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CURRENT VALUED MEASURES AND GEÖCZE AREA[1]

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Abstract. If f is a continuous mapping of finite Geöcze area from a polyhedral region $X \subset R^k$ into R^n, $2 \leqq k \leqq n$, then, under suitable hypotheses, one can associate with f, by means of the Cesari-Weierstrass integral, a current valued measure T over the middle space of f. In particular, if either $k=2$ or the $k+1$-dimensional Hausdorff measure of $f(X)$ is zero, then T is essentially the same as a current valued measure defined by H. Federer and hence serves to describe the tangential properties of f and the multiplicities with which f assumes its values. Further, the total variation of T is equal to the Geöcze area of f.

1. **Introduction.** Suppose f is a continuous mapping of finite Geöcze area, $V(f)$, from a polyhedral region $X \subset R^k$ into R^n, $2 \leqq k \leqq n$. If f belongs to the class $\mathscr{T}^*(k, n)$ defined by T. Nishiura [13], then (Theorem 1) we can associate with f, by means of the Cesari-Weierstrass integral, a current valued measure T over the middle space of f.

Suppose $\{f_i\}$ is a sequence of quasi-linear maps of X into R^n converging uniformly to f with bounded areas and let $f = l \circ m$ be the monotone-light factorization of f with middle space M. With each f_i we associate a current valued measure T_i over M defined by letting

$$T_i(g)(\varphi) = f_{i\#}[X \wedge (g \circ m)](\varphi) = \int_X (g \circ m) f_i^{\#}(\varphi)$$

whenever g is a continuous real valued function on M and φ is a differential k-form of class ∞ on R^n.

If $f \in \mathscr{T}^*(k, n)$ and there is a sequence $\{f_i\}$ of quasi-linear maps as above such that the sequence $\{T_i\}$ converges weakly to T, then we show in Theorem 3 that T is essentially the same as the current valued measure considered by H. Federer [7] and, in particular, shares its representation as the indefinite integral with respect to k-dimensional Hausdorff measure over M of a k-vector valued density v which describes the tangential properties of f and the multiplicity with which f assumes its values. Further, the total variation measure $\|T\|$ of T, taken with respect to mass, is equal to the Geöcze area measure μ induced on M by f.

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Let $\Lambda(k, n)$ denote the set of all k-tuples $\lambda=(\lambda_1, \ldots, \lambda_k)$ of integers such that $1 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_k \leqq n$. With each $\lambda \in \Lambda(k, n)$ associate the projection $p^\lambda: R^n \to R^k$ defined by

$$p^\lambda(y) = (y_{\lambda_1}, \ldots, y_{\lambda_k}) \quad \text{for } y = (y_1, \ldots, y_n) \in R^n.$$

In Theorem 4 we show that, if $f \in \mathscr{T}^*(k, n)$ and $\{f_i\}$ is any sequence of quasi-linear maps of X into R^n converging uniformly to f and such that

$$V(p^\lambda \circ f) = \lim_{i \to \infty} V(p^\lambda \circ f_i) \quad \text{for } \lambda \in \Lambda(k, n),$$

then the sequence $\{T_i\}$ of associated current valued measures converges weakly to T.

If $V(f) < \infty$ and either $k=2$ or $H_n^{k+1}(f(X))=0$, where H_n^m is m-dimensional Hausdorff measure in R^n, we note (Theorem 5) that the hypotheses of Theorem 3 are satisfied. In a subsequent paper we will show that, if either $k=2$ or $H_n^{k+1}(f(X))=0$, then the Lebesgue area and the Geöcze area of f coincide. From this one infers readily that T coincides with the current valued measure defined in [10] and that Theorem 4 represents a strengthening of the main result of [10].

In Theorem 2 we show that the current valued measure T associated with $f \in \mathscr{T}^*(k, n)$ possesses a representation as the indefinite integral with respect to Geöcze area measure over M of a k-vector valued function θ. In case either $k=2$ or $H_n^{k+1}(f(X))=0$, we show in Theorem 7 that $\theta(z)=v(z)/|v(z)|$ for μ almost every $z \in M$. Here $|v(z)|$ denotes the Euclidean norm of the k-vector $v(z)$.

2. **The current valued measure T.** Suppose k and n are integers, $2 \leqq k \leqq n$. Let $\Lambda(k, n)$ denote the set of all k-tuples $\lambda=(\lambda_1, \ldots, \lambda_k)$ of integers such that $1 \leqq \lambda_1 < \cdots < \lambda_k \leqq n$. Let $e_1, \ldots, e_n$ be the usual basis in R^n. Then, denoting exterior multiplication by $\wedge$, the k-vectors $e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_k}$, $\lambda \in \Lambda(k, n)$, form the usual basis for the space $\bigwedge_k (R^n)$ of k-vectors in R^n.

For each $\lambda \in \Lambda(k, n)$, let $p^\lambda: R^n \to R^k$ be defined by

$$p^\lambda(y) = (y_{\lambda_1}, \ldots, y_{\lambda_k}) \quad \text{for } y = (y_1, \ldots, y_n) \in R^n.$$

Consider each p^λ as projecting R^n onto the k-dimensional coordinate hyperspace of R^n determined by $e_{\lambda_1}, \ldots, e_{\lambda_k}$.

