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A CLASS OF COMPLETE ORTHOGONAL SEQUENCES OF BROKEN LINE FUNCTIONS

BY
J. L. SOX, JR.

Abstract. A class of orthonormal sets of continuous broken line functions is defined. Each member is shown to be complete in $L_2(0, 1)$ and pointwise convergence theorems are obtained for the Fourier expansions relative to these sets.

1. **Introduction.** It was shown in [2] that each sequence of points which is dense in [0, 1] determines a complete orthonormal set of step functions in $L_2(0, 1)$. In this paper we prove that each such sequence of points also determines a complete orthonormal set of continuous broken line functions similar to that constructed by Franklin [1]. The Fourier expansion of a function $f \in L_2(0, 1)$ relative to a set of this class is found to converge at each point of continuity of f and is shown to converge uniformly on [0, 1] when f is continuous on this interval.

2. **Definitions.** Suppose that $A = \{a_n\}_{n=1}^{\infty}$ is a sequence of distinct points in (0, 1) which is dense in [0, 1] and let $\{h_n\}_{n=0}^{\infty}$ be the set of linear functions defined by

$$\begin{aligned} h_0(x) \equiv 1, \qquad h_1(x) &= x, \qquad x \in [0, 1]; \\ h_{n+1}(x) &= 0, \qquad x \in [0, a_n), \\ &= x - a_n, \qquad x \in [a_n, 1]. \end{aligned}$$

Since it is evident that no h_i is a linear combination of the other functions in the set, we see that the h_i are linearly independent on [0, 1]. Thus, one can employ the Gram-Schmidt process to construct an orthonormal sequence $\{u_n(x)\}$ such that each u_n is a linear combination of the h_i, $i \leqq n$. Because of the triangular nature of this construction, each h_n can also be expressed as a linear combination of the u_i, $i \leqq n$.

3. **Completeness of $\{u_n\}$.** To prove that the sequence of functions $\{u_n\}$ is complete in $L_2(0, 1)$, one needs an obvious property of the sequence A which is given in Lemma 1. In this lemma and throughout this paper the term "adjacent points" of a finite subset $A_N \subset A$ will be used to denote successive elements of the subset when its elements are arranged in order of magnitude; i.e. a_m and a_n are adjacent points of A_N if and only if there is no $a_k \in A_N$ such that $a_m < a_k < a_n$ or $a_n < a_k < a_m$.

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LEMMA 1. *Let* $A=\{a_1, a_2, \dots\}$ *be a sequence of distinct points of* $(0, 1)$ *which is dense in* $[0, 1]$. *Then for each* $\delta>0$ *there is an integer* N_δ *such that if* $N>N_\delta$, (i) *any pair of adjacent points* a_m *and* a_n *in the subset* $A_N=\{a_1, a_2, \dots, a_N\}$ *satisfy* $|a_m-a_n| <\delta$; (ii) $d(x, A_N)<\delta$ *for* $x \in [0, 1]$. ($d(x, A_N)$ *is the distance from* x *to the* A_N *defined in the usual manner.*)

THEOREM 1. *The orthonormal sequence of functions* $\{u_n\}$ *is complete in* $L_2(0, 1)$.

Proof. Let $0<a_{i_1}<a_{i_2}<\cdots<a_{i_N}<1$ be the points of $\{0, a_1, a_2, \dots, a_N, 1\}$ arranged in order of magnitude. If P_N is any continuous polygonal function (broken line function) which is linear on each subinterval $[a_{i_{k-1}}, a_{i_k}]$ of the partition of $[0, 1]$ determined by these points, it is clear that P_N can be expressed as a linear combination of the h_i, $i \leqq N$. Thus since each h_i, $i \leqq N$, is a linear combination of the u_i, $i \leqq N$, any such P_N is a linear combination of the u_i, $i \leqq N$.

Now suppose that F is any continuous function on $[0, 1]$ and let δ be a positive number such that $|F(x_1)-F(x_2)|<\varepsilon/2$ when $x_1, x_2 \in [0, 1]$ and $|x_1-x_2|<\delta$. By Lemma 1 we can choose an integer N_δ such that if $N>N_\delta$, the norm of the partition of $[0, 1]$ determined by the points of A_N is less than δ. Therefore, the broken line function P_N which equals F at each point of this partition and is linear elsewhere in $[0, 1]$ satisfies $|P_N(x)-F(x)|<\varepsilon$ for $x \in [0, 1]$. It follows from the preceding remarks that there is a linear combination of u_i, $i \leqq N$, say T_N, such that $|T_N(x)-F(x)|<\varepsilon$ if $x \in [0, 1]$ or such that

$$\|T_N-F\|_2^2 = \int_0^1 [T_N-F]^2\, dx < \varepsilon^2.$$

Since the set of continuous functions on $[0, 1]$ is dense in $L_2(0, 1)$, we conclude from the last inequality that the set of linear combinations of the u_i is also dense in this space. This statement, of course, implies that the sequence $\{u_n\}$ is complete in $L_2(0, 1)$.

