. \tag{2.12}$$

This inequality is sharp for each μ.

Proof. If $f(z)=z+\sum_{k=2}^\infty b_kz^k$ is chosen in $\mathscr{F}_\alpha$ so that $zg'(z)=f(z)$, then, as was shown by Keogh and Merkes [4],

$$|b_3-\nu b_2^2|\leqq\cos\alpha\max\{1,|1-2\cos\alpha\, e^{-i\alpha}(2\nu-1)|\} \tag{2.13}$$

for any complex number ν and it is sharp. (2.12) follows immediately from (2.13) by making appropriate substitutions. Both (2.12) and (2.13) were obtained by Ziegler [14] using methods like those of Keogh and Merkes [4].

LEMMA 4. *If* $g(z)$ *is in* $\mathscr{G}_\alpha$, *then*

$$\begin{aligned}&-2\cos^2\alpha\arcsin(r\cos\alpha)+\sin2\alpha\ln((1-r^2\cos^2\alpha)^{1/2}-r\sin\alpha)\\ &\leqq\arg\{g'(z)\}\\ &\leqq 2\cos^2\alpha\arcsin(r\cos\alpha)+\sin2\alpha\ln((1-r^2\cos^2\alpha)^{1/2}+r\sin\alpha).\end{aligned} \tag{2.14}$$

Lemma 4 is equivalent to Theorem 2 of Singh [12], and is obtained as a consequence to a more general problem in [14].

3. **Principal results.** Let $f(z)$ be regular in Δ and let $\{f, z\}$ denote the Schwarzian derivative of $f(z)$,

$$\{f, z\} = [f''(z)/f'(z)]' - \tfrac{1}{2}[f''(z)/f'(z)]^2.$$

Nehari has shown that $f(z)$ is univalent in Δ whenever $|\{f, z\}| \leqq 2/(1-|z|^2)^2$, $z \in \Delta$ [8]. This result is used to obtain a range of α for which $f(z)$ in $\mathscr{G}_\alpha$ is always univalent in Δ.

THEOREM 1. *If $g(z)$ is in $\mathscr{G}_\alpha$ and $|z| = r < 1$, then*

$$|\{g, z\}| \leqq (6|\sin\alpha|\cos\alpha + 2\cos\alpha)/(1-r^2)^2 \tag{3.1}$$

and $g(z)$ is univalent in Δ for $0 < \cos\alpha \leqq x_0$, where x_0 is the unique positive root of $9x^3+9x^2+x-1=0$; $.256 < x_0 < .257$.

Proof. If $g(z) \in \mathscr{G}_\alpha$ and $g_*(z) = z + a_2^* z^2 + a_3^* z^3 + \cdots$ is defined by (2.11), then

$$a_2^* = \tfrac{1}{2}\{(1-|a|^2)g''(a)/g'(a) - \bar{a}(e^{-2i\alpha}+1)\}$$

and

$$a_3^* = \tfrac{1}{6}\{(1-|a|^2)^2 g'''(a)/g'(a) - 2\bar{a}(1-|a|^2)(e^{-2i\alpha}+2)g''(a)/g'(a) \\ + \bar{a}^2(e^{-2i\alpha}+1)(e^{-2i\alpha}+2)\}.$$

Consequently,

$$a_3^* - (a_2^*)^2 = (1-|a|^2)^2\{g, a\}/6 - (1-e^{-2i\alpha})(1-|a|^2)\bar{a}W/6 - (1-e^{-4i\alpha})\bar{a}^2/12 \tag{3.2}$$

where

$$W = g''(a)/g'(a) - \bar{a}(1+e^{-2i\alpha})/(1-|a|^2) = 2a_2^*/(1-|a|^2).$$

The relationship between $\mathscr{G}_\alpha$ and $\mathscr{F}_\alpha$ and the coefficient bounds given in [6] enable us to conclude that $|W| \leqq (2\cos\alpha)/(1-|a|^2)$ and Lemma 3 implies that $|a_3^* - (a_2^*)^2| \leqq \tfrac{1}{3}\cos\alpha$. Using these bounds in (3.2) and replacing a by z, $|z| = r$, yields the following:

$$|\{g, z\}| \leqq \frac{4r|\sin\alpha|\cos\alpha + 2r^2|\sin\alpha|\cos\alpha + 2\cos\alpha}{(1-r^2)^2} \leqq \frac{6|\sin\alpha|\cos\alpha + 2\cos\alpha}{(1-r^2)^2}.$$

Thus, by Nehari's test, $g(z)$ is univalent whenever $3|\sin\alpha|\cos\alpha + \cos\alpha \leqq 1$ or, equivalently, whenever $9\sin^2\alpha\cos^2\alpha \leqq (1-\cos\alpha)^2$. Evidently, $g(z)$ is univalent when $\alpha = 0$ or when

$$9\cos^3\alpha + 9\cos^2\alpha + \cos\alpha - 1 \leqq 0.$$

$\alpha = 0$ implies that $g(z)$ is in $\mathscr{K}$, hence is univalent, and, since the equation $9x^3+9x^2+x-1=0$ has only one positive root x_0, it follows that $g(z)$ is likewise univalent for $0 < \cos\alpha \leqq x_0$. A calculation shows $.256 < x_0 < .257$.