Suppose f is a continuous mapping of finite Geöcze area $V(f)$ from a polyhedral region $X \subset R^k$ into R^n. With each simple polyhedral region $\pi \subset X$ we associate the k-vector

$$u(f, \pi) = \sum_{\lambda \in \Lambda(k,n)} u(f^\lambda, \pi) e_\lambda$$

where $u(f^\lambda, \pi) = \int_{R^k} O(f^\lambda, \pi, y)\, dy$ for $\lambda \in \Lambda(k, n)$. Here $f^\lambda = p^\lambda \circ f$ and $O(f^\lambda, \pi, y)$ denotes the topological index of $y \in R^k$ with respect to the mapping $f^\lambda | \pi: \pi \to R^k$ if $y \in R^k - f^\lambda(\text{Bdry } \pi)$ and $O(f^\lambda, \lambda, y)=0$ if $y \in f^\lambda(\text{Bdry } \pi)$.

Let $\mathscr{G}$ denote the set of all finite collections G of nonoverlapping simple polyhedral regions $\pi\subset X$. Define on $\mathscr{G}$ a real valued function δ by

$$\delta(G) = \max\{\operatorname{diam} f(\pi) : \pi \in G\} + \max\left\{V(f^\lambda) - \sum_{\pi\in G} |u(f^\lambda, \pi)| : \lambda \in \Lambda(k, n)\right\}.$$

If $f \in \mathscr{T}^*(k, n)$, that is, if $\inf\{\delta(G) : G \in \mathscr{G}\}=0$, then, according to [13, §5.9], $u(f, \pi)$ is quasi-additive with respect to δ and mass $(\|\cdot\|)$ in $\bigwedge_k (R^n)$.

Let $f=l \circ m$ denote the monotone-light factorization of f with middle space M. Let d_f denote the usual metric induced on M by f and let $C(M)$ denote the space of continuous real valued functions on M.

THEOREM 1. *If $f \in \mathscr{T}^*(k, n)$, then, for each $g \in C(M)$ and continuous differential k-form φ in R^n,*

$$(*) \qquad T(g)(\varphi) = \lim_{\delta(G)\to 0} \sum_{\pi\in G} \frac{1}{|\pi|} \int_\pi g(m(x))\varphi(f(x))\cdot u(f, \pi)\, dx$$

exists. Here $|\pi|$ denotes k-dimensional Lebesgue measure of π and $\varphi(f(x))\cdot u(f, \pi)$ denotes evaluation of the k-covector $\varphi(f(x))$ at $u(f, \pi)$.

Proof. The essential elements of a proof of the above statement can be found in [13] and [3]. We will give a complete proof here in our present notation.

Suppose $g \in C(M)$, φ is a continuous k-form on R^n, and $\varepsilon>0$.

For any pair of simple polyhedral regions π', $\pi\subset X$ let $s(\pi', \pi)=1$ if $\pi'\subset\pi$ and $s(\pi', \pi)=0$ otherwise. According to [13, §5.9] there is a $\delta>0$ such that, if $G \in \mathscr{G}$ with $\delta(G)<\delta$, then there is a $\gamma>0$ such that

$$\sum_{\pi\in G} \left\| u(f, \pi) - \sum_{\pi_1\in G_1} s(\pi_1, \pi) u(f, \pi_1)\right\| < \varepsilon$$

and

$$\sum_{\pi_1\in G_1} \left[1 - \sum_{\pi\in G} s(\pi_1, \pi)\right] \|u(f, \pi_1)\| < \varepsilon$$

whenever $G_1 \in \mathscr{G}$ with $\delta(G_1)<\gamma$.

Suppose G and G_1 are as above. Since, for any simple polyhedral region $\pi\subset X$, the diameter of $m(\pi)$ relative to d_f does not exceed the diameter of $f(\pi)$, we can assume that $\delta(G)$ is so small that

$$\sup\{\|g(m(x))\varphi(f(x)) - g(m(x'))\varphi(f(x'))\| : x, x' \in \pi\} < \varepsilon$$

for all $\pi \in G$. Here $\|\cdot\|$ denotes comass in the space $\bigwedge^k (R^n)$ of k-covectors in R^n.

Letting

$$M(g) = \sup\{|g(z)| : z \in M\}$$

and

$$M(\varphi \circ f) = \sup\{\|\varphi(f(x))\| : x \in X\},$$

we have

$$\begin{aligned}
&\left|\sum_{\pi\in G}\frac{1}{|\pi|}\int_\pi g(m(x))\varphi(f(x))\cdot u(f,\pi)\,dx-\sum_{\pi_1\in G_1}\frac{1}{|\pi_1|}\int_{\pi_1} g(m(x))\varphi(f(x))\cdot u(f,\pi_1)\,dx\right|\\
&\quad\leqq \sum_{\pi\in G}\left|\frac{1}{|\pi|}\int_\pi g(m(x))\varphi(f(x))\cdot u(f,\pi)\,dx\right.\\
&\qquad\qquad\left.-\sum_{\pi_1\in G_1}\frac{s(\pi_1,\pi)}{|\pi_1|}\int_{\pi_1} g(m(x))\varphi(f(x))\cdot u(f,\pi_1)\,dx\right|\\
&\qquad+\sum_{\pi_1\in G_1}\left[1-\sum_{\pi\in G}s(\pi_1,\pi)\right]\left|\frac{1}{|\pi_1|}\int_{\pi_1} g(m(x))\varphi(f(x))\cdot u(f,\pi_1)\,dx\right|\\
&\quad\leqq \sum_{\pi\in G}\left|\frac{1}{|\pi|}\int_\pi g(m(x))\varphi(f(x))\cdot\left\{u(f,\pi)-\sum_{\pi_1\in G_1}s(\pi_1,\pi)u(f,\pi_1)\right\}dx\right|\\
&\qquad+\sum_{\pi\in G}\sum_{\pi_1\in G_1}s(\pi_1,\pi)\left|\frac{1}{|\pi|}\int_\pi g(m(x))\varphi(f(x))\cdot u(f,\pi_1)\,dx\right.\\
&\qquad\qquad\left.-\frac{1}{|\pi_1|}\int_{\pi_1} g(m(x))\varphi(f(x))\cdot u(f,\pi_1)\,dx\right|\\
&\qquad+M(g)M(\varphi\circ f)\sum_{\pi_1\in G_1}\left[1-\sum_{\pi\in G}s(\pi_1,\pi)\right]\|u(f,\pi_1)\|\\
&\quad\leqq M(g)M(\varphi\circ f)\sum_{\pi\in G}\left\|u(f,\pi)-\sum_{\pi_1\in G_1}s(\pi_1,\pi)u(f,\pi_1)\right\|\\
&\qquad+\varepsilon\sum_{\pi_1\in G_1}\|u(f,\pi_1)\|+\varepsilon M(g)M(\varphi\circ f)\\
&\quad\leqq\left[2M(g)M(\varphi\circ f)+\binom{n}{k}V(f)\right]\varepsilon,
\end{aligned}$$

