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ON REPLACING PROPER DEHN MAPS WITH PROPER EMBEDDINGS

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Abstract. In this paper we develop algebraic and geometric conditions which imply that a given proper Dehn map can be replaced by an embedding. The embedding, whose existence is implied by our theorem, retains most of the algebraic and geometric properties required in the original proper Dehn map.

I. **Preliminaries.** In this paper all spaces will be simplicial complexes and all maps will be piecewise linear. The boundary, closure, and interior of a subset X of a space Y will be denoted by bd (X), cl (X) and int (X), respectively. A *proper map* f, taking a space X into a space Y, is a map such that

$$f\,\mathrm{bd}\,(X) = \mathrm{bd}\,(Y) \cap f(X).$$

A Dehn map f, taking a compact, bounded, two-manifold F into a 3-manifold M, is a map such that

(1) $\mathrm{bd}\,(F) = f^{-1}f\,\mathrm{bd}\,(F)$;

(2) $f|\mathrm{bd}\,(F)$ is a homeomorphism.

All unlabeled homomorphisms are natural maps induced by inclusion.

We shall consider the problem of replacing a proper Dehn map of a compact, connected, orientable, bounded surface F into a compact, (necessarily) bounded, irreducible, orientable 3-manifold M with an embedding. As will be pointed out later, the requirement of orientability may not be strictly necessary.

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In [3] Papakyriakopoulos proved Dehn's lemma, that is, a proper Dehn map f of a disk $\mathscr{D}$ into a 3-manifold M can be replaced by a proper embedding g of $\mathscr{D}$ into M such that $g\,\mathrm{bd}\,(\mathscr{D}) = f\,\mathrm{bd}\,(\mathscr{D})$. A natural question to consider is the following: Let $f: F \to M$ be a proper Dehn map of a compact, connected, bounded, orientable surface F into a compact 3-manifold M. Does there always exist a proper embedding $g: F_1 \to M$ such that

(1) $g(\mathrm{bd}\,(F_1)) \subseteq f(\mathrm{bd}\,(F))$;

(2) genus $(F_1) \leqq$ genus (F)? (Question 2, p. 25 in [3].)

The above seems to be a very difficult question and we content ourselves with

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the following restricted question: What are sufficient conditions on the f above so that the required F_1 and g will exist, genus $(F_1)=$ genus (F), and g bd $(F_1)=f$ bd (F)?

Since F_1 is to be homeomorphic to F, we may drop the subscript 1.

We shall ask also that $g_*\pi_1(F)\subset f_*\pi_1(F)$.

It seems natural to ask that for each component $\mathscr{C}$ of bd (F), $f(\mathscr{C})$ is not null-homotopic in M. We will now obtain sufficient conditions for the existence of the embedding g by

(1) assuming that g exists;

(2) abstracting algebraic and geometric properties of g.

We observe that if $g(F)$ (which is necessarily a two-sided surface in M) is not *incompressible* in M (that is, $\pi_1(g(F))\to\pi_1(M)$ is not 1-1), it will follow from the loop theorem in [4] that the genus of $g(F)$ could have been reduced. Thus it seems natural to require that $f_*\colon \pi_1(F)\to\pi_1(M)$ is 1-1.

We observe that each component of f bd (F) separates a regular neighborhood of itself in bd (M) since M is orientable.

We consider two possibilities as:

Case 1. $g(F)$ separates M;

Case 2. $g(F)$ does not separate M.

Case 1. If an embedded incompressible surface $g(F)$ separates M into two 3-submanifolds M_1 and M_2, we obtain the group diagram shown in Figure 1 where the h_i are homomorphisms induced by inclusion for $i=1,\dots,4$.

Since $g_*\colon \pi_1(F)\to\pi_1(M)$ is an injection, h_1 and h_2 are monomorphisms. If h_i was an epimorphism for $i=1$ or 2, it would follow from 3.1 in [1] that M_i was the

$$\begin{tikzcd}
 & \pi_1(g(F)) = g_*\pi_1(F) \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\
\pi_1(M_1) \arrow[dr, "h_3"'] & & \pi_1(M_2) \arrow[dl, "h_4"] \\
 & \pi_1(M) = \pi_1(M_1) \ast_{\pi_1(g(F))} \pi_1(M_2) &
\end{tikzcd}$$

FIGURE 1

product of F with the unit interval since M is irreducible. This case is not so interesting as $g(F)$ could then be deformed to lie in the boundary of M. Note that in this case bd $(g(F))$ separates bd (M) into two sets $S_i=$ bd $(M_i)\cap$ bd (M) for $i=1,2$. We observe that bd (M_i) is homeomorphic to $S_i\cup g(F)$ and that if $S_i\cup g(F)$ is connected and we assume a base point on bd $(g(F))$, the image of the map $\pi_1(S_i\cup g(F))\to\pi_1(M)$ is contained in $\pi_1(M_i)$ for $i=1,2$.