Robertson has shown that $g(z)$ in $\mathscr{G}_\alpha$ is univalent whenever $0<\cos\alpha\leqq x_1$, where $.231<x_1<.232$ [10]. The following corollary is an immediate consequence of Theorem 1 and also appears in [10].

COROLLARY 4. *If* $g(z)\in\mathscr{K}$ *and* $|z|=r<1$, *then* $|\{g,z\}|\leqq 2/(1-r^2)^2$.

Kaplan [3] has shown that if $g(z)$ is regular in Δ and has a nonvanishing derivative there, then $g(z)$ maps $|z|=r<1$ onto a close-to-convex curve if and only if

$$\arg[z_2g'(z_2)]-\arg[z_1g'(z_1)]\geqq -\pi \tag{3.3}$$

for all z_1 and z_2 satisfying $z_2=e^{i\theta}z_1$, $0<\theta<2\pi$, and $|z_2|=|z_1|=r$. We use (3.3) to determine the radius of close-to-convexity of $\mathscr{G}_\alpha$ for each α. The techniques used are similar to those employed by Krzyż to obtain the radius of close-to-convexity for the class of univalent functions $\mathscr{S}$ [5].

THEOREM 2. *If* $\alpha\neq 0$; r_0 *is the radius of convexity of* $\mathscr{G}_\alpha$;

$$x_0=(1-r^2\cos 2\alpha)/2r|\sin\alpha|,\qquad r\in(r_0,1);$$
$$\theta_0=2\arccos x_0,\qquad 0<\theta_0<\pi;$$

and

$$\begin{aligned}\Delta(r)=\;&\theta_0+2\cos^2\alpha\arctan\left\{\frac{r^2\sin\theta_0}{1-r^2\cos\theta_0}\right\}\\ &-2\cos^2\alpha\arcsin\left\{r\cos\alpha\left[\frac{2(1-\cos\theta_0)}{1-2r^2\cos\theta_0+r^4}\right]^{1/2}\right\}\\ &+\sin 2\alpha\ln\{[1-2r^2(\sin^2\alpha\cos\theta_0+\cos^2\alpha)+r^4]^{1/2}\\ &\qquad\qquad -r\sin\alpha[2(1-\cos\theta_0)]^{1/2}\}\\ &-\sin 2\alpha\ln\{1-r^2\};\end{aligned}\tag{3.4}$$

then the radius of close-to-convexity of $\mathscr{G}_\alpha$ *is the unique root of the equation* $\Delta(r)=-\pi$, *where* r *is in* $(r_0,1)$.

Proof. Let

$$\Delta(r,\theta)=\inf_{g(z)\in\mathscr{G}_\alpha}\arg\left\{\frac{z_2g'(z_2)}{z_1g'(z_1)}\right\},$$

where z_1 and z_2 are any two points satisfying $z_2=e^{i\theta}z_1$, $0<\theta<2\pi$ and $|z_1|=r$, and the argument is chosen so as to vary continuously from an initial value of zero. If $g(z)$ is in $\mathscr{G}_\alpha$ and C_r is the image of $|z|=r$ under the mappings $g(z)$, then it follows from (3.3) that C_r will be a close-to-convex curve if $\Delta(r,\theta)\geqq-\pi$.

If $\zeta=(z-z_1)/(1-\bar z_1z)$ and $\zeta_0=(z_2-z_1)/(1-\bar z_1z_2)$ and $g_*(\zeta)$ is defined by

$$g_*'(\zeta)=\frac{g'((\zeta+z_1)/(1+\bar z_1\zeta))}{g'(z_1)(1+\bar z_1\zeta)^{e^{-2i\alpha}+1}},\qquad g_*(0)=0,$$

then it follows from Lemma 2 that $g_*(\zeta)\in\mathcal{G}_\alpha$. Thus we have

$$g'_*(\zeta_0)=\frac{g'(z_2)}{g'(z_1)}\left\{\frac{1-\bar{z}_1z_2}{1-|z_1|^2}\right\}^{e^{-2i\alpha}+1}$$

and, consequently,

$$\Delta(r,\theta)=\arg\left\{\frac{z_2}{z_1}\left[\frac{1-|z_2|^2}{1-\bar{z}_1z_2}\right]^{e^{-2i\alpha}+1}\right\}+\inf_{g_*(\zeta)\in\mathcal{G}_\alpha}\arg\{g'_*(\zeta_0)\}. \tag{3.5}$$

