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THE GLIDING HUMPS TECHNIQUE FOR *FK*-SPACES

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Abstract. The gliding humps technique has been used by various authors to establish the existence of bounded divergent sequences in certain summability domains. The purpose of this paper is to extend these results and to obtain analogous ones for sequence spaces other than c and m. This serves to unify and improve many known results and to obtain several new ones—applications include extensions to theorems of Dawson, Lorentz-Zeller, Snyder-Wilansky and Yurimyae. Improving another result of Wilansky allows us to consider countable collections of sequence spaces—applications including the proof of a conjecture of Hill and Sledd and extensions to theorems of Berg and Brudno. A related result of Petersen is also considered and a simple proof using the Baire category theorem is given.

1. **Introduction.** Using the gliding humps technique, Wilansky and Zeller [19] have shown that if a convergence domain c_A (defined below) contains c as a nonclosed subspace, then c_A contains a bounded divergent sequence. This result was extended by Meyer-König and Zeller in [13] and [14], where c_A was replaced by an arbitrary *FK*-space. The main result of this paper is Theorem 1 which leads to analogous results for sequence spaces other than c and m. Applications include extensions to theorems of Agnew [1], Dawson [7], Lorentz and Zeller [11], Snyder and Wilansky [17], and Yurimyae [20]. In §4 a result of Wilansky is improved and this allows us to consider countable collections of sequence spaces. Applications include the proof of a conjecture of Hill and Sledd [8] and extensions to theorems of Berg [5] and Brudno [6]. In the final section a related result of Petersen [15] is considered and a simple proof using the Baire category theorem is given.

2. **Notation.** ω denotes the space of all complex-valued sequences and any vector subspace E of ω is a *sequence space*. A sequence space E with a complete, metrizable, locally convex topology τ is called an *FK-space* if the inclusion map $(E, \tau) \to \omega$ is continuous when ω is endowed with the topology of coordinatewise convergence. An *FK*-space whose topology is normable is called a *BK-space*. The following well-known *BK*-spaces will be important in the sequel:

m, the space of all bounded sequences;

c, the space of all convergent sequences;

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c_0, the space of all null sequences;

l^p, $1 \leqq p < \infty$, the space of all absolutely p-summable sequences;

$bv = \{x \in \omega : \sum_{j=1}^{\infty} |x_j - x_{j+1}| < \infty\}$, the space of all sequences of bounded variation;

$cs = \{x \in \omega : \sum_{j=1}^{\infty} x_j$ converges$\}$, the space of all summable sequences; and

$bs = \{x \in \omega : \sup |\sum_{j=1}^{n} x_j| < \infty\}$.

As usual, l^1 is replaced by l and $\|\cdot\|_\infty$ denotes the norm on m.

$x \in \omega$ is called *ultimately almost periodic* if, to each $\varepsilon > 0$, there correspond positive integers n, N such that every interval $(k, k+n)$, $k = 1, 2, \ldots$, contains an integer t satisfying

$$|x_r - x_{r+t}| < \varepsilon \quad \text{for all } r > N.$$

uap denotes the space of all ultimately almost periodic sequences and it should be noted that $c \subset uap \subset m$.

3. The main result.

THEOREM 1. *Let E be an FK-space with $c \cap E$ not closed in E. Then E contains*

(i) *a bounded sequence which is not ultimately almost periodic*;

(ii) *a null sequence which is not of bounded variation*;

(iii) *a null sequence which is not absolutely p-summable for any $p \geqq 1$.*

Proof. (i) follows as in Theorem 1 of [14] with only minor modifications.

(ii) As in the proof of Theorem 1 of [14], we may suppose that the topology of E is given by a family of seminorms $\{p_n\}_{n=1}^{\infty}$ with the property that

$$|x_m| \leqq p_m(x) \leqq p_{m+1}(x) \qquad (x \in E;\ m = 1, 2, \ldots). \tag{1}$$

Now c_0 is of codimension 1 in c so it follows from [9, §15.8 (3)] that $c_0 \cap E$ is not closed in E. We then have

(2) given $\varepsilon > 0$, $\eta > 0$ and a positive integer m, there exists $x \in c_0 \cap E$ such that $p_m(x) < \varepsilon$ and $\|x\|_\infty = \eta$.

