TANGENTIAL LIMITS OF FUNCTIONS ORTHOGONAL TO INVARIANT SUBSPACES

BY

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Abstract. For any inner function φ, let $M^\perp$ be the orthogonal complement of φH^2, in H^2, where H^2 is the usual Hardy space. The relationship between the tangential convergence of all functions in $M^\perp$ and the finiteness of certain sums and integrals involving φ is studied. In particular, it is shown that the tangential convergence of all functions in $M^\perp$ is a stronger condition than the tangential convergence of φ, itself.

1. In this paper we study the boundary behavior of certain functions that are holomorphic in the unit disc U. For each $m > 0$, each $\gamma \geq 1$, and each point ζ on the boundary ∂U of U, we let

$$R(m, \zeta, \gamma) = \{z : 1 - |z| \geq m|\arg(\bar{\zeta}z)|^\gamma; \ 0 < |z| < 1\},$$

where $\arg(\bar{\zeta}z)$ is always restricted to the interval $(-\pi, \pi]$. For the sake of convenience, set $R(m, \gamma) = R(m, 1, \gamma)$. Using Cargo's terminology [3], we say that a function f defined on U has a T_γ-limit at ζ if there exists a finite number L such that, for each $m > 0$, $f(z) \to L$ as $z \to \zeta$ through $R(m, \zeta, \gamma)$. Let us note that f has a T_1-limit at ζ if and only if f has a classical nontangential limit at ζ.

Let

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \bar{a}_n z},$$

where $0 < |a_n| < 1$ and $\sum (1 - |a_n|) < \infty$. B is holomorphic in U and is called a Blaschke product. In [7], results of Cargo [3] and of Ahern and Clark [1] are used to obtain

THEOREM I. *Let B be a Blaschke product with zeros $\{a_n\}$ such that*

(1)
$$\sum_{n=1}^{\infty} (1 - |a_n|^2)/|1 - \bar{\zeta}a_n|^\gamma < \infty$$

for some $\zeta \in \partial U$ and some $\gamma \geq 1$. Then for any nonnegative integer k with $k \leq \gamma - 1$, the kth derivative of B, as well as the kth derivative of any subproduct of B, has a $T_{\gamma/(k+1)}$-limit at ζ. In the case $k = 0$, the limits are of modulus one. Moreover, for each

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$m > 0$, *there exists a constant C such that the kth derivative of B, as well as the kth derivative of any subproduct of B, is bounded in modulus by C in the region $R(m, \zeta, \gamma/(k+1))$.*

Linden and Somadasa show in [5] that the converse to Theorem I with $\gamma > 1$ and $k = 0$ does not hold. This has been expanded [7] to give

THEOREM II. *Fix $\gamma > 1$. For each $t > 1$, there exists a Blaschke product B with zeros $\{a_n\}$ such that B and all its subproducts have T_γ-limits of modulus one at $\zeta = 1$, but $\sum_{n=1}^{\infty} (1 - |a_n|^2)/|1 - a_n|^t = \infty$.*

Let H^2 denote the usual Hardy class of functions holomorphic in U. A well-known theorem of Beurling (see, for example, [8, p. 342]) states that any closed invariant subspace of H^2 is of the form φH^2, where φ is an inner function. That is, φ is a bounded, holomorphic function in U whose radial limits are of modulus one at almost all points of ∂U. In particular, any Blaschke product is inner. For any inner function φ, we let $(\varphi H^2)^\perp$ be the orthogonal complement of φH^2 in H^2.

In §2, we give a result analogous to Theorem I relating $(BH^2)^\perp$ with the expression (1) for a Blaschke product B. In §3, a partial converse is obtained in contrast to Theorem II. Both of these results are shown to be best possible. §4 is devoted to extending the work of the preceding sections to the case of a general closed invariant subspace of H^2.

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2. We start with

THEOREM 1. *Let B be a Blaschke product with zeros $\{a_n\}$ such that*

$$(2) \qquad \sum_n (1 - |a_n|^2)/|1 - \zeta a_n|^{2\gamma} < \infty$$

for some $\zeta \in \partial U$ and $\gamma \geq 1$. Then for each $f \in (BH^2)^\perp$ and each nonnegative integer k with $k \leq \gamma - 1$, the kth derivative of f has a $T_{\gamma/(k+1)}$-limit at ζ.