Let x be a point in bd (F). We shall say that a proper Dehn map $f\colon F\to M$ *splits* M if the following three conditions hold:

(1) $f_*\pi_1(F,x)\to\pi_1(M,f(x))$ is 1-1.

(2) $f\,\mathrm{bd}\,(F)$ separates $\mathrm{bd}\,(M)$ into two compact, bounded, (not necessarily connected) 2-manifolds S_1 and S_2 such that

$S_1 \cup S_2 = \mathrm{bd}\,(M)$;

$S_1 \cap S_2 = f\,\mathrm{bd}\,(F)$;

$S_i \cup f(F)$ is connected for $i=1, 2$.

(3) Let $\bar{S}_i$ be the closed surface defined by joining a surface homeomorphic to S_i to F along the boundaries of the surfaces so as to make the natural map $\gamma_i\colon \bar{S}_i \to M$ defined by $\gamma_i|F=f$ and $\gamma_i|S_i=$ the natural inclusion map continuous for $i=1, 2$. There exist proper subgroups A_1 and A_2 of $\pi_1(M, f(x))$ such that the diagram in Figure 2 is commutative.

$$\begin{tikzcd}
 & & \pi_1(F,x) \arrow[dll] \arrow[drr] \arrow[dl, "f_*"'] \arrow[dr, "f_*"] \arrow[dd, "f_*"] & & \\
\pi_1(\bar{S}_1,x) \arrow[dr, "\gamma_{1*}"] & & & & \pi_1(\bar{S}_2,x) \arrow[dl, "\gamma_{2*}"'] \\
 & A_1 \arrow[dr] & & A_2 \arrow[dl] & \\
 & & \pi_1(M,f(x)) = A_1 \underset{f_*\pi_1(F,x)}{*} A_2 & &
\end{tikzcd}$$

FIGURE 2

We note that the requirement in (2) above, that $S_i \cup f(F)$ is connected, is not necessary in our proof. This requirement was introduced to simplify Figure 2.

If L_j is a component of $S_i \cup f(F)$ which does not meet $f(F)$, one requires that the image of the homomorphism $\pi_1(L_j) \to \pi_1(M)$ be conjugate to a subgroup of A_i where $i=1$ or 2.

Case 2. Suppose $g(F)$ does not separate M and $g\,\mathrm{bd}\,(F)$ does not separate at least one component of $\mathrm{bd}\,(M)$ which it meets. Let λ be a simple loop in $\mathrm{bd}\,(M)$ which meets $g\,\mathrm{bd}\,(F)$ in a single point x and which crosses $g\,\mathrm{bd}\,(F)$ at that point. We suppose that $g\,\mathrm{bd}\,(F)$ meets each component of $\mathrm{bd}\,(M)$. Let $A \subset \pi_1(M, x)$ be the subgroup of $\pi_1(M, x)$ generated by loops based at x and not crossing $g(F)$. It is easily shown that $\pi_1(M, x)$ is generated by the elements of A together with $[\lambda]$. Let S be the compact, bounded, 2-manifold obtained by cutting $\mathrm{bd}\,(M)$ along $g\,\mathrm{bd}\,(F)$. Let $\bar{S}$ be the connected, closed 2-manifold obtained by joining two copies F', F'' of F to S along their boundaries so that the map γ defined by

(1) $\gamma|F' = g|F$;

(2) $\gamma|F'' = g|F$;

(3) $\gamma|S=$ the natural inclusion map;

is continuous.