For, if not, there exist $\varepsilon > 0$, $\eta > 0$ and a positive integer m such that $\|x\|_\infty = \eta \Rightarrow p_m(x) \geqq \varepsilon$ whenever $x \in c_0 \cap E$. Then, for $0 \neq x \in c_0 \cap E$, we have $p_m(\eta x / \|x\|_\infty) \geqq \varepsilon$ so that $\|x\|_\infty \leqq (\eta/\varepsilon) p_m(x)$ for *every* $x \in c_0 \cap E$. It follows that $c_0 \cap E$ is closed in E, a contradiction.

With $\varepsilon = \varepsilon_1 = 1/2^2$, $\eta = \eta_1 = 1$ and $m = m_1 = 1$, find $x^{(1)}$, say, as in (2). Let $n > 1$ and suppose that $m_1, \ldots, m_{n-1}$ and $x^{(1)}, \ldots, x^{(n-1)}$ have been chosen. With $\varepsilon_n = 1/2^{n+1}$ and $\eta_n = 1/n$, choose $m_n > m_{n-1}$ so that

$$|x_k^{(i)}| \leqq 1/2^{n+1} \qquad (k \geqq m_n;\ 1 \leqq i < n), \tag{3}$$

and choose $x^{(n)} \in c_0 \cap E$ so that

$$p_{m_n}(x^{(n)}) < \varepsilon_n \quad \text{and} \quad \|x^{(n)}\|_\infty = \eta_n. \tag{4}$$

In this way we obtain by (1) a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of elements of $c_0 \cap E$ with $p_1(x^{(n)}) \leqq \cdots \leqq p_n(x^{(n)}) < 1/2^{n+1}$ $(n=1, 2, \ldots)$. Let $x = \sum_{n=1}^{\infty} x^{(n)}$, the series obviously converging in E. We complete the proof by showing that $x \in c_0 \backslash bv$.

In the interval $m_j \leqq k \leqq m_{j+1}$ we have

$$(5)\qquad \begin{aligned} |x_k^{(i)}| &\leqq 1/2^{j+1} && \text{if } i < j \text{ (by (3))}, \\ &\leqq 1/i && \text{if } i = j \text{ (by (4))}, \\ &\leqq 1/2^{i+1} && \text{if } i > j \text{ (by (4))}; \end{aligned} \qquad (j = 1, 2, \ldots)$$

so that

$$|x_k| \leqq \sum_{i=1}^{\infty} |x_k^{(i)}| \leqq \sum_{i=1}^{j-1} 1/2^{j+1} + 1/j + \sum_{i=j+1}^{\infty} 1/2^{i+1} \to 0 \quad \text{as } j \to \infty,$$

and so $x \in c_0$.

Finally, we show that $x \notin bv$ by exhibiting a subsequence y of x that is not of bounded variation. For each positive integer j, there exists $n_j \in (m_j, m_{j+1})$ such that $|x_{n_j}^{(j)}| = 1/j$ and so, by (5),

$$|x_{n_j}| \geqq 1/j - \sum_{i=1}^{j-1} 1/2^{j+1} - \sum_{i=j+1}^{\infty} 1/2^{i+1} = 1/j - j/2^{j+1}.$$

Now again by (4) and (5),

$$|x_{m_j}| \leqq \sum_{i=1}^{j-1} 1/2^{j+1} + 1/2^{j+1} + \sum_{i=j+1}^{\infty} 1/2^{i+1} = (j+2)/2^{j+1}.$$

Let

$$\begin{aligned} y_j &= x_{m_j} && \text{if } j \text{ odd}, \\ &= x_{n_j} && \text{if } j \text{ even}, \end{aligned} \qquad (j = 1, 2, \ldots),$$

then y is a subsequence of x and it is easy to see that $y \notin bv$. This completes the proof of (ii); (iii) can be proved similarly and the details are left to the reader.

It is not difficult to see that none of the converse results to (i), (ii) or (iii) is true in general.

4. **Applications.** In this section we shall be concerned with matrix transformations $y = Ax$, where $x, y \in \omega$, $A = \{a_{ij}\}_{i,j=1}^{\infty}$ is an infinite matrix with complex coefficients and $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$ $(i = 1, 2, \ldots)$.