Before beginning the proof, we note the result, which can be found in [4, Chapter VIII], that an inner function φ is regular at a point of ∂U if and only if each $f \in (\varphi H^2)^\perp$ is regular there. In particular, the conclusion of Theorem 1 holds for any finite Blaschke product. So, in our proof, we will assume that B is an infinite Blaschke product.

Proof. Put $M = BH^2$. Let $B_1 = 1$, for $n = 2, 3, \ldots$, let B_n be the subproduct of B with zeros $a_1, \ldots, a_{n-1}$, and set

$$h_n(z) = (1 - |a_n|^2)^{1/2} B_n(z)/(1 - \bar{a}_n z).$$

It is shown in [1] that $\{h_n\}$ is an orthonormal basis for $M^\perp$. Thus, replacing B by the Blaschke product with zeros, $\{\zeta a_n\}$ shows that there is no loss in generality in

assuming that $\zeta = 1$. Then, (2) implies that we can assume that arg a_n is never zero, also without loss of generality.

Let us pick any $f \in M^\perp$. Then for some $\{c_n\} \in l^2$,

$$f(z) = \sum_{n=1}^{\infty} c_n h_n(z)$$

for all $z \in U$. Fix $m > 0$ and an integer k with $0 \leq k \leq \gamma - 1$. We wish to show first that the series $\sum |h_n^{(k)}(z)|^2$ converges uniformly in $R(m, \gamma/(k+1))$.

By Leibniz's rule,

$$(3) \qquad h_n^{(k)}(z) = (1 - |a_n|^2)^{1/2} \sum_{j=0}^{k} \binom{k}{j} B_n^{(k-j)}(z) \frac{j! \, \bar{a}_n^j}{(1 - \bar{a}_n z)^{j+1}}.$$

We know from Theorem I that there is a constant C such that, for each n, $|B_n^{(k-j)}(z)| \leq C$ for all z in $R(m, 2\gamma/(k+1))$, $j = 0, 1, \ldots, k$. So,

$$|h_n^{(k)}(z)|^2 \leq C' \sum_{j=0}^{2k} (1 - |a_n|)/|1 - \bar{a}_n z|^{j+2}$$

for all $z \in R(m, 2\gamma/(k+1))$ and some constant C'. On the other hand, it is shown in [3] that (2) implies

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{w_n |\arg a_n|^{2\gamma}} < \infty$$

for some sequence $\{w_n\}$ such that $0 < w_n \leq 1$ and $w_n \to 0$ as $n \to \infty$. Fix $j = 0, 1, \ldots, 2k$, and put

$$S_{j,n} = \{z : |z - a_n| < w_n^{1/(j+2)} |\arg a_n|^{2\gamma/(j+2)}\},$$

$n = 1, 2, \ldots$. If $z \in U - S_{j,n}$, then

$$(1 - |a_n|)/|1 - \bar{a}_n z|^{j+2} \leq (1 - |a_n|)/|z - a_n|^{j+2}$$
$$\leq (1 - |a_n|)/w_n |\arg a_n|^{2\gamma}.$$

It is also shown in [3] that $R(m, \gamma/(k+1))$ intersects at most finitely many of the $S_{j,n}$. Thus,

$$|h_n^{(k)}(z)|^2 \leq (2k+1)C'(1 - |a_n|)/w_n |\arg a_n|^{2\gamma}$$

for all $z \in R(m, \gamma/(k+1))$ and all n greater than some fixed integer. Therefore, by the Weierstrass M-test, $\sum |h_n^{(k)}(z)|^2$ converges uniformly in $R(m, \gamma/(k+1))$.

We can now conclude that $\{h_n^{(k)}(1)\}$ is square summable and that $\{h_n^{(k)}(z)\}$ approaches $\{h_n^{(k)}(1)\}$ in the l^2 norm as $z \to 1$ through $R(m, \gamma/(k+1))$. But,

$$f^{(k)}(z) = \sum_{n=1}^{\infty} c_n h_n^{(k)}(z).$$

So, as $z \to 1$ through $R(m, \gamma/(k+1))$, $f^{(k)}(z)$ has a finite limit since $\{h_n^{(k)}(z)\}$ approaches $\{h_n^{(k)}(1)\}$ weakly. This completes the proof.