Now $\gamma_*\pi_1(\bar{S}, x)$ is a subgroup of A. Let $\alpha=\gamma^{-1}(\lambda)$. Then α is a simple arc from

F' to F''. It is a consequence of Van Kampen's theorem that $\pi_1(M, x)$ is the group generated by A and $[\lambda]$ and having all relations of the form

$$(**)\qquad [(\gamma\alpha)g(l)(\gamma\alpha)^{-1}] = [\lambda][g(l)][\lambda]^{-1}$$

where λ is parameterized appropriately, and l is an arbitrary loop on F.

We shall say that a proper Dehn map $f\colon F \to M$ *cuts* M if the following three conditions hold:

(1) $f_*\colon \pi_1(F) \to \pi_1(M)$ is 1-1.

(2) $f\,\mathrm{bd}\,(F)$ meets each component of bd (M) and fails to separate at least one of them.

(3) Let α, λ, x, $\bar{S}$ be as above. Let A be a subgroup of $\pi_1(M, x)$. Let $\gamma\colon \bar{S} \to M$ be defined by

$$\gamma|F' = f; \qquad \gamma|F'' = f; \qquad \gamma|S = \text{the natural inclusion map.}$$

Now we require that $\gamma_*\pi_1(\bar{S}, x) \subset A$.

We also require that $\pi_1(M, x)$ is generated by the elements of A together with $[\lambda]$ and that the relations $[(\gamma\alpha)f(l)(\gamma\alpha)^{-1}] = [\lambda][f(l)][\lambda]^{-1}$, where l is allowed to be any loop on F based at x, completely determine the presentation of $\pi_1(M, x)$.

II. **Our main theorem.** We are now in a position to conveniently state our result.

THEOREM. *Let M be a compact, connected, orientable, irreducible 3-manifold. Let F be a compact, connected, orientable 2-manifold with nonvacuous boundary. Let $f\colon F \to M$ be a proper Dehn map.*

(1) *Suppose f splits M. Then there is a proper embedding $g\colon F \to M$ splitting M such that g bd $(F) = f$ bd (F).*

(2) *Suppose f cuts M. Then there is a proper embedding $g\colon F \to M$ cutting M such that g bd $(F) = f$ bd (F).*

Furthermore, in both cases $g_*\pi_1(F) = f_*\pi_1(F)$.

The idea of the proof is very simple. We construct a complex X having the same fundamental group as M. The X we construct will contain an incompressible embedding of F and also an embedding of bd (M). We will then construct a map G from M to X such that $G_*\colon \pi_1(M) \to \pi_1(X)$ is an isomorphism and $G|\mathrm{bd}\,(M)$ is a homeomorphism. Using Lemma 1.1 in [6], we will be able to assume $G^{-1}(F)$ is an incompressible surface in M. Theorem 1 in [2] will show that G may be taken to be a homeomorphism on this surface. At this point in the proof it will be obvious that $G^{-1}(F)$ is the desired embedding of F. We point out that the only places we used the orientability of M were:

(1) to satisfy the conditions of Lemma 1.1 in [6];

(2) to insure that f bd (F) separated a regular neighborhood of itself in bd (M).

We also point out that the assumption that F be orientable is necessary for similar reasons and that if we prove an analog to Lemma 1.1 in [6] for nonorientable 3-manifolds and nonorientable 2-manifolds we can easily state a generalization of our theorem covering these cases.

Proof. We divide the proof into two cases.

Case 1. f separates M.

Let (M_{A_i}, ρ_i) be the covering space of M associated with the subgroup A_i of $\pi_1(M, f(x))$ for $i=1, 2$. Since $f_*\pi_1(F, x) \subset A_i$ for $i=1, 2$, we can find maps f^i, δ^i for $i=1, 2$ which complete the diagram of maps given in Figure 3.

$$\begin{tikzcd}
 & & M_{A_i} \arrow[d, "\rho_i"] \\
F \arrow[r] \arrow[urr, "f^i"] \arrow[rr, bend right, "f"'] & \bar{S}_i \arrow[ur, "\delta^i"] \arrow[r, "\gamma_i"] & M
\end{tikzcd}$$

FIGURE 3

Let $X_1 = M_{A_1} \cup F \times [0, \frac{1}{2}]$, $f^1(v) = (v, 0)$ for $v \in F$.

Let $X_2 = M_{A_2} \cup F \times [\frac{1}{2}, 1]$, $f^2(v) = (v, 1)$ for $v \in F$.

Let $X = X_1 \cup X_2$.