For any subspace E of ω and any matrix A, E_A denotes the set $\{x \in \omega : Ax \text{ exists and } Ax \in E\}$. If E is an *FK*-space then by [21, Theorem 4.10] E_A may be topologized so that it too becomes an *FK*-space, and it will always be assumed that E_A carries this topology. Of special interest will be the spaces l_A, the *absolute summability domain* of A; and c_A, the *convergence domain* of A. If $x \in c_A$, we say that x is *limited by the matrix* A and write "$\lim_A x$" in place of "$\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} x_j$". A is called *conservative* if $c \subseteq c_A$; *regular* if $c \subseteq c_A$ and $\lim_A x = \lim_{j\to\infty} x_j$ for each $x \in c$. $\rho_A^{(i)} = \{a_{ij}\}_{j=1}^{\infty}$ denotes the *ith row of* A, $i = 1, 2, \ldots$, and $m \cap c_A$ is called the *bounded convergence domain of* A.

Yurimyae [20, Theorem 8] has shown that if l is a nonclosed subspace of l_A, then $l_A \cap (bs \backslash l)$ is nonempty; this can be substantially improved as follows:

THEOREM 2(i). *Let E be an FK-space with $l \cap E$ not closed in E; then E contains a summable sequence which is not absolutely summable.*

Proof. Consider the one-to-one mapping S of ω onto itself given by

$$S(x) = (x_1, x_1+x_2, x_1+x_2+x_3, \ldots) \qquad (x \in \omega).$$

S maps E onto an FK-space, say F. $l \cap E$ is not closed in E and so $bv \cap F$, which corresponds to $l \cap F$ under S, is not closed in F. If $c \cap F = bv \cap F$, then $c \cap F$ is not closed in F and Theorem 1(ii) is contradicted. It follows that $F \cap (c \backslash bv)$ is nonempty and then that $E \cap (cs \backslash l)$ is nonempty, which gives the desired result.

It would be extremely interesting to find necessary and sufficient conditions on the matrix A in order that l be a closed subspace of l_A.

Only minor modifications to the above argument are needed to prove the following results:

THEOREM 2(ii). *If $bv \cap E$ is not closed in E, then $E \cap (cs \backslash bv)$ is nonempty.*

(iii) *If $p > 1$ and $l^p \cap E$ is not closed in E, then $E \cap (cs \backslash l^p)$ is nonempty.*

In a forthcoming paper we treat the related problem of determining when $E \cap F$ is closed in F, where E and F are arbitrary FK-spaces with the so-called BS-property.

Our next result improves a theorem of Dawson [7, Theorem 2].

THEOREM 3. *Let $\{A^n\}_{n=1}^{\infty}$ be a collection of matrices, each matrix limiting every sequence of bounded variation. Then there is a summable sequence, not of bounded variation, which is limited by each of the matrices A^n.*

Proof. By the corollary to Lemma 5 and Corollary 1 to Theorem 5 of [2], $\bigcap_{n=1}^{\infty} c_{A^n}$ is an FK-space with a separable strong dual. As in the proof of Corollary 3 to Theorem 5 of [2], bv is a nonclosed subspace of $\bigcap_{n=1}^{\infty} c_{A^n}$. The desired result now follows from Theorem 2(ii) by putting $E = \bigcap_{n=1}^{\infty} c_{A^n}$.

In exactly the same way we can establish the following improvement of a result of Lorentz and Zeller (see [11, Lemma 5]). We note, however, that a quicker proof can be obtained by simply adding A to the collection $\{A^n\}_{n=1}^{\infty}$, where the matrix A consists entirely of "ones".

THEOREM 4. *Let $\{A^n\}_{n=1}^{\infty}$ be a collection of matrices, each matrix limiting every absolutely summable sequence. Then there is a summable sequence, not absolutely summable, which is limited by each of the matrices A^n.*

We note here that the analogous result with l replaced by l^p in Theorem 4 still holds. This, however, requires a different proof and will be discussed elsewhere.

The proofs of Theorems 3 and 4 depend in an essential way on the separability of the strong dual space of a convergence domain; for general *FK*-spaces it turns out that we have to impose a monotonicity condition:

THEOREM 5. *Let* $\{F^n\}_{n=1}^{\infty}$ *be a decreasing sequence of FK-spaces and E be a BK-space such that* $E\subseteq\bigcap_{n=1}^{\infty} F^n$. *Then E is closed in* $\bigcap_{n=1}^{\infty} F^n$ (*if and*) *only if E is closed in* F^m *for some positive integer m.*