For any $z \in U$ and any nonnegative integer k, let $K_{z,k}$ be the function in $(BH^2)^\perp$ satisfying

$$(4) \qquad\qquad f^{(k)}(z) = (f, K_{z,k})$$

for all $f \in (BH^2)^\perp$. (For a general discussion of reproducing kernels, one can see [6].) Since $\{h_n\}$ is an orthonormal basis for $(BH^2)^\perp$, $K_{z,k} = \sum d_n h_n$ where $\|K_{z,k}\|^2 = \sum |d_n|^2$. As in [1], we see by setting $f = h_n$ in (4) that $\bar{d}_n = h_n^{(k)}(z)$. Thus, what we did in the proof above was to show that $\|K_{z,k}\|$ had a limit $\|K_{1,k}\|$ as $z \to 1$ through $R(m, \gamma/(k+1))$, $m > 0$.

The special case of Theorem 1 corresponding to $\gamma = 1$ can be found, along with its converse, in [1]. Let us now prove that Theorem 1 for general $\gamma \geqq 1$ is best possible in a natural sense.

EXAMPLE 1. *Choose a number $\gamma \geqq 1$, and let k be any nonnegative integer with $k \leqq \gamma - 1$. For each $t > \gamma/(k+1)$, there exists a Blaschke product B with zeros $\{a_n\}$ such that (2) holds with $\zeta = 1$, but $f^{(k)}$ does not have a T_t-limit at $\zeta = 1$ for each f in some dense G_δ subset of $(BH^2)^\perp$.*

Proof. We can assume without loss of generality that $t \leqq 2\gamma/(k+1)$. Let us choose a positive number α such that $1/(2k+2)\alpha < t - \gamma/(k+1)$. Then, $1 + 2\gamma\alpha < (2k+2)t\alpha$, and we can choose a number β so that $1 + 2\gamma\alpha < \beta < (2k+2)t\alpha$. Also, we require that β be chosen so that $\beta > 2kt\alpha$. Now put $\arg a_n = n^{-\alpha}$ and $|a_n| = 1 - n^{-\beta}$ for $n = 2, 3, \ldots$.

We have

$$\sum_{n=2}^{\infty} (1 - |a_n|)/|\arg a_n|^{2\gamma} = \sum_{n=2}^{\infty} 1/n^{\beta - 2\gamma\alpha} < \infty$$

since $\beta > 1 + 2\gamma\alpha$. Now (2) with $\zeta = 1$ follows since $\sin|\arg a_n| \leqq |1 - a_n|$.

Let $K_{z,k}$ be the function given by (4), and fix $m > 0$. Because of the uniform boundedness theorem, we shall be done if we can show that $\|K_{z,k}\|$ is unbounded in $R(m, t)$.

Define $z_n \in R(m, t)$ by $\arg z_n = \arg a_n$, $|z_n| = 1 - m|\arg a_n|^t$, for $n = 2, 3, \ldots$. We shall show that $|h_n^{(k)}(z_n)| \to \infty$ as $n \to \infty$, using formula (3). A simple calculation gives

$$\frac{1 - \bar{a}_n z_n}{(1 - |a_n|)^{1/(2j+2)}} = \frac{n^{-\beta} + mn^{-t\alpha} - mn^{-(\beta+t\alpha)}}{n^{-\beta/(2j+2)}},$$

and so as $n \to \infty$, $(1 - |a_n|^2)^{1/2}/(1 - \bar{a}_n z_n)^{j+1}$ approaches 0 for $j = 0, 1, \ldots, k-1$ (if $k > 0$) and approaches ∞ for $j = k$ since $2kt\alpha < \beta < (2k+2)t\alpha$. Furthermore, (2) holds and $t \leqq 2\gamma/(k+1)$. So Theorem I implies that $|B_n(z_n)| \to 1$ since $|B_n(z_n)| \geqq |B(z_n)|$, and that $|B_n^{(k-j)}(z_n)|$ remains bounded as $n \to \infty$, $j = 0, 1, \ldots, k-1$ (if $k > 0$). Thus, $B_n^{(k-j)}(z_n)(1 - |a_n|^2)^{1/2}/(1 - \bar{a}_n z_n)^{j+1}$ approaches 0 for $j = 0, 1, \ldots, k-1$ (if $k > 0$) and approaches ∞ for $j = k$. So, $|h_n^{(k)}(z_n)| \to \infty$ as $n \to \infty$. Therefore, $\|K_{z_n,k}\| \to \infty$ as $n \to \infty$.

