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# OPERATOR AND DUAL OPERATOR BASES IN LINEAR TOPOLOGICAL SPACES(1)

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**Abstract.** A net $\{S_d : d \in D\}$ of continuous linear projections of finite range on a Hausdorff linear topological space $V$ is said to be a Schauder operator basis—S.O.B. —(resp. Schauder dual operator basis—S.D.O.B.) provided it is pointwise bounded and converges pointwise to the identity operator on $V$, and $S_e S_d = S_d$ (resp. $S_d S_e = S_d$) whenever $e \geqq d$.

S.O.B.'s and S.D.O.B.'s are natural generalizations of finite dimensional Schauder bases of subspaces. In fact, a sequence of operators is both a S.O.B. and S.D.O.B. iff it is the sequence of partial sum operators associated with a finite dimensional Schauder basis of subspaces.

We show that many duality-theory results concerning Schauder bases can be extended to S.O.B.'s or S.D.O.B.'s. In particular, a space with a S.D.O.B. is semi-reflexive if and only if the S.D.O.B. is shrinking and boundedly complete.

Several results on S.O.B.'s and S.D.O.B.'s were previously unknown even in the case of Schauder bases. For example, Corollary IV.2 implies that the strong dual of an evaluable space which admits a shrinking Schauder basis is a complete barrelled space.

I. **Introduction.** Let $V$ be a Hausdorff linear topological space. A basis of subspaces for $V$ is a sequence $\{M_n\}_{n=1}^{\infty}$ of subspaces of $V$ such that for every $x$ in $V$, there is a unique sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in M_n$ such that $x = \sum_{n=1}^{\infty} x_n$.

Bases of subspaces were introduced by Grinblyum in [6] and have been extensively studied by McArthur and his students. Note that a summation basis can be considered to be a basis of subspaces $\{M_n\}_{n=1}^{\infty}$ where each $M_n$ is one dimensional.

It is known that many of the theorems concerning summation bases have analogues in the theory of bases of subspaces. For example, the weak basis theorems of [1], [2], and [3] hold true for bases of subspaces if each $M_n$ is closed [10]; the characterization of reflexivity of a Banach space in terms of a boundedly complete and shrinking basis given by James in [7] is true for a basis of subspaces if each $M_n$ is reflexive [12].

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Most of the proofs of theorems involving a basis of subspaces do not use the $M_n$'s but consider instead a sequence of linear projections defined in terms of the $M_n$'s. Let $\{M_n\}_{n=1}^{\infty}$ be a basis of subspaces for $V$. For each $n$, define the partial sum operator $S_n$: $V \to V$ by

$$S_n(x) = \sum_{i=1}^{n} x_i,$$

where $\sum_{i=1}^{\infty} x_i$ is the expansion of $x$ in terms of the elements of the $M_i$'s. Each $S_n$ is a linear projection of $V$ onto the span of $\bigcup_{i=1}^{n} M_i$. Further, for each $n$ and $m$, $S_n S_m = S_{\min(n,m)}$; and for each $x$ in $V$, $\lim_{n\to\infty} S_n(x) = x$. If each $S_n$ is continuous, $\{M_n\}_{n=1}^{\infty}$ is called a Schauder basis of subspaces for $V$ or a Schauder decomposition for $V$. In this paper we abstract the properties of the partial sum operators associated with a basis of subspaces to get the following definition:

Definition I.1. Let $V$ be a Hausdorff linear topological space and let $\{S_d : d \in D\}$ be a net of linear projections of finite range on $V$. $V$ is said to be an operator basis —O.B.—(resp. dual operator basis—D.O.B.) for $V$ provided

(1) for each $x$ in $V$, the net $\{S_d(x) : d \in D\}$ is bounded and converges to $x$;

(2) $S_e S_d = S_d$ whenever $e \geqq d$ (resp. $S_d S_e = S_d$ whenever $e \geqq d$);

(3) if $\{x_j\}$ is a net in $V$ which converges to $x$ and if $\lim_j S_d(x_j) = y_d$ uniformly on $d$ in $D$, and if $\lim_d y_d = x$, then $S_d(x) = y_d$ for all $d$ in $D$.

If each $S_d$ is continuous, $\{S_d\}$ is said to be a Schauder O.B.—S.O.B.—(resp. Schauder D.O.B.—S.D.O.B.). If, in addition, $\{S_d\}$ is uniformly bounded on bounded sets, $\{S_d\}$ is said to be a u.b.S.O.B. (resp. u.b.S.D.O.B.). If $\{S_d\}$ is equicontinuous, $\{S_d\}$ is said to be an e-S.O.B. (resp. e-S.D.O.B.).