Define a map $\rho \colon X \to M$ by

(1) $\rho(s) = \rho_i(s)$ for $s \in M_{A_i}$, $i=1, 2$;

(2) $\rho(v, t) = f(v)$ for $v \in F$ and $t \in [0, 1]$.

Then ρ_* is easily shown to be an isomorphism. Since M is irreducible, it follows from 1.1.5 in [7] that $\pi_j(M) = 0$ for $j \geqq 2$.

Now $\pi_j(M_{A_i}) = \pi_j(X_i) = 0$ for $j \geqq 2$, $i=1, 2$, and by Theorem 5, p. 199 in [5], $\pi_j(X) = 0$ for $j \geqq 2$. Let

$$S^* = (\delta^1(\bar{S}_1) - f^1(\text{int}(F))) \cup (\delta^2(\bar{S}_2) - f^2(\text{int}(F))) \cup (\text{bd}(F) \times [0, 1]).$$

Now S^* is homeomorphic to bd (M) and $\rho S^* = \text{bd}(M)$. In fact,

$$\rho | (S^* - (\text{bd}(F) \times [0, 1]))$$

is a homeomorphism. Also,

$$\rho_* \pi_1(S^*, f^1(x)) = \pi_1(\text{bd}(M), f(x)) \subset \pi_1(M, f(x)).$$

We now define a map $G \colon (\text{bd}(M), f(x)) \to (S^*, f^1(x))$ so that $\rho_* G_* \colon \pi_1(\text{bd}(M), f(x)) \to \pi_1(\text{bd}(M), f(x))$ is the identity. Since $\pi_j(X) = 0$ for $j \geqq 2$, it is a simple matter to extend G to M so that $G_* = \rho_*^{-1} \colon \pi_1(M) \to \pi_1(X)$ and $G^{-1}(S^*) = \text{bd}(M)$. This is done by first extending G to the 1-skeleton of M, then to the 2-skeleton, and finally to all 3-simplexes.

It is a consequence of Lemma 1.1 in [6] that $G^{-1}(F \times \{\frac{1}{2}\})$ can be taken to be an incompressible surface in M. Let F_1 be a component of this surface. Since G_* is an isomorphism, $(G|F_1)_* \colon \pi_1(F_1) \to \pi_1(F \times \{\frac{1}{2}\})$ is a monomorphism. Observe that F_1 is not closed since $\pi_1(F)$ is free. It follows from Theorem 1 in [2] that either

(a) $G|F_1$ is homotopic, leaving bd (F_1) fixed, to a covering map, or

(b) F_1 is an annulus or möbius band and $G|F_1$ is homotopic to a map carrying F_1 to bd $(F\times\{\frac{1}{2}\})$.

We observe that $G|\mathrm{bd}\,(F_1)$ is a homeomorphism by construction. Thus if F_1 is an annulus so is $F\times\{\frac{1}{2}\}$ and $G|F_1$ is homotopic to a homeomorphism leaving bd (F_1) fixed. Since F_1 is assumed to separate a regular neighborhood of itself and M is orientable, F_1 is not a möbius band. It follows that $G|F_1$ is homotopic to a covering map leaving bd (F_1) fixed. Since $G|\mathrm{bd}\,(F_1)$ is a homeomorphism, $G|F_1$ is homotopic to a homeomorphism leaving bd (F_1) fixed.

It follows that we may assume that $G|F_1$ is a homeomorphism. Since F is connected, $G^{-1}(F\times\{\frac{1}{2}\})=F_1$. Thus we can find an embedding $g\colon F\to F_1$. Let $M_i=G^{-1}(X_i)$ for $i=1, 2$.

Now, by construction, $G_*\pi_1(M_i, f(x))\subseteq\pi_1(X_i, Gf(x))=A_i$ for $i=1, 2$; and since $G|g(F)$ is a homeomorphism, $G_*\pi_1(g(F), f(x))=\pi_1(F\times\{\frac{1}{2}\}, Gf(x))\subset\pi_1(X, Gf(x))$.

It follows from Proposition 2 in [1] that $G_*\pi_1(M_i, f(x))=\pi_1(X_i, Gf(x))$ for $i=1, 2$ and that the splitting induced by the embedding g is the same as that induced by our original map f.

Case 2. $f(F)$ cuts M.

Let (M_A, ρ_1) be the covering space of M associated with $A\subset\pi_1(M, x)$.