3. We now prove a partial converse to Theorem 1 with $k=0$. Actually, we prove a little more.

THEOREM 2. *Let B be a Blaschke product with zeros $\{a_n\}$ and choose any $\zeta \in \partial U$, $\gamma \geq 1$. Then, if for some $m > 0$, each function in $(BH^2)^\perp$ is bounded in $R(m, \zeta, \gamma)$, it follows that*

$$\sum_{n=1}^{\infty} (1 - |a_n|^2)/|1 - \zeta a_n|^t < \infty$$

for all $t < \gamma + 1$.

Proof. It is clear that there is no loss of generality in assuming that $a_n \to \zeta$ as $n \to \infty$ and that $\zeta = 1$. Fix $m > 0$ such that each function in $(BH^2)^\perp$ is bounded in $R(m, \gamma)$. The uniform boundedness theorem implies that $\|K_z\|$ is bounded as z ranges over $R(m, \gamma)$, where $K_z = K_{z,0}$. We have

$$\|K_z\|^2 = \sum_{n=1}^{\infty} |B_n(z)|^2 (1 - |a_n|^2)/|1 - \bar{a}_n z|^2.$$

On the other hand, $K_z(\lambda) = (1 - \bar{B}(z)B(\lambda))/(1 - \bar{z}\lambda)$ and so

$$\|K_z\|^2 = (1 - |B(z)|^2)/(1 - |z|^2)$$

since $\|K_z\|^2 = K_z(z)$. So, $\|K_z\|$ being bounded in $R(m, \gamma)$ implies that $|B(z)| \to 1$ as $z \to 1$ through $R(m, \gamma)$. Thus, there exists a constant C such that

$$(5) \qquad \sum_{n=1}^{\infty} (1 - |a_n|^2)/|1 - \bar{a}_n z|^2 \leq C$$

for all $z \in R(m, \gamma)$. In particular, $(1 - |a_n|^2)^{-1} \leq C$ for each n with $a_n \in R(m, \gamma)$ and so, there is a number n_0 such that $1 - |a_n| < m|\arg a_n|^\gamma$ (and $|\arg a_n|^{\gamma-1} < 1/m$ if $\gamma > 1$) for all $n \geq n_0$.

Next, let us put $A = \{n : \arg a_n > 0 \text{ and } n \geq n_0\}$. Denote by ∂R^+ that part of $\partial R(m, \gamma)$ which lies in the upper half-plane. For any $n \in A$ and $z \in \partial R^+$ we have, by the law of cosines,

$$\begin{aligned} |1 - \bar{a}_n z|^2 &= (1 - |a_n| \, |z|)^2 + 4|a_n| \, |z| \sin^2 \tfrac{1}{2}(\arg a_n - \arg z) \\ &\leq (1 - |a_n| + m|\arg z|^\gamma)^2 + (\arg a_n - \arg z)^2 \\ &\leq (m|\arg a_n|^\gamma + m|\arg z|^\gamma)^2 + (\arg a_n - \arg z)^2. \end{aligned}$$

For any $n \in A$, set

$$E_n = \{u > 0 : \arg a_n - m|\arg a_n|^\gamma \leq u \leq \arg a_n\}.$$

Then whenever $z \in \partial R^+$ has $\arg z \in E_n$, we get

$$|1 - \bar{a}_n z|^2 \leq 5m^2 |\arg a_n|^{2\gamma} \qquad (n \in A).$$

Setting $u = \arg z$, we can think of $z \in \partial R^+$ as a function of u. Choose any p with $0 < p < 1$ and assume that $m \leqq \pi^{-\gamma}$. Then (5) implies

$$\int_0^\pi \sum_{n \in A} \frac{1 - |a_n|^2}{|1 - \bar{a}_n z|^2} u^{-p} \, du \leqq C\pi^{1-p}/(1-p)$$

where z is considered to be a function of u. So,

$$\sum_{n \in A} \int_{E_n} \frac{1 - |a_n|^2}{|1 - \bar{a}_n z|^2} u^{-p} \, du \leqq C\pi^{1-p}/(1-p)$$

and then

$$\sum_{n \in A} \int_{E_n} \frac{1 - |a_n|^2}{5m^2 |\arg a_n|^{2\gamma}} u^{-p} \, du \leqq C\pi^{1-p}/(1-p).$$