Some comments are in order about the various parts of Definition I.1. In most of the work that follows, the assumption that each $S_d$ has finite dimensional range is not essential. In particular, this condition could be replaced in §II by the assumption of closed range and in §III by the assumption of reflexive range. However, it seems doubtful that such a general theory has any useful applications and the assumption of finite range considerably simplifies the proofs of several theorems. Note that $\{I\}$, where $I$ is the identity operator on $V$, satisfies all the conditions of Definition I.1 other than the finite range condition.

The pointwise boundedness condition in (1) does not arise in the definition of basis of subspaces because a convergent sequence is automatically bounded. This mild-seeming condition has powerful implications. Roughly speaking, one can say that the structure of a space with a S.O.B. or S.D.O.B. is determined by the structure of its bounded sets.

Condition (2) for O.B.'s says that the ranges of the $S_d$'s are directed by inclusion. Condition (2) for D.O.B.'s says that the null spaces of the $S_d$'s are directed by containment. Note that the sequence of partial sum operators associated with a basis of subspaces satisfies both of these conditions.

Condition (3) replaces the uniqueness of expansion condition for a basis of subspaces. The relationship between (3) and the uniqueness of expansion condition is made precise in Theorem II.1.

In §II, we consider conditions on $V$ which will guarantee that an O.B. or D.O.B. for $V$ is a S.O.B. or S.D.O.B. for $V$. The corresponding results for bases of subspaces are in [10].

§§III and IV, which can be read independently of §II, contain applications of S.O.B.'s and S.D.O.B.'s to the duality theory of locally convex spaces. Note that if $\{S_d : d \in D\}$ is a S.O.B. (resp. S.D.O.B.) for $V$, then $\{S_d^* : d \in D\}$ is a S.D.O.B. (resp. S.O.B.) for $V^*$ if $V^*$ is endowed with the weak* topology. (This relationship justifies the use of "dual" in "dual operator basis".)

It is shown in §III that a locally convex space with a S.D.O.B. is semireflexive if and only if the S.D.O.B. is shrinking (Definition III.5) and boundedly complete (Definition III.2). The corresponding theorem for Schauder bases in Banach spaces was proved by James in [7]. Retherford [11] established the "if" part of the theorem for Schauder bases in locally convex spaces, and he proved the "only if" part for Schauder bases in complete, reflexive spaces.

Theorems III.7, III.8, and III.10 discuss the duality between shrinking S.O.B.'s and boundedly complete S.D.O.B.'s. In [4], Dubinsky and Retherford prove III.7 and III.8 and a restricted variant of III.10 for Schauder bases in locally convex spaces.

The other results of §III are structure theorems for locally convex spaces with a S.O.B. or S.D.O.B. These theorems generalize the best known results for Schauder bases. A typical example is Theorem III.3, which shows that a space with a boundedly complete e-S.D.O.B. is complete. In [4], Dubinsky and Retherford show that a space with a boundedly complete e-Schauder basis is sequentially complete.

In §IV we deal with the structure of a space which admits a boundedly complete S.D.O.B. (Corollary IV.3 and Theorem IV.4). However, the most interesting and useful result of this section is Theorem IV.1, which guarantees the evaluability of a "large enough" subspace of $V^*$ when $V$ admits a u.b.S.D.O.B.

Throughout this work we use the notation and terminology of [9]. We also use the following notation: If $S$ is a linear operator, $R(S)$ denotes the range of $S$ and $\ker(S)$ denotes the null space of $S$. "$I$" denotes the identity operator, "sp $A$" denotes the linear span of $A$ and "cl $A$" denotes the closure of $A$. If $V$ is locally convex, the canonical embedding of $V$ into $V^{**}$ is denoted by "$\wedge$".

## II. Weak basis theorems.

Theorem II.1 justifies the assertion that O.B.'s and D.O.B.'s generalize the concept of finite dimensional bases of subspaces.