because

$$\sum_{\pi_1\in G_1}\|u(f,\pi_1)\|\leqq\binom{n}{k}\sum_{\pi_1\in G_1}|u(f,\pi_1)|\leqq\binom{n}{k}V(f).$$

Since the above holds for any $G_1\in\mathscr{G}$ with $\delta(G_1)<\gamma$, the theorem follows.

The formula (*) above defines a linear mapping of $C(M)$ into the space $E_k(R^n)$ of k-dimensional currents in R^n. (The notation concerning forms and currents is that of [9].)

Since, for $g\in C(M)$ and $\varphi\in E^k(R^n)$,

$$|T(g)(\varphi)|\leqq\binom{n}{k}M(g)M(\varphi\circ f)V(f),$$

this mapping possesses a unique extension, also denoted by T, to the class of all bounded Borel measurable real valued functions on M such that Lebesgue's bounded convergence theorem holds. In particular, one obtains a countably additive current valued function T on the class of all Borel subsets of M which will be referred to as the current valued measure associated with f.

Assume throughout the remainder of this section that $f \in \mathscr{T}^*(k, n)$ so that the current valued measure T is defined.

For each simple polyhedral region $\pi \subset X$ and $\lambda \in \Lambda(k, n)$ let

$$u^+(f^\lambda, \pi) = \tfrac{1}{2}[|u(f^\lambda, \pi)| + u(f^\lambda, \pi)]$$

and

$$u^-(f^\lambda, \pi) = \tfrac{1}{2}[|u(f^\lambda, \pi)| - u(f^\lambda, \pi)].$$

For each set U open in X let

$$V^\pm(f^\lambda | U) = \sup \sum_{\pi \in S} u^\pm(f^\lambda, \pi)$$

where the supremum is taken over all finite collections, S, of nonoverlapping simple polyhedral regions $\pi \subset U$.

According to [13, §6] the function defined for all Borel sets $B \subset M$ by

$$\mu(B) = \inf\{V(f | m^{-1}(A)) : A \text{ open in } M \text{ and } A \supset B\}$$

is a finite Borel measure over M, and finite Borel measures μ^λ, $\mu^\lambda_\pm$ over M can be defined analogously using the functions $V(f^\lambda | \cdot)$ and $V^\pm(f^\lambda | \cdot)$.

We note from [13, §6] that

$$\mu(M) = V(f) = \lim_{\delta(G)\to 0} \sum_{\pi \in G} |u(f, \pi)|,$$

$$\mu^\lambda = \mu^\lambda_+ + \mu^\lambda_- \quad \text{for } \lambda \in \Lambda(k, n), \qquad \mu(m(\text{Bdry } X)) = 0$$

and that

$$\mu(m(\pi)) = \mu([m(\pi)]^\circ) = V(f | \pi)$$

for each simple polyhedral region $\pi \subset X$. Here $[m(\pi)]^\circ$ denotes interior of $m(\pi)$ relative to M.

For $\lambda \in \Lambda(k, n)$, let $\nu^\lambda = \mu^\lambda_+ - \mu^\lambda_-$ and define a k-vector valued measure over M by $\nu = \sum_{\lambda \in \Lambda(k,n)} \nu^\lambda e_\lambda$. Then, for any simple polyhedral region $\pi \subset X$,

$$\begin{aligned}|u(f^\lambda, \pi) - \nu^\lambda(m(\pi))| &\leqq |u^+(f^\lambda, \pi) - \mu^\lambda_+(m(\pi))| + |u^-(f^\lambda, \pi) - \mu^\lambda_-(m(\pi))| \\ &\leqq \mu^\lambda(m(\pi)) - |u(f^\lambda, \pi)|.\end{aligned}$$

Let θ^λ denote the Radon-Nikodym derivative of ν^λ with respect to μ for $\lambda \in \Lambda(k, n)$ and let $\theta = \sum_{\lambda \in \Lambda(k,n)} \theta^\lambda e_\lambda$. Then, by [3, 5ii], $|\theta(z)| = 1$ for μ almost every $z \in M$, where $|\cdot|$ denotes the Euclidean norm on $\bigwedge_k (R^n)$.

THEOREM 2. *For $g \in C(M)$ and $\varphi \in E^k(R^n)$,*

$$T(g)(\varphi) = \int_M g(z)\varphi(l(z)) \cdot \theta(z)\, d\mu.$$

Proof. The essential elements of a proof of this statement can be found in [3] and [13]. Because of the difference in viewpoint and notation, a short proof is included here.