An elementary computation shows that there is a constant $C_0 > 0$ such that $\int_{E_n} u^{-p} \, du \geqq C_0 |\arg a_n|^{\gamma - p}$ for large enough n. So we conclude that

$$\sum_{n \in A} (1 - |a_n|^2)/|\arg a_n|^{\gamma + p} < \infty.$$

Applying the preceding work to the Blaschke product with zeros $\{\bar{a}_n\}$ and noting that $\arg a_n \neq 0$ whenever $n \geqq n_0$, we get

$$\sum_{n = n_0}^\infty (1 - |a_n|^2)/|\arg a_n|^{\gamma + p} < \infty.$$

The desired result follows since $\sin |\arg a_n| \leqq |1 - a_n|$. If $m > \pi^{-\gamma}$, integrate (5) over $(0, m^{-1/\gamma})$ instead of $(0, \pi)$. Then proceed as above.

Comparing Theorem 2 in the case $\gamma > 1$ with Theorem II, we see that the tangential limit behavior of the functions in $(BH^2)^\perp$ tells us more about the zeros of B than does the tangential limit behavior of B, itself, along with all its subproducts. We shall now prove that Theorem 2 with $\gamma > 1$ is best possible. The theorem of Ahern and Clark [1] referred to just before the last example shows that Theorem 2 can be slightly improved in the case $\gamma = 1$. This improvement also can be gotten from formula (5).

EXAMPLE 2. *For any $\gamma > 1$, there exists a Blaschke product B with zeros $\{a_n\}$ such that each function in $(BH^2)^\perp$ has a T_γ-limit at $\zeta = 1$, but*

(6) $$\sum_{n = 1}^\infty (1 - |a_n|^2)/|1 - a_n|^{\gamma + 1} = \infty.$$

Proof. For $n = 2, 3, \ldots$, let $\arg a_n = n^{-\alpha}$ and $|a_n| = 1 - n^{-\beta}$, where $\alpha = 1/(\gamma - 1)$ and $\beta = 2\gamma/(\gamma - 1)$. Then,

$$\sum_{n = 2}^\infty (1 - |a_n|)/|\arg a_n|^{\gamma + 1} = \sum_{n = 2}^\infty 1/n^{\beta - (\gamma + 1)\alpha} = \infty$$

since $\beta - (\gamma + 1)\alpha = 1$. So, (6) follows from an argument given in [3].

Fix any $m>0$ small enough that $1-m2^{-\alpha}\geq\frac{3}{4}$ and $m<\alpha/4$. Then for each $j=2,3,\ldots$, define $z_j\in R(m,\gamma)$ by $\arg z_j=\arg a_j$ and $|z_j|=1-m|\arg a_j|^\gamma$. We are going to show that the H^2 norm of K_z is bounded in $R(m,\gamma)$ by first considering $z=z_j$.

By the law of cosines,

$$|1-\bar{a}_n z_j|^2 = (1-|a_n|\,|z_j|)^2+4|a_n|\,|z_j|\sin^2\tfrac{1}{2}(\arg a_n-\arg z_j)$$
$$\geq (n^{-\beta}+mj^{-\gamma\alpha}-mj^{-\gamma\alpha}n^{-\beta})^2+\tfrac{1}{2}(n^{-\alpha}-j^{-\alpha})^2.$$