4. **Convergence of the Fourier** $\{u_n\}$ **expansion.** Since $\{u_n\}$ is a complete orthonormal sequence in $L_2(0, 1)$, each $f \in L_2(0, 1)$ has the norm-convergent Fourier expansion

$$f(x) \sim \sum c_k u_k(x) \tag{1}$$

where

$$c_k = \int_0^1 f u_k\, dx.$$

We next investigate the pointwise convergence of this expansion.

THEOREM 2. *The Fourier* $-u_n$ *expansion of* $f \in L_2(0, 1)$ *converges to* $f(x)$ *at each point* $x \in [0, 1]$ *at which* f *is continuous.*

Proof. Let $S_N(x, f)$ denote the Nth partial sum of (1). Since each u_i is a linear combination of the h_k, $k \leqq i$, S_N itself is a linear combination of the h_i, $i \leqq N$, and thus is a continuous broken line function which is linear on each subinterval of the partition of $[0, 1]$ determined by the points of $A_N = \{a_1, a_2, \ldots, a_N\}$. Suppose $0 < a_{i_1} < a_{i_2} < \cdots < a_{i_N} < 1$ are the points of A_N arranged in order of magnitude and let $K_0, K_1, \ldots, K_N$ denote the characteristic functions of the intervals $[0, a_{i_1})$, $[a_{i_1}, a_{i_2}), \ldots, [a_{i_N}, 1]$. Then

$$S_N(x, f) = \sum_0^N c_i u_i = \sum_0^N (\alpha_i + \beta_i h_i) K_i$$

where the α's and β's are constants. To determine α_i and β_i we use the well-known fact that if T_N is any linear combination of the u_i, $i \leqq N$, $\int_0^1 (f - T_N)^2\, dx$ assumes its minimum value when $T_N = S_N$. Thus α_i and β_i must have values which minimize

$$\int_0^1 \left[f - \sum_0^N (\alpha_i + \beta_i h_i) K_i \right]^2 dx$$

and when the partial derivatives of this integral with respect to α_m and β_m are equated to 0, one has for each $m = 0, 1, 2, \ldots, N$,

$$\int_0^1 \left[f - \sum_0^N (\alpha_i + \beta_i h_i) K_i \right] K_m\, dx = 0, \tag{2}$$

$$\int_0^1 \left[f - \sum_0^N (\alpha_i + \beta_i h_i) K_i \right] K_m h_m\, dx = 0. \tag{3}$$

Now if $I = [a_{i_m}, a_{i_{m+1}})$, we obtain from (2) and (3) respectively

$$\int_I f\, dx = \alpha_m |I| + \beta_m \frac{|I|^2}{2} \tag{4}$$

and

$$\int_I f h_m\, dx = \alpha_m \frac{|I|^2}{2} + \beta_m \frac{|I|^3}{3}. \tag{5}$$

Thus

$$\alpha_m = \frac{2}{|I|^2} \int_I (2|I| - 3h_m) f\, dx$$

and

$$\beta_m = \frac{6}{|I|^3} \int_I (2h_m - |I|) f\, dx.$$

Since

$$\int_I (2|I| - 3h_m)\, dx = \frac{|I|^2}{2}$$

and

$$\int_I (2h_m - |I|)\, dx = 0,$$

we have if $x_0 \in I$,

$$\begin{aligned} |S_N(x_0, f) - f(x_0)| &= |\alpha_m + \beta_m h_m(x_0) - f(x_0)| \\ &= \left| \frac{2}{|I|^2} \int_I [2|I| - 3h_m(x)][f(x) - f(x_0)]\, dx \right. \\ &\qquad \left. + \frac{6h_m(x_0)}{|I|^3} \int_I [2h_m(x) - |I|][f(x) - f(x_0)]\, dx \right|. \end{aligned} \tag{6}$$

If x_0 is a point of continuity of f, there exists a positive number δ such that $|f(x)-f(x_0)|<\varepsilon$ when $x \in I$ and $|I|<\delta$. By Lemma 1 there is an integer N_δ such that if $N>N_\delta$, $|I|<\delta$ and since $|h_m(x)| \leqq |I|$ when $x \in I$, we find from (6) if $N>N_\delta$,

$$|S_N(x_0,f)-f(x_0)| < 10\varepsilon+18\varepsilon = 28\varepsilon.$$

THEOREM 3. *If f is continuous on* [0, 1], *the Fourier* $-u_n$ *expansion* (1) *converges uniformly to* $f(x)$ *on* [0, 1].

Proof. If $\varepsilon>0$, there exists a $\delta>0$ such that $|f(x_1)-f(x_2)|<\varepsilon/28$ when $x_1, x_2 \in [0, 1]$ and $|x_1-x_2|<\delta$. By Lemma 1 we can choose an integer N_δ such that if $N>N_\delta$, the norm of the partition of [0, 1] determined by A_N is less than δ. Then from equation (6) of the preceding proof we see that $|S_N(x_0,f)-f(x_0)|<\varepsilon$ for any $x_0 \in [0, 1]$.

In closing it should be pointed out that if the set A involved in the definition of $\{u_n\}$ is taken to be the particular set described in §7(B) of [2], the resulting $\{u_n\}$ is the orthonormal sequence of functions defined by Franklin [1].

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27607