Suppose $g \in C(M)$, $\varphi \in E^k(R^n)$, and $\varepsilon > 0$. Let $\psi(z) = g(z)\varphi(l(z))$ for $z \in M$. Then ψ is a continuous k-covector valued function on M. Since, for any polyhedral region $\pi \subset X$, the diameter of $m(\pi)$ does not exceed the diameter of $f(\pi)$ we have

$$\max_{\pi \in G} \sup \{\|\psi(m(x)) - \psi(m(x'))\| : x', x \in \pi\} < \varepsilon$$

for $G \in \mathscr{G}$ with $\delta(G)$ sufficiently small.

For such a G consider

$$\Delta = \left| \sum_{\pi \in G} \frac{1}{|\pi|} \int_\pi \psi(m(x)) \cdot u(f, \pi)\, dx - \int_M \psi(z) \cdot \theta(z)\, d\mu \right|.$$

Since $\mu(m(\pi)) = \mu([m(\pi)]^\circ)$ for $\pi \in G$ we have

$$\begin{aligned}
\Delta &\leqq \sum_{\pi \in G} \left| \frac{1}{|\pi|} \int_\pi \psi(m(x)) \cdot u(f, \pi)\, dx - \int_{m(\pi)} \psi(z) \cdot \theta(z)\, d\mu \right| \\
&\quad + \left| \int_{M - \bigcup_{\pi \in G} m(\pi)} \psi(z) \cdot \theta(z)\, d\mu \right| \\
&\leqq \varepsilon \sum_{\pi \in G} \|u(f, \pi)\| + M(\psi) \sum_{\pi \in G} \|u(f, \pi) - \nu(m(\pi))\| \\
&\quad + \varepsilon \sum_{\pi \in G} \|\nu(m(\pi))\| + M(\psi) \binom{n}{k} \mu\left(M - \bigcup_{\pi \in G} m(\pi)\right) \\
&\leqq 2\varepsilon \binom{n}{k} V(f) + M(\psi) \sum_{\lambda \in \Lambda(k,n)} \sum_{\pi \in G} |u(f^\lambda, \pi) - \nu^\lambda(m(\pi))| \\
&\quad + M(\psi) \binom{n}{k} \left[\mu(M) - \sum_{\pi \in G} \mu(m(\pi))\right] \\
&\leqq 2\varepsilon \binom{n}{k} V(f) + M(\psi) \sum_{\lambda \in \Lambda(k,n)} \left[V(f^\lambda) - \sum_{\pi \in G} |u(f^\lambda, \pi)|\right] \\
&\quad + M(\psi) \binom{n}{k} \left[V(f) - \sum_{\pi \in G} |u(f, \pi)|\right],
\end{aligned}$$

and the theorem follows.

By Lebesgue's bounded convergence theorem the conclusion of the above theorem remains valid if g is any bounded Borel measurable function on M.

COROLLARY. $\mu \leqq \|T\| \leqq \binom{n}{k}\mu$.

Proof. If B is a Borel set in M and $\varphi \in E^k(R^n)$, then

$$\begin{aligned}
|T(B)(\varphi)| &= \left| \int_B \varphi(l(z)) \cdot \theta(z)\, d\mu \right| \\
&\leqq M(\varphi) \int_B \|\theta(z)\|\, d\mu \leqq M(\varphi) \binom{n}{k} \mu(B).
\end{aligned}$$

Thus $\|T\| \leqq \binom{n}{k}\mu$.

Suppose A is open in M. For each $G \in \mathscr{G}$ let

$$G(A) = G \cap \{\pi : \pi \subset m^{-1}(A)\}.$$

Then, by [13, §5.7],

$$\mu(A) = \lim_{\delta(G)\to 0} \sum_{\pi\in G(A)} |u(f, \pi)|.$$

Since

$$\begin{aligned}\sum_{\pi\in G} |u(f, \pi)-\nu(m(\pi))| &\leqq \sum_{\lambda\in\Lambda(k,n)} \sum_{\pi\in G} |u(f^\lambda, \pi)-\nu^\lambda(m(\pi))| \\ &\leqq \sum_{\lambda\in\Lambda(k,n)} \left[V(f^\lambda)-\sum_{\pi\in G} |u(f^\lambda, \pi)|\right] \leqq \binom{n}{k}\delta(G),\end{aligned}$$

and $\mu(m(\pi))=\mu([m(\pi)]^\circ)$ for $\pi\in G$, we have

$$\mu(A) = \lim_{\delta(G)\to 0} \sum_{\pi\in G(A)} |\nu(m(\pi))| \leqq \limsup_{\delta(G)\to 0} \sum_{\pi\in G(A)} M(T([m(\pi)]^\circ)) \leqq \|T\|(A).$$

If B is a Borel set in M and A is open with $B\subset A$ then

$$\mu(B)-\|T\|(B) \leqq \mu(A)-\|T\|(B) \leqq \|T\|(A-B) \leqq \binom{n}{k}\mu(A-B).$$

3. Representation.

THEOREM 3. *Suppose $f\in\mathscr{T}^*(k, n)$ and T is the current valued measure associated with f.*