In particular, $(1-|a_j|)/|1-\bar{a}_j z_j|^2\leq 1/m^2$ since $\beta=2\gamma\alpha$. If $j<n\leq 2j$, an elementary calculation, using the factorization of x^k-y^k (k a positive integer), shows that $2^\alpha\alpha^{-1}n^{1-\alpha}(n^\alpha-j^\alpha)\geq n-j$ for all rational $\alpha>0$ and so for all $\alpha>0$. Also, note that $\beta-2\alpha=2$. So,

$$\sum_{n=j+1}^{2j}\frac{1-|a_n|}{|1-\bar{a}_n z_j|^2}\leq 2\sum_{n=j+1}^{2j}\frac{j^{2\alpha}n^{-2}}{(n^\alpha-j^\alpha)^2}\leq\frac{2^{2\alpha+1}}{\alpha^2}\sum_{n=j+1}^{2j}\frac{j^{2\alpha}n^{-2\alpha}}{(n-j)^2}$$
$$\leq\frac{2^{2\alpha+1}}{\alpha^2}\sum_{n=j+1}^{2j}1/(n-j)^2\leq\frac{2^{2\alpha+1}}{\alpha^2}\sum_{n=1}^{\infty}1/n^2.$$

Likewise, letting p be the greatest integer in $j/2$, we get

$$\sum_{n=p+1}^{j-1}(1-|a_n|)/|1-\bar{a}_n z_j|^2\leq(2^{\beta+1}/\alpha^2)\sum_{n=1}^{\infty}1/n^2.$$

Also,

$$\sum_{n=2j+1}^{\infty}\frac{1-|a_n|}{|1-\bar{a}_n z_j|^2}\leq 2\sum_{n=2j+1}^{\infty}\frac{j^{2\alpha}n^{-2}}{((2j)^\alpha-j^\alpha)^2}\leq\frac{2}{(2^\alpha-1)^2}\sum_{n=1}^{\infty}1/n^2;$$

and similarly,

$$\sum_{n=2}^{p}(1-|a_n|)/|1-\bar{a}_n z_j|^2\leq 2(1-2^{-\alpha})^{-2}\sum_{n=1}^{\infty}1/n^2.$$

Thus, we conclude that

(7)
$$\sum_{n=2}^{\infty}(1-|a_n|)/|1-\bar{a}_n z|^2$$

is bounded for $z=z_j$, where $2\leq j<\infty$.

For $j=2,3,\ldots$, let w_j be the value of $z\in R(m,\gamma)$ which minimizes $|1-\bar{a}_j z|=|a_j|\cdot|\bar{a}_j^{-1}-z|$. The existence and uniqueness of each w_j is ensured by the convexity of $R(m,\gamma)\cap\{z:\operatorname{Im}z>0\}$. Since $\partial R(m,\gamma)$ is tangent to ∂U, the law of sines implies that there is an integer n_0 such that $|1-\bar{a}_j z_j|\leq 2|1-\bar{a}_j w_j|$ for all $j\geq n_0$. Next we observe that w_j lies on ∂R^+ between z_j and 1. By l'Hospital's rule, when j is sufficiently large,

$$\frac{|1-\bar{a}_j z_j|}{|1-\bar{a}_j z_{j+1}|}\leq\frac{4mj^{-\gamma\alpha}}{j^{-\alpha}-(j+1)^{-\alpha}}=\frac{4mj^{-1}}{1-(1+1/j)^{-\alpha}}\sim\frac{4m}{\alpha}<1$$

since $\alpha(\gamma-1)=1$ and $m<\alpha/4$. So, there is an integer $n_1\geq n_0$ such that w_j lies between

z_j and z_{j+1} for all $j \geqq n_1$. Pick $z \in \partial R^+$ and suppose z lies between z_j and z_{j+1}, $j \geqq n_1$. Then the preceding work gives

$$\sum_{n=2}^{\infty} \frac{1-|a_n|}{|1-\bar{a}_n z|^2} \leqq \sum_{n=2}^{j-1} \frac{1-|a_n|}{|1-\bar{a}_n z_j|^2} + \frac{4(1-|a_j|)}{|1-\bar{a}_j z_j|^2} + \sum_{n=j+1}^{\infty} \frac{1-|a_n|}{|1-\bar{a}_n z_{j+1}|^2}.$$

So, (7) is bounded on ∂R^+.