Proof. Let $\|\cdot\|$ be a norm defining the topology of E and, for each positive integer n, let $\{p_j^n\}_{j=1}^{\infty}$ be a sequence of seminorms defining the topology of F^n. As usual, we may assume that $p_j^n \leqq p_k^n$ whenever $j\leqq k$; $n=1, 2, \ldots$; and since $F^n\supseteq F^{n+1}$ for $n=1, 2, \ldots$, we may further assume that $p_j^n\leqq p_j^m$ whenever $n\leqq m$; $j=1, 2, \ldots$. The *FK*-topology on $\bigcap_{n=1}^{\infty} F^n$ is given by the family of seminorms $\{p_j^n\}_{j,n=1}^{\infty}$ and so, if E is closed in $\bigcap_{n=1}^{\infty} F^n$, we can find $k>0$ and positive integers j, m such that $\|x\|\leqq kp_j^m(x)$ for each $x\in E$. On the other hand, since $E\subseteq F^m$, it follows from the closed graph theorem (see [21, Theorem 4.5]) that

$$p_j^m(x) \leqq k'\|x\| \quad (x\in E) \qquad \text{for some } k' > 0.$$

Thus E is a topological subspace of F^m and, being complete, is closed. The converse result is obvious.

Theorem 5 was stated only in terms of *FK*-spaces but can clearly be extended to *FH*-spaces (see [18] for the definition of *FH*-spaces). It then improves a result of Wilansky [18, p. 67, Theorem 2.2]. A slightly sharper version is valid for *FK*-spaces with the *BS*-property and this will be discussed in another paper.

We note here that the monotonicity condition of Theorem 5 cannot be dropped. To see this, let x be the sequence given by $x_n=(-1)^n$, $n=1, 2, \ldots$; then $x\cdot bv$, taking coordinatewise products, is a *BK*-space containing l. Now l is closed in neither of the spaces bv, $x\cdot bv$ but we do have $l=bv\cap x\cdot bv$.

In [8], Hill and Sledd conjectured that *uap* could not be the bounded convergence domain of any regular matrix. That this is the case, even for nonregular matrices, was established by Berg in [5]. The next two results extend these ideas to countable collections of matrices.

THEOREM 6. *Let* $\{A^n\}_{n=1}^{\infty}$ *be a collection of conservative matrices with* $c_{A^n}\supseteq c_{A^{n+1}}$, $n=1, 2, \ldots$, *and such that each* A^n *limits at least one bounded divergent sequence. Then there is a bounded sequence, not ultimately almost periodic, which is limited by each of the matrices* A^n.

Proof. By Theorem 1 of [19], c is a nonclosed subspace of c_{A^n}, $n=1, 2, \ldots$. Theorem 5 shows that c is a nonclosed subspace of $\bigcap_{n=1}^{\infty} c_{A^n}$ and the conclusion follows at once from Theorem 1(i).

For regular matrices the monotonicity condition on the sequence $\{A^n\}_{n=1}^{\infty}$ can be relaxed slightly to obtain the following improvement of a result due to Brudno [6]. (Brudno's theorem is given in English as Theorem 4.3.3 of [15].) For the proof we need the following remarkable "consistency theorem", first stated by Mazur and Orlicz in [12] and first proved by Brudno in [6].

Let A and B be regular matrices with $c_A \cap m \subseteq c_B$; then $\lim_B x = \lim_A x$ whenever $x \in c_A \cap m$.

For a survey of this result, the reader is referred to the paper [4].

THEOREM 7. *Let $\{A^n\}_{n=1}^{\infty}$ be a collection of regular matrices with $c_{A^n} \supseteq c_{A^{n+1}} \cap m$, $n=1, 2, \ldots$, and such that each A^n limits at least one bounded divergent sequence. Then there is a bounded sequence, not ultimately almost periodic, which is limited by each of the matrices A^n.*

Proof. We introduce a new family $\{B^n\}_{n=1}^{\infty}$ of matrices as follows. Set $B^1 = A^1$ and suppose that $B^1, \ldots, B^n$ have been defined. The ith row of B^{n+1} is given by

$$\begin{aligned} \rho_{B^{n+1}}^{(i)} &= \rho_{A^{n+1}}^{(k)} \quad \text{if } i = 2k-1, \\ &= \rho_{B^n}^{(k)} \quad \text{if } i = 2k, \end{aligned} \qquad (k = 1, 2, \ldots).$$

It is easy to see that $c_{A^n} \supseteq c_{B^n} \supseteq c_{B^{n+1}}$ $(n=1, 2, \ldots)$. Furthermore, the Brudno-Mazur-Orlicz consistency theorem and an elementary induction argument give $c_{B^n} \supseteq m \cap c_{A^n}$ $(n=1, 2, \ldots)$. Thus the B_n's satisfy the hypotheses of Theorem 6 and so $m \cap \bigcap_{n=1}^{\infty} c_{A^n}$, which is equal to $m \cap \bigcap_{n=1}^{\infty} c_{B^n}$, contains a sequence which is not ultimately almost periodic.