If there is a sequence $\{f_i\}$ of quasi-linear maps $f_i\colon X\to R^n$ converging uniformly to f with bounded areas and such that

$$T(g)(\varphi) = \lim_{i\to\infty} f_{i\#}[X \wedge (g\circ m)](\varphi)$$

for all $g\in C(M)$ and $\varphi\in E^k(R^n)$, then

1. *T is rectifiable current valued.*

2. *There is a Baire function $v\colon M\to\bigwedge_k(R^n)$ such that, for $\|T\|$ almost every $z\in M$, $v(z)$ is a simple k-vector, $|v(z)|$ is an integer,*

$$\varphi(l(z))\cdot v(z) = \lim_{r\to 0+} \frac{T(\Delta(z, r))(\varphi)}{\alpha(k)r^k} \quad \textit{for } \varphi\in E^k(R^n),$$

and

$$|v(z)| = \lim_{r\to 0+} \frac{\|T\|(\Delta(z, r))}{\alpha(k)r^k},$$

where $\alpha(k)$ is the k-dimensional Lebesgue measure of $\{x\in R^k : |x|\leqq 1\}$ and, for $r>0$, $\Delta(z, r)$ is the component of $l^{-1}(\{y : |y-l(z)|<r\})$ that contains z.

3. *For each Borel set $B\subset M$,*

$$\begin{aligned}T(B)(\varphi) &= \int_B \varphi(l(z))\cdot v(z)\, dH_f^k \\ &= \int_{R^n} \varphi(y)\cdot\Big\{\sum_{z\in l^{-1}(y)\cap B} v(z)\Big\}\, dH_n^k \quad \textit{for } \varphi\in E^k(R^n),\end{aligned}$$

and

$$\|T\|(B) = \int_B |v(z)|\, dH_f^k = \int_{R^n} \left(\sum_{z \in l^{-1}(y) \cap B} |v(z)| \right) dH_n^k$$

where H_f^k and H_n^k denote k-dimensional Hausdorff measure in M and R^n respectively.

Proof. Let F denote the class of Borel sets $B \subset M$ for which $T(B)$ is rectifiable. Then F is closed under countable disjoint union and proper subtraction.

Using the arguments of [7, §§3.2 and 3.4] one shows that $A \in F$ whenever A is an open subset of M with $A \cap m(\text{Bdry } X) = \varnothing$. Since

$$\|T\|(m(\text{Bdry } X)) \leqq \binom{n}{k} \mu(m(\text{Bdry } X)) = 0,$$

statement 1 follows.

From the hypothesis that, for $g \in C(M)$, $T(g)$ is the weak limit of the $f_{i\#}[X \wedge (g \circ m)]$ it follows readily that, for any open $V \subset M$ with $V \cap m(\text{Bdry } X) = \varnothing$, we have spt $\partial T(V) \subset l(\text{Bdry } V)$. This condition, together with $\|T\|(m(\text{Bdry } X)) = 0$ allows one to use, with obvious modifications, the arguments of [7, §§2.1 and 2.2] to prove statements 2 and 3.

We now investigate conditions under which the hypotheses of Theorem 3 are satisfied.

Suppose that f is a continuous mapping from a polyhedral region $X \subset R^k$ into R^k with $V(f) < \infty$.

For each polyhedral region $\sigma \subset X$ and $y \in R^k$ let

$$N(f, \sigma, y) = \sup \sum_{\pi \in S} |O(f, \pi, y)|$$

and

$$N^{\pm}(f, \sigma, y) = \sup \sum_{\pi \in S} O^{\pm}(f, \pi, y)$$

where the suprema are taken over all finite collections S of nonoverlapping simple polyhedral regions $\pi \subset \sigma$. Then for each σ the functions $N(f, \sigma, y)$, $N^{\pm}(f, \sigma, y)$ are nonnegative, integer valued, lower semicontinuous functions on R^k, $V(f|\sigma) = \int_{R^k} N(f, \sigma, y)\, dy$ and $N(f, \sigma, y) = N^+(f, \sigma, y) + N^-(f, \sigma, y)$ for almost all $y \in R^k$.

For each σ let

$$\begin{aligned} n(f, \sigma, y) &= N^+(f, \sigma, y) - N^-(f, \sigma, y) \quad \text{if } N(f, \sigma, y) < \infty, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

For each simple polyhedral region $\pi \subset X$ let

$$v(f, \pi) = \int_{R^k} |O(f, \pi, y)|\, dy.$$

Lemma. *Suppose $f: X \to R^k$ as above and $\{f_i\}$ is a sequence of continuous mappings $f_i: X \to R^k$ converging uniformly to f with $V(f_i) < \infty$ for $i = 1, 2, \ldots$.*

If G is any finite collection of nonoverlapping simple polyhedral regions $\pi \subset X$, then

(a)
$$\limsup_{i\to\infty} \sum_{\pi\in G} |u(f,\pi)-u(f_i,\pi)| \leqq 2\Big[V(f)-\sum_{\pi\in G} v(f,\pi)\Big]+2\limsup_{i\to\infty}[V(f_i)-V(f)],$$

and

(b)
$$\limsup_{i\to\infty}\Big[V(f_i)-\sum_{\pi\in G} v(f_i,\pi)\Big] \leqq V(f)-\sum_{\pi\in G} v(f,\pi)+\limsup_{i\to\infty}[V(f_i)-V(f)].$$

Proof. For $\pi \in G$, consider

$$\begin{aligned}|u(f,\pi)-u(f_i,\pi)| \leqq{}& \left|u(f,\pi)-\int_{R^k} n(f,\pi,y)\,dy\right| \\ &+\left|\int_{R^k} (n(f,\pi,y)-n(f_i,\pi,y))\,dy\right| \\ &+\left|\int_{R^k} n(f_i,\pi,y)\,dy-u(f_i,\pi)\right|.\end{aligned}$$