Let us choose any $z \in R(m, \gamma)$ with $z = x+iy$, $x > 0$, $y > 0$. Then, let $z' = x+iy'$ and $z'' = x''+iy$ be on $\partial R(m, \gamma)$ with $y' \geqq y$ and $x'' \geqq x$. For any a_n, put $\bar{a}_n^{-1} = b_n + ic_n$. Either $b_n \geqq (x+x'')/2$ or $c_n \geqq (y+y')/2$. If $b_n \geqq (x+x'')/2$, then $|\bar{a}_n^{-1} - z| \geqq |\bar{a}_n^{-1} - z''|$ and so $|1-\bar{a}_n z| \geqq |1-\bar{a}_n z''|$. Likewise, $|1-\bar{a}_n z| \geqq |1-\bar{a}_n z'|$ if $c_n \geqq (y+y')/2$. Finally, choose any $z \in R(m, \gamma)$ with $z = x+iy$, $x > 0$, $y < 0$. Then, $\bar{z} = x+i(-y) \in R(m, \gamma)$ and $|1-\bar{a}_n z| \geqq |1-\bar{a}_n \bar{z}|$. We conclude that (7) is bounded in $R(m, \gamma)$, and so, $\|K_z\|$ is bounded in $R(m, \gamma)$. (That $\|K_z\|$ is bounded for $z \in R(m, \gamma)$ with $y = 0$ can be seen from the continuity of $\|K_z\|^2$.)

Let z approach 1 through $R(m, \gamma)$. Since the closed unit ball in a Hilbert space is weakly compact, there is a weak limit point given by, say, $K_1 \in (BH^2)^{\perp}$. We note that K_1 is unique since it is determined by (h_n, K_1), $n = 2, 3, \ldots$, and each h_n is continuous on the closure of U. Therefore, $K_z \to K_1$ weakly as $z \to 1$ through $R(m, \gamma)$ and we are done.

Let us note that the Blaschke product in the last example has the property that $\|K_z\|$ does not approach $\|K_1\|$ as $z \to 1$ through $R(m, \gamma)$ for some $m > 0$ even though each $f \in (BH)^{\perp}$ has a T_{γ}-limit. This is seen by checking that $(1-|a_j|)/|1-\bar{a}_j z_j|^2 \to 1/m^2$ as $j \to \infty$, for then $\|K_{z_j}\|$ cannot approach $\|K_1\|$ if m is chosen small enough.

So it is natural to ask whether we can get something more closely resembling a converse to Theorem 1 with $k = 0$ by assuming that $\|K_z\|$ has a T_{γ}-limit. A slight alteration of the last example shows that nothing significant can be attained by this approach. Suppose $\gamma > 1$ and $t > \gamma + 1$. Then by choosing $(\gamma-1)^{-1} < \alpha < (2\gamma-t)^{-1}$ and $2\gamma\alpha < \beta \leqq 1 + t\alpha$, we get a Blaschke product B such that $\|K_z\|$ has a T_{γ}-limit at $\zeta = 1$ but $\sum (1-|a_n|^2)/|1-a_n|^t = \infty$. The proof is very similar to that of Example 2.

4. In this section we extend our results to arbitrary closed invariant subspaces of H^2. Since the proofs are quite similar to those given in the last two sections, we will only give brief sketches.

If φ is an inner function, then $\varphi = MBs$ where M is a monomial, B is a Blaschke product with zeros $\{a_n\}$ ($a_n \neq 0$), and s is a singular inner function. That is,

$$s(z) = \exp\left\{-\int_0^{2\pi} (e^{i\theta}+z)/(e^{i\theta}-z) \, d\sigma(\theta)\right\},$$

where σ is a finite positive singular measure. The function s, in turn, can be factored into the product of two singular inner functions, s_c and s_a, corresponding to, respectively, a continuous measure σ_c and a purely atomic measure σ_a with $\sigma = \sigma_c + \sigma_a$.

THEOREM 3. *Let $\varphi = MBs$ be an inner function. Suppose that*

$$(8) \qquad \sum_n (1 - |a_n|^2)/|1 - \zeta a_n|^{2\gamma} < \infty$$

and

$$(9) \qquad \int_0^{2\pi} |1 - \zeta e^{i\theta}|^{-2\gamma} \, d\sigma(\theta) < \infty$$

for some fixed $\zeta \in \partial U$, $\gamma \geqq 1$. Then for each $f \in (\varphi H^2)^\perp$ and each nonnegative integer k with $k \leqq \gamma - 1$, the kth derivative of f has a $T_{\gamma/(k+1)}$-limit at ζ.