A brief calculation shows

$$|\zeta_0|=r[2(1-\cos\theta)/(1-2r^2\cos\theta+r^4)]^{1/2} \tag{3.6}$$

and

$$\begin{aligned}\arg\left\{\left[\frac{1-|z_1|^2}{1-\bar{z}_1z_2}\right]^{e^{-2i\alpha}+1}\right\}&=2\cos^2\alpha\arctan\frac{r^2\sin\theta}{1-r^2\cos\theta}\\&\quad-\sin 2\alpha\ln\left[\frac{1-r^2}{(1-2r^2\cos\theta+r^4)^{1/2}}\right].\end{aligned} \tag{3.7}$$

Using (2.14), (3.6) and (3.7) to evaluate (3.5) enables us to write

$$\begin{aligned}\Delta(r,\theta)&=\theta+2\cos^2\alpha\arctan\left[\frac{r^2\sin\theta}{1-r^2\cos\theta}\right]\\&\quad-2\cos^2\alpha\arcsin\left\{r\cos\alpha\left[\frac{2(1-\cos\theta)}{1-2r^2\cos\theta+r^4}\right]^{1/2}\right\}\\&\quad+\sin 2\alpha\ln\{[1-2r^2(\cos\theta\sin^2\alpha+\cos^2\alpha)+r^4]^{1/2}\\&\qquad\qquad-r\sin\alpha\,[2(1-\cos\theta)]^{1/2}\}\\&\quad-\sin 2\alpha\ln(1-r^2).\end{aligned} \tag{3.8}$$

The expression for $\Delta(r,\theta)$ given in (3.8) is sharp since, letting $f_*(z)$ be the function in $\mathcal{G}_\alpha$ which gives equality in (2.14) and, for fixed z_1 and z_2, letting $f(z)$ in $\mathcal{G}_\alpha$ be defined by

$$f'(z)=\frac{f'_*((z-z_1)/(1-\bar{z}_1z))}{f'_*(-z_1)(1-\bar{z}_1z)^{e^{-2i\alpha}+1}},\qquad f(0)=0,$$

it follows that

$$\arg\{z_2f'(z_2)/z_1f'(z_1)\}=\Delta(r,\theta).$$

If $\Delta(r)=\inf_{0<\theta<2\pi}\Delta(r,\theta)$, then it is evident that $\Delta(r)$ is a decreasing function of r and that the radius of close-to-convexity of $\mathcal{G}_\alpha$ is the solution r_1 of the equation $\Delta(r)=-\pi$, provided such a solution exists. The remainder of this proof is concerned with demonstrating the existence of r_1 and characterizing r_1 as the root of an equation depending only on α.

If r_0 is the radius of convexity of $\mathcal{G}_\alpha$, then $\Delta(r)\geqq 0$ for $r\leqq r_0$ and consequently $r_1>r_0$. Thus we may assume $r\in(r_0,1)$ throughout the rest of this argument.

Differentiating (3.8) with respect to θ yields

$$
\begin{aligned}
\frac{\partial\Delta(r,\theta)}{\partial\theta} &= 1+\frac{2\cos^2\alpha\, r^2(\cos\theta-r^2)}{1-2r^2\cos\theta+r^4} \\
&\quad -\frac{2r\cos\alpha\sin\theta\,[1-2r^2(\cos\theta\sin^2\alpha+\cos^2\alpha)+r^4]^{1/2}}{[2(1-\cos\theta)]^{1/2}(1-2r^2\cos\theta+r^4)}.
\end{aligned}
\tag{3.9}
$$

If $x=\cos(\theta/2)$, $0<\theta<2\pi$, then $\cos\theta=2x^2-1$, $(\sin\theta)/[2(1-\cos\theta)]^{1/2}=x$, and (3.9) can be expressed as

$$\partial\Delta(r,\theta)/\partial\theta = (p(x)-2rx\cos\alpha\, k(x))/g(x),$$

where

$$
\begin{aligned}
p(x) &= (1+r^2)(1+r^2-2r^2\cos^2\alpha)-4r^2x^2\sin^2\alpha,\\
k(x) &= [1-2r^2\cos 2\alpha+r^4-4r^2x^2\sin^2\alpha]^{1/2},\\
g(x) &= (1+r^2)^2-4r^2x^2.
\end{aligned}
$$