We have

$$\begin{aligned}&\left|u(f,\pi)-\int_{R^k} n(f,\pi,y)\,dy\right| = \left|\int_{R^k} (O(f,\pi,y)-n(f,\pi,y))\,dy\right| \\ &\leqq \int_{R^k} (N^+(f,\pi,y)-O^+(f,\pi,y))\,dy+\int_{R^k} (N^-(f,\pi,y)-O^-(f,\pi,y))\,dy \\ &= V(f|\pi)-v(f,\pi),\end{aligned}$$

and, similarly,

$$\left|u(f_i,\pi)-\int_{R^k} n(f_i,\pi,y)\,dy\right| \leqq V(f_i|\pi)-v(f_i,\pi).$$

Let

$$B^\pm(\pi,i) = \{y\in R^k : N^\pm(f,\pi,y) > N^\pm(f_i,\pi,y)\}.$$

Then

$$\begin{aligned}&\left|\int_{R^k} (n(f,\pi,y)-n(f_i,\pi,y))\,dy\right| \\ &\quad\leqq \int_{R^k} |N^+(f,\pi,y)-N^+(f_i,\pi,y)|\,dy+\int_{R^k} |N^-(f,\pi,y)-N^-(f_i,\pi,y)|\,dy \\ &\quad\leqq \int_{B^+(\pi,i)} N^+(f,\pi,y)\,dy+\int_{B^-(\pi,i)} N^-(f,\pi,y)\,dy \\ &\qquad+\int_{R^k-B^+(\pi,i)} (N^+(f_i,\pi,y)-N^+(f,\pi,y))\,dy \\ &\qquad+\int_{R^k-B^-(\pi,i)} (N^-(f_i,\pi,y)-N^-(f,\pi,y))\,dy \\ &\quad\leqq 2\int_{B^+(\pi,i)} N^+(f,\pi,y)\,dy+2\int_{B^-(\pi,i)} N^-(f,\pi,i)+V(f_i|\pi)-V(f|\pi).\end{aligned}$$

Thus

$$\sum_{\pi\in G}|u(f,\pi)-u(f_i,\pi)| \leqq V(f_i)-\sum_{\pi\in G} v(f,\pi)+V(f_i)-\sum_{\pi\in G} v(f_i,\pi)$$
$$+2\sum_{\pi\in G}\left\{\int_{B^+(\pi,i)} N^+(f,\pi,y)\,dy+\int_{B^-(\pi,i)} N^-(f,\pi,y)\,dy\right\}.$$

By [12, §2.2], $v(f,\pi)\leqq\liminf_{i\to\infty} v(f_i,\pi)$ and $N^\pm(f,\pi,y)\leqq\liminf_{i\to\infty} N^\pm(f_i,\pi,y)$ for $\pi\in G$ and $y\in R^k$. Thus, given $\varepsilon>0$, we have $\sum_{\pi\in G} v(f,\pi)-\varepsilon<\sum_{\pi\in G} v(f_i,\pi)$ for i sufficiently large. Hence

$$V(f_i)-\sum_{\pi\in G} v(f_i,\pi) < V(f_i)-V(f)+V(f)-\sum_{\pi\in G} v(f,\pi)+\varepsilon$$

and (b) is proved.

Now, for each $\pi\in G$, the functions $N^+(f,\pi,y)$ and $N^-(f,\pi,y)$ are integrable, the k-dimensional Lebesgue measure of $\bigcup_{i=1}^\infty B^\pm(\pi,i)$ is finite and

$$\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} B^\pm(\pi,j)\subset\{y : N^\pm(f,\pi,y)=\infty\}.$$

Thus, since G is finite,

$$\limsup_{i\to\infty}\sum_{\pi\in G}|u(f,\pi)-u(f_i,\pi)| \leqq 2\left[V(f)-\sum_{\pi\in G} v(f,\pi)\right]+2\limsup_{i\to\infty}[V(f_i)-V(f)]$$

and (a) is proved.

THEOREM 4. *Suppose f is a continuous mapping of a polyhedral region $X\subset R^k$ into R^n with $f\in\mathscr{T}^*(k,n)$ and let T denote the current valued measure associated with f. If $\{f_i\}$ is a sequence of quasi-linear maps $f_i: X\to R^n$ converging uniformly to f with $V(f^\lambda)=\lim_{i\to\infty} V(f_i^\lambda)$ for $\lambda\in\Lambda(k,n)$, then*

$$T(g)(\varphi) = \lim_{i\to\infty} f_{i\#}[X\wedge(g\circ m)](\varphi)$$

for all $g\in C(M)$ and $\varphi\in E^k(R^n)$.

Proof. Suppose $g\in C(M)$, $\varphi\in E^k(R^n)$, and $\varepsilon>0$. Note that

$$f_{i\#}[X\wedge(g\circ m)](\varphi) = \int_X g(m(x))\varphi(f_i(x))\cdot Jf_i(x)\,dx$$

where $Jf_i(x)=\sum_{\lambda\in\Lambda(k,n)} Jf_i^\lambda(x)e_\lambda$. Here Jf_i^λ denotes the ordinary Jacobian of f_i^λ. Also, from [1, 8.9iii] we have $u(f_i^\lambda,\pi)=\int_\pi Jf_i^\lambda(x)\,dx$ whenever π is a simple polyhedral region in X.