Proof. It is clear that (9) holds with σ replaced by either σ_a or σ_c. So, a theorem in [7] analogous to Theorem I states that $s_c^{(j)}$ and $s_a^{(j)}$ have $T_{2\gamma/(j+1)}$-limits, $j = 0, 1, \ldots, k$. It is easy to check with the aid of the Cauchy formula that $(MH^2)^\perp$ is just the space of all polynomials of degree less than the degree of M. Also, a lemma in [2] states that if φ_1 and $\dot\varphi_2$ are inner functions, then

$$(\varphi_1 \varphi_2 H^2)^\perp = (\varphi_1 H^2)^\perp \oplus \varphi_1 (\varphi_2 H^2)^\perp.$$

So this, along with Theorem I and Theorem 1, implies that we need only show that the conclusion holds for functions in $(s_c H^2)^\perp$ and $(s_a H^2)^\perp$.

It is proven in [2] that if $f \in (s_c H^2)^\perp$, then

$$f(z) = \int_0^{2\pi} c(t) 2^{1/2} s_{c,t}(z)(1 - e^{-it}z)^{-1} \, d\sigma_c(t),$$

where $c \in L^2(d\sigma)$ and $s_{c,t}$ is the inner function given by

$$s_{c,t}(z) = \exp \left\{ -\int_0^t (e^{i\theta} + z)/(e^{i\theta} - z) \, d\sigma_c(\theta) \right\}.$$

Then, the conclusion for $(s_c H^2)^\perp$ follows from (9) with σ replaced by σ_c in the same way that the conclusion of Theorem 1 followed from (2) using the basis $\{h_n\}$. The proof is so similar that it will be omitted. Likewise, it is shown in [2] that if $f \in (s_a H^2)^\perp$ and σ_a consists of masses r_n at points θ_n, then

$$f(z) = \int_0^\infty c(t) 2^{1/2} s_{a,t}(z)(1 - \exp(-i\theta_{N+1})z)^{-1} \, d\tau(t),$$

where N is the greatest integer in t, τ is the measure on $(0, \infty)$ given by $d\tau(t) = r_{N+1} \, dt$ on $(N, N+1]$, $N = 0, 1, \ldots,$ and $s_{a,t}$ is the inner function given by

$$s_{a,t}(z) = \exp \left\{ -\sum_{j=1}^N r_j \frac{\exp(i\theta_j) + z}{\exp(i\theta_j) - z} - (t - N) r_{N+1} \frac{\exp(i\theta_{N+1}) + z}{\exp(i\theta_{N+1}) - z} \right\}.$$

The proof for $(s_a H^2)^\perp$ follows as in the previous cases.

THEOREM 4. *Let $\varphi = MBs$ be an inner function and choose any $\zeta \in \partial U$, $\gamma \geqq 1$. Then, if for some $m > 0$, each function in $(\varphi H^2)^\perp$ is bounded in $R(m, \zeta, \gamma)$, it follows that*

(10)
$$\sum_n (1 - |a_n|^2)/|1 - \zeta a_n|^t < \infty$$

and

(11)
$$\int_0^{2\pi} |1 - \zeta e^{i\theta}|^{-t} \, d\sigma(\theta) < \infty$$

for all $t < \gamma + 1$.

Proof. Each function in $(BH^2)^\perp$, $(s_c H^2)^\perp$, and $(s_a H^2)^\perp$ is bounded in $R(m, \zeta, \gamma)$ since all of these spaces are contained in $(\varphi H^2)^\perp$. So (10) holds by Theorem 2. In the case of $(s_c H^2)^\perp$, we have for the function K_z given by $f(z) = (f, K_z)$ that

$$\|K_z\|^2 = 2 \int_0^{2\pi} |s_{c,t}(z)|^2/|1 - e^{-it}z|^2 \, d\sigma_c(t).$$

Then, (11) with σ replaced by σ_c follows. The proof is very similar to the proof of Theorem 2 and will be omitted. Likewise, in the case of $(s_a H^2)^\perp$, we have

$$\|K_z\|^2 = 2 \int_0^\infty |s_{a,t}(z)|^2/|1 - \exp(-i\theta_{N+1})z|^2 \, d\tau(t);$$

and (11) with σ replaced by σ_a follows.

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