The interested reader is invited to contrast these last two results with Theorem 3 of [23].

We close this section with some further remarks concerning Theorem 1. First we observe that it is possible, using (2), to establish the existence in $m \cap E$ of an uncountable family $\{x^{(\alpha)}\}$ of points such that $\|x^{(\alpha)} - x^{(\beta)}\|_{\infty} > 1$ whenever $\alpha \neq \beta$. Thus we have

THEOREM 8. *Let E be an FK-space such that $c \cap E$ is not closed in E; then $m \cap E$ is a nonseparable subspace of m.*

Following Snyder [16], we say that an *FK*-space E is *conull* provided that $\psi^r \to 0$ weakly in E, where ψ^r denotes the sequence $(0, \ldots, 0, 1, 1, \ldots)$ with r zero coordinates. We next show that Theorem 8 is stronger than a recent result of Snyder and Wilansky [17, Corollary 7 to Theorem 11].

COROLLARY 1. *Let E be a conull FK-space; then $m \cap E$ is nonseparable in m.*

Proof. By Theorem 5 of [16], we see that $c \cap E$ is not conull. Again by Theorem 5 of [16], $c \cap E$ cannot be closed in E; the desired conclusion now follows at once from Theorem 8.

The next result was obtained by Agnew in [1] for a special class of conservative matrices and by Zeller in [22] in the form given here.

COROLLARY 2. *Let A be a conservative matrix which limits at least one bounded divergent sequence; then $c_A \cap m$ is nonseparable in m.*

Proof. Theorem 1 of [19] shows that c is not closed in c_A; the conclusion follows at once from Theorem 8.

$x \in \omega$ is called *almost convergent* (see [10]) if the limit

$$\lim_{p\to\infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p}$$

exists uniformly in n. ac denotes the space of all almost convergent sequences and we have the following inclusion relationships: $c \subset uap \subset ac \subset m$. In view of this it seems natural to ask whether Theorem 1(i) can be improved slightly by replacing "*uap*" by "*ac*". It turns out in fact that such an improvement is not possible; to see this, we recall Theorem 1 of [19] and note that Lorentz [10, p. 174] has given an example of a regular matrix A, limiting a bounded divergent sequence, with $m \cap c_A = ac \cap c_A$.

On the other hand, Berg [5] has shown that if a conservative matrix limits every periodic sequence, then $m \cap c_A$ properly contains $ac \cap c_A$. This naturally raises the following question, which will be discussed elsewhere. If E is an *FK*-space containing the periodic sequences and is such that $c \cap E$ is not closed in E, must the set $E \cap (m \backslash ac)$ be nonempty?

5. **The Baire category theorem and a result of Petersen.** Related to Theorems 6 and 7 is a rather difficult result of Petersen [15, Theorem 4.4.1]. We improve Petersen's result by giving an entirely different proof based upon the Baire category theorem. First we need the following well-known lemma [24, Chapter 17, IV].

LEMMA. *Let $\{F^n\}_{n=1}^{\infty}$ be a collection of FK-spaces such that $F^m \neq \bigcup_{n=1}^{\infty} F^n$, $m = 1, 2, \ldots$; then $\bigcup_{n=1}^{\infty} F^n$ is not an FK-space.*

THEOREM 9. *Let $\{A^n\}_{n=1}^{\infty}$ be a set of matrices such that each matrix A^n, $n = 2, 3, \ldots$, limits a bounded divergent sequence not limited by A^r, $r = 1, 2, \ldots, n-1$. If A is a matrix with $c_A \supseteq m \cap \bigcup_{n=1}^{\infty} c_{A^n}$, then A limits a bounded sequence which is not limited by any A^n.*

Proof. Putting $F^n = m \cap c_{A^n}$, $n = 1, 2, \ldots$, we see that the hypotheses of the lemma are satisfied. It follows that $m \cap \bigcup_{n=1}^{\infty} c_{A^n}$ is not an *FK*-space and so does not coincide with $m \cap c_A$.

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