Let $G\in\mathscr{G}$ with $\delta(G)$ so small that

$$\left|T(g)(\varphi)-\sum_{\pi\in G}\frac{1}{|\pi|}\int_\pi g(m(x))\varphi(f(x))\cdot u(f,\pi)\,dx\right| < \varepsilon,$$

and

$$\max\{|g(m(x))\varphi(f(x))-g(m(x'))\varphi(f(x'))| : x,x'\in\pi\} < \varepsilon$$

for $\pi \in G$. Then

$$\left| \sum_{\pi \in G} \frac{1}{|\pi|} \int_\pi g(m(x))\varphi(f(x)) \cdot u(f, \pi)\, dx - \int_X g(m(x))\varphi(f_i(x)) \cdot Jf_i(x)\, dx \right|$$
$$\leq \sum_{\pi \in G} \left| \frac{1}{|\pi|} \int_\pi g(m(x))\varphi(f(x)) \cdot [u(f, \pi) - u(f_i, \pi)]\, dx \right|$$
$$+ \sum_{\pi \in G} \left| \int_\pi g(m(x))\varphi(f(x)) \cdot \left[\frac{u(f_i, \pi)}{|\pi|} - Jf_i(x)\right] dx \right|$$
$$+ \sum_{\pi \in G} \left| \int_\pi g(m(x))[\varphi(f(x)) - \varphi(f_i(x))] \cdot Jf_i(x)\, dx \right|$$
$$+ \left| \int_{X - \bigcup_{\pi \in G} \pi} g(m(x))\varphi(f_i(x)) \cdot Jf_i(x)\, dx \right|$$
$$\leq M(g)M(\varphi \circ f) \sum_{\pi \in G} \|u(f, \pi) - u(f_i, \pi)\| + \varepsilon \sum_{\pi \in G} \left\{ |u(f_i, \pi)| + \int_\pi |Jf_i(x)|\, dx \right\}$$
$$+ M(g)M(\varphi \circ f - \varphi \circ f_i) \sum_{\pi \in G} V(f_i|\pi) + M(g)M(\varphi \circ f_i) V\left(f_i \Big| X - \bigcup_{\pi \in G} \pi\right)$$
$$\leq M(g)M(\varphi \circ f) \sum_{\lambda \in \Lambda(k,n)} \sum_{\pi \in G} |u(f^\lambda, \pi) - u(f_i^\lambda, \pi)| + 2\varepsilon V(f_i)$$
$$+ M(g)M(\varphi \circ f - \varphi \circ f_i)V(f_i) + M(g)M(\varphi \circ f_i) \sum_{\lambda \in \Lambda(k,n)} \left[V(f_i^\lambda) - \sum_{\pi \in G} v(f_i^\lambda, \pi)\right].$$

By the preceding lemma we have

$$\limsup_{i \to \infty} \sum_{\pi \in G} |u(f^\lambda, \pi) - u(f_i^\lambda, \pi)| \leq 2[V(f^\lambda) - \sum_{\pi \in G} |u(f^\lambda, \pi)|] < 2\delta(G),$$

and

$$\limsup_{i \to \infty} \left[V(f_i^\lambda) - \sum_{\pi \in G} v(f_i^\lambda, \pi)\right] \leq V(f^\lambda) - \sum_{\pi \in G} |u(f^\lambda, \pi)| < \delta(G)$$

for $\lambda \in \Lambda(k, n)$.

Since $\{f_i\}$ converges uniformly to f we have

$$\lim_{i \to \infty} M(\varphi \circ f - \varphi \circ f_i) = 0$$

and the theorem follows.

REMARK. Theorem 4 was originally proved under the hypothesis that given any $\varepsilon > 0$ there is a $G \in \mathscr{G}$ such that

$$\delta(G) + \max \left\{ \mathscr{L}_k\left(f^\lambda\left(\bigcup_{\pi \in G} \text{Bdry } \pi\right)\right) : \lambda \in \Lambda(k, n) \right\} < \varepsilon.$$

Here $\mathscr{L}_k$ denotes k-dimensional Lebesgue measure in R^k. It was shown in [11, §3] that this hypothesis is satisfied if either $k=2$ or $H_n^{k+1}(f(X))=0$.

The proof of Theorem 4 under the weaker hypothesis $f \in \mathscr{T}^*(k, n)$ was suggested by T. Nishiura.

THEOREM 5. *If f is a continuous mapping from a polyhedral region $X \subset R^k$ into R^n, $2 \leqq k \leqq n$, $V(f) < \infty$, and either $k=2$ or $H_n^{k+1}(f(X))=0$, then $f \in \mathscr{T}^*(k, n)$ and there is a sequence $\{f_i\}$ of quasi-linear mappings $f_i \colon X \to R^n$ converging uniformly to f with $V(f^\lambda)=\lim_{i\to\infty} V(f_i^\lambda)$ for $\lambda \in \Lambda(k, n)$.*

Proof. That, under these conditions, $f \in \mathscr{T}^*(k, n)$ was proved in [11]. The proofs of [4, Theorems 3.16 and 5.7] consist of constructions, under the given conditions, of sequences of quasi-linear maps having the desired properties.

4. **Densities.** Throughout this section assume that $f \colon X \to R^n$ satisfies the hypotheses of Theorem 3. We will show that $\mu = \|T\|$ and describe the relation between the functions v and θ.

For each $z \in M$, let

$$\mu'(z) = \limsup_{r\to 0+} \frac{\mu(\Delta(z, r))}{\alpha(k)r^k}.$$

It is readily shown that μ' is Borel measurable.

THEOREM 6. *For each Borel set $B \subset M$, $\int_B \mu'(z)\, dH_f^k \leqq \mu(B)$.*

Proof. Using the arguments of [5, Lemma 6.1] and the definition of μ we find that, if A is a Borel subset of M and $A \subset \{z : \mu'(z) > c\}$, $c > 0$, then $\mu(A) \geqq cH_f^k(A)$. Adapting the proof of [5, Theorem 6.2], the theorem follows.