Since $g(x)>0$ for $r\in[0,1)$ and $x\in(-1,1)$, the zeros of $\partial\Delta(r,\theta)/\partial\theta$ will be identical with the zeros of $p_1(x)=p(x)-2rx\cos\alpha\, k(x)$. For $x\in(-1,1)$,

$$p(x) \geqq (1-r^4)-(1-r^2)2r^2\sin^2\alpha \geqq (1-r^2)^2 > 0$$

and $k(x)>0$, hence $p_1(x)$ has no zeros in the interval $(-1,0]$. Restricting x to $(0,1)$ or, equivalently, θ to $(0,\pi)$, we see that the zeros of $p_1(x)$ are identical with the zeros of $p_2(x)$ where

$$p_2(x) = p(x)^2-4r^2x^2\cos^2\alpha\, k(x)^2 = a_0-a_1x^2+a_2x^4,$$

with

$$
\begin{aligned}
a_0 &= (1+r^2)^2(1-r^2\cos 2\alpha)^2,\\
a_1 &= 4r^2[(1-r^2\cos 2\alpha)^2+(1+r^2)^2\sin^2\alpha],\\
a_2 &= 16r^4\sin^2\alpha.
\end{aligned}
$$

The positive roots of $p_2(x)$ are

$$x_0 = (1-r^2\cos 2\alpha)/2r|\sin\alpha| \quad\text{and}\quad x_1 = (1+r^2)/2r.$$

Since $x_1>1$ for $r\in(0,1)$, $p_2(x)$ will have a zero in $(0,1)$ if and only if $x_0<1$.

$$
\begin{aligned}
p_2(0) &= (1+r^2)^2(1-r^2\cos 2\alpha)^2 > 0,\\
p_2(1) &= (1-r^2)^2(1-2r^2+r^4\cos^2 2\alpha)\\
&= (1-r^2)^2(1-2r\cos\alpha+r^2\cos 2\alpha)(1+2r\cos\alpha+r^2\cos 2\alpha).
\end{aligned}
$$

From (2.10) and the relationship between $\mathscr{G}_\alpha$ and $\mathscr{F}_\alpha$, it follows that $1-2r\cos\alpha+r^2\cos 2\alpha<0$ for $r>r_0$. Also, since $|\alpha|<\pi/2$, $1+2r\cos\alpha+r^2\cos 2\alpha\geqq 1-r^2>0$. Consequently, $p_2(1)<0$ and $p_2(0)>0$ together imply that $x_0<1$ for $r\in(r_0,1)$.

Thus we have shown that x_0 is the only zero of $p_1(x)$ when $x \in (-1, 1)$ or equivalently, putting $\theta_0 = 2 \arccos x_0$, θ_0 is the only zero of $\partial\Delta(r, \theta)/\partial\theta$ for $0 < \theta < 2\pi$. An examination of the sign of $\partial\Delta(r, \theta)/\partial\theta$ for θ near θ_0 shows that $\Delta(r, \theta)$ assumes its minimum value at θ_0, hence

$$\Delta(r) = \inf_{\theta} \Delta(r, \theta) = \Delta(r, \theta_0)$$

and letting $\theta = \theta_0$ in (3.8) yields (3.4). A brief calculation shows $\Delta(r) \to -\infty$ as $r \to 1$. Since $\Delta(r)$ is a continuous, decreasing function, unbounded below, and $\Delta(r_0) = 0$, it follows that there exists a unique solution r_1 to the equation $\Delta(r) = -\pi$ and this value r_1 is the radius of close-to-convexity of $\mathcal{G}_\alpha$. For any fixed value of α, r_1 can be approximated by successive substitutions in (3.4).

4. **Summarizing remarks.** As an aid in interpretation of the results given in Theorem 2, the authors obtained approximate solutions to the equation $\Delta(r) = -\pi$, (3.4). [The calculation was done on a Burroughs B5500 by Mr. Joseph Chicosky, an undergraduate student at the University of Delaware, using the Newton-Raphson method.] Taking symmetries in α into account the calculations were restricted to positive values of α between $\alpha = 20°$ and $\alpha = 70°$ at intervals of 1°. A partial table of these calculations follows.

α	*Radius of close-to-convexity of* $\mathcal{G}_\alpha$
20°	.99880847
25°	.99706432
30°	.99490938
35°	.99289071
40°	.99147945
43°	.99105705
44°	.99099574
45°	.99097524
46°	.99099574
47°	.99105705
50°	.99147945
55°	.99289071
60°	.99490938
65°	.99706432
70°	.99880847

These estimates are symmetric about $\alpha = 45°$ and the minimum of these estimates occurs at $\alpha = 45°$. Additional calculations taken at intervals of .1° in the interval $\alpha = 43°$ to $\alpha = 47°$ supports these observations.

These observations together with the corresponding behavior for the radius of convexity of $\mathcal{G}_\alpha$, i.e. the radii of convexity of $\mathcal{G}_\alpha$ are symmetric about $\alpha = 45°$ and the minimum radius occurs there, makes it reasonable to expect that the radius of close-to-convexity of $\mathcal{G}$ occurs at $\alpha = 45°$ and is approximately .99097524.

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