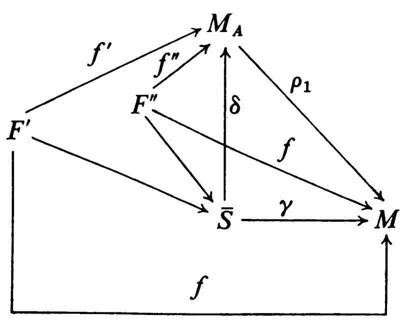

FIGURE 4

Since $\gamma_*\pi_1(\bar{S}, x)\subset A$, one can find the maps f', f'' and δ necessary to complete the diagram of maps in Figure 4. Let X be the complex defined by $X=M_A\cup F\times[0, 1]$ where we make the identifications $(v, 0)=f'(v)$ for v in F and $(v, 1)=f''(v)$ for v in F.

Define a map $\bar{\rho}\colon X\to M$ by

(1) $\bar{\rho}|M_A=\rho_1$;

(2) $\bar{\rho}(s, t)=f(s)$ for s in F and t in $[0, 1]$.

Now $\bar{\rho}_*$ has been constructed to be an isomorphism. One sees this by picking the elements of $\pi_1(M_A)\subset\pi_1(X)$ and an element representing the loop $\bar{\rho}^{-1}(\lambda)$ as generators for $\pi_1(X)$. The relations are the natural ones among the elements of $\pi_1(M_A)$ together with a collection of relations which are carried by $\bar{\rho}$ to the relations mentioned in (**).

Now $\pi_j(M_A)=0$ for $j \geqq 2$ since $\pi_j(M)=0$ by 1.1.5 in [7]. We wish to show that $\pi_j(X)=0$ for $j \geqq 2$. Let X_l be the space obtained by cutting X along $F \times \{\frac{1}{2}\}$. Let F_1 and F_2 be disjoint surfaces in X_l coming from $F \times \{\frac{1}{2}\}$. We can form an infinite cyclic covering (X^*, ρ) of X by taking infinitely many copies X_l^k of X_l and pasting F_1^k to F_2^{k+1} for k an integer and F_i^k the copy of F_i contained in X_l^k for $i=1, 2$. We claim that $\pi_j(X^*)=0$ for $j \geqq 2$ and thus $\pi_j(X)=0$ for $j \geqq 2$. We observe that it follows from Theorem 5, p. 199 in [5] that $\pi_j(\bigcup_{k=l}^{l+N} X_l^k)=0$ where l is an integer and N is a positive integer for $j \geqq 2$ since M_A is a deformation retract of X_l. However, since a sphere is compact, the image of a sphere in X^* meets only finitely many of the F_i^k and thus lies in a set of the form $\bigcup_{k=l}^{l+N} X_l^k$ where l is an integer and N is a positive integer. It follows that $\pi_j(X^*)=0$ for $j \geqq 2$ and thus that $\pi_j(X)=0$ for $j \geqq 2$.

Let $\bar{S}'$ be the closure of $\bar{S}-(F' \cup F'')$. Then the surface $\delta(\bar{S}') \cup \operatorname{bd}(F) \times [0, 1]$ is homeomorphic to bd (M).

As was done in Case 1 we can find a map $G: M \to X$ such that

(1) $G \operatorname{bd}(M)=\delta(\bar{S}') \cup \operatorname{bd}(F) \times [0, 1]$.

(2) $G|\operatorname{bd}(M)$ is a homeomorphism.

(3) $G^{-1}G \operatorname{bd}(M)=\operatorname{bd}(M)$.

(4) $\bar{\rho}_* G_*: \pi_1(M) \to \pi_1(M)$ is the identity.

It now follows from Lemma 1.1 in [6] that there is a map H homotopic to G such that

(a) $H|\operatorname{bd}(M)=G|\operatorname{bd}(M)$.

(b) H satisfies the three conditions on G given above.

(c) $H^{-1}(F \times \{\frac{1}{2}\})$ is an incompressible surface $\mathscr{F}$ in M.

It follows from Theorem 1 in [2] that $H|\mathscr{F}$ may be taken to be a homeomorphism. Thus $\mathscr{F}=F$.

Note $\pi_1(M) \supset \pi_1(\mathscr{F})=f_*\pi_1(F)$.

This completes the proof of our theorem.

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