According to Theorem 3, for $\|T\|$ almost every $z \in M$, there is a simple k-covector ω with $|\omega| = 1$ such that

$$|v(z)| = \omega \cdot v(z) = \lim_{r\to 0+} \frac{T(\Delta(z, r))(\omega)}{\alpha(k)r^k}.$$

For $r > 0$,

$$T(\Delta(z, r))(\omega) = \int_{\Delta(z,r)} \omega \cdot \theta(z)\, d\mu \leqq \int_{\Delta(z,r)} |\omega|\, |\theta(z)|\, d\mu = \mu(\Delta(z, r)).$$

Thus, for $\|T\|$ almost every $z \in M$, $|v(z)| \leqq \mu'(z)$ and, hence, for any Borel set $B \subset M$,

$$\|T\|(B) = \int_B |v(z)|\, dH_f^k \leqq \int_B \mu'(z)\, dH_f^k \leqq \mu(B).$$

By the corollary to Theorem 2, $\mu = \|T\|$.

Thus, by Theorem 3,

$$1 \leqq \mu'(z) = \lim_{r\to 0+} \frac{\mu(\Delta(z, r))}{\alpha(k)r^k} < \infty$$

for μ almost every $z \in M$, and, hence

$$\lim_{r\to 0+} \frac{\mu(\Delta(z, 5r))}{\mu(\Delta(z, r))} \leqq 5^k$$

for μ almost every $z \in M$.

For $z \in M$ and $r>0$, let $c(z, r)$ denote the closure of $\Delta(z, r)$ in M and let

$$V = \{c(z, r) : z \in M, 0 < r < 1, \mu(\text{Bdry } c(z, r)) = 0, \text{ and } \mu(c(z, 5r)) < (5^k+1)\mu(\Delta(z, r))\}.$$

Note that $\inf\{r : c(z, r) \in V\}=0$ for μ almost every $z \in M$.

For $S \subset M$, let $\operatorname{diam}_f(S)$ denote the diameter of S in M, and, for $S \in V$, let

$$\hat{S} = \bigcup \{S' : S' \in V, S' \cap S \neq \varnothing, \text{ and } \operatorname{diam}_f(S') \leqq \tfrac{4}{3} \operatorname{diam}_f(S)\}.$$

If $S=c(z, r)$, $S'=c(z', r')$, $S \cap S' \neq \varnothing$, and $\operatorname{diam}_f(S') \leqq \frac{4}{3} \operatorname{diam}_f(S)$, then, for any $z'' \in c(z', r')$, we have

$$|l(z'')-l(z)| \leqq \operatorname{diam}_f(S')+\operatorname{diam}_f(S) \leqq (1+\tfrac{4}{3})(2r) < 5r,$$

and hence, since S' is connected, $S' \subset c(z, 5r)$.

Thus, for $S \in V$, we have $\mu(\hat{S})<(5^k+1)\mu(S)$. Referring to [8, Theorem 2.8.7] we have the following

LEMMA. *Suppose A is a Borel set in M, W is open in M, $F \subset V$, and*

$$\inf\{r : c(z, r) \in F\} = 0$$

for all $z \in A$. Then there exists a countable disjointed subfamily G of F such that

$$\bigcup_{S \in G} S \subset W \quad \text{and} \quad \mu\Big(A \cap W - \bigcup_{S \in G} S\Big) = 0.$$

In view of this lemma and [8, Theorem 2.9.7] we find that, for $\lambda \in \Lambda(k, n)$,

$$\gamma^\lambda_\pm(z) = \lim_{r \to 0+} \frac{\mu^\lambda_\pm(\Delta(z, r))}{\mu(\Delta(z, r))}$$

exists and is finite for μ almost every $z \in M$, and since $\mu^\lambda_\pm \leqq \mu$,

$$\mu^\lambda_\pm(B) = \int_B \gamma^\lambda_\pm(z)\, d\mu$$

for each Borel set $B \subset M$.

Thus,

$$\theta^\lambda(z) = \gamma^\lambda_+(z)-\gamma^\lambda_-(z) = \lim_{r \to 0+} \frac{\nu^\lambda(\Delta(z, r))}{\mu(\Delta(z, r))}$$

for μ almost every $z \in M$.

We summarize the results of this section in

THEOREM 7. *If $f: X \to R^n$ satisfies the hypotheses of Theorem* 3, *then $\mu=\|T\|$ and $v(z)=\mu'(z)\theta(z)$ for μ almost every $z \in M$.*

Proof. Let $\{e^\lambda : \lambda \in \Lambda(k, n)\}$ denote the basis of $\bigwedge^k (R^n)$ dual to $\{e_\lambda : \lambda \in \Lambda(k, n)\}$. For each $\lambda \in \Lambda(k, n)$,

$$e^\lambda \cdot v(z) = \lim_{r \to 0+} \frac{T(\Delta(z, r))(e^\lambda)}{\alpha(k)r^k}$$

for μ almost every $z \in M$.

Now, by Theorem 2,

$$T(\Delta(z,r))(e^{\lambda}) = \int_{\Delta(z,r)} e^{\lambda}\cdot\theta(z)\,d\mu = \nu^{\lambda}(\Delta(z,r)),$$

and, hence,

$$e^{\lambda}\cdot v(z) = \lim_{r\to 0+} \frac{\nu^{\lambda}(\Delta(z,r))}{\alpha(k)r^{k}} = \mu'(z)\theta^{\lambda}(z).$$

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