THEOREM II.1. *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of linear operators on a linear topological space $V$. Then $\{S_n\}_{n=1}^{\infty}$ is both an O.B. and D.O.B. for $V$ iff $\{S_n\}_{n=1}^{\infty}$ are the partial sum operators associated with a finite dimensional basis of subspaces for $V$.*

**Proof.** To go one way, suppose that $\{S_n\}_{n=1}^{\infty}$ is both an O.B. and D.O.B. for $V$.

Let $M_1 = R(S_1)$, $M_n = R(S_n - S_{n-1})$ $(n > 1)$. Note that each $M_n$ is finite dimensional. $\{M_n\}_{n=1}^{\infty}$ is representing, because if $x \in V$,

$$x = \lim_n S_n(x) = S_1(x) + \sum_{n=2}^{\infty} (S_n - S_{n-1})(x).$$

We show that this representation of $x$ is unique. Suppose that $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in M_n$. Since $\sum_{i=1}^{n} x_i - S_n(\sum_{i=1}^{j} x_i)$ is either 0 or $\sum_{i=j+1}^{n} x_i$ according as $j \geqq n$ or $j < n$, it follows from the convergence of $\sum_{n=1}^{\infty} x_n$ that

$$\lim_j S_n\left(\sum_{i=1}^{j} x_i\right) = \sum_{i=1}^{n} x_i$$

uniformly on $n$. We conclude from I.1(3) that $S_n(x) = \sum_{i=1}^{n} x_i$ for all $n$. Thus $S_1(x) = x_1$ and $(S_n - S_{n-1})(x) = x_n$ for $n > 1$.

To go the other way, suppose that $\{S_n\}_{n=1}^{\infty}$ are the partial sum operators associated with a finite dimensional basis of subspaces for $V$. $\{S_n\}_{n=1}^{\infty}$ obviously satisfies conditions (1) and (2) of I.1. Let $\{x_j\}$, $x$, and $\{y_n\}_{n=1}^{\infty}$ be as in (3) of I.1. Since $R(S_n)$ is finite dimensional for each $n$, $R(S_n - S_{n-1})$ is finite dimensional for each $n > 1$. In particular, these subspaces are closed. Thus $y_1 \in R(S_1)$ and for $n > 1$,

$$y_n - y_{n-1} = \lim_j (S_n - S_{n-1})(x_j) \in R(S_n - S_{n-1}).$$

But

$$x = \lim_n y_n = y_1 + \sum_{n=2}^{\infty} y_n - y_{n-1},$$

so that $y_1 = S_1(x)$ and $y_n - y_{n-1} = (S_n - S_{n-1})(x)$ for $n > 1$. From this it follows that $y_n = S_n(x)$, for all $n$. Q.E.D.

The following lemma provides most of the machinery for deriving the so-called "weak basis" theorems for O.B.'s and D.O.B.'s. The lemma is a straightforward generalization of Lemma 2 in [10].

LEMMA II.2. *Let* $(V, T)$ *be a* [*locally convex*] *linear topological space. Let* $\{S_d : d \in D\}$ *be a* [*weak*] *O.B. or a* [*weak*] *D.O.B. for* $V$. *Then there is a linear topology* $T'$ *for* $V$ *such that*

(1) $T \subset T'$;

(2) $\{S_d : (V, T') \to (V, T) : d \in D\}$ *is equicontinuous*;

(3) *if* $(V, T)$ *is locally convex, so is* $(V, T')$;

(4) *if* $(V, T)$ *is metrizable, so is* $(V, T')$;

(5) *if* $(V, T)$ *is complete or quasi-complete* (*bounded Cauchy nets are convergent*) *or sequentially complete, then so is* $(V, T')$.

**Proof.** Let $L$ be a local base of closed, circled $T$ neighborhoods of 0. If $T$ is locally convex, let each member of $L$ be convex. Let $L' = \{U' : U \in L\}$, where

$$U' = \bigcap_{d \in D} S_d^{-1}(U).$$

McArthur's proof of Lemma 2 in [10] shows that $L'$ is a local base for a linear topology, $T'$, on $V$, and that (1), (2), and (3) are satisfied. (4) is satisfied because if $T$ is metrizable, $L$ can be chosen to be countable, so that $L'$ is countable. We show that (5) holds. Suppose that $(V, T)$ is complete (resp. quasi-complete, resp. sequentially complete). Let $\{x_a : a \in A\}$ be a $T'$-Cauchy net (resp. bounded $T'$-Cauchy net, resp. $T'$-Cauchy sequence). By (1), $\{x_a\}$ is $T$-Cauchy (and $T$-bounded if $\{x_a\}$ is $T'$-bounded) and thus $T$-converges to, say, $x$. Since $\{x_a\}$ is $T'$-Cauchy, it follows from the definition of $T'$ that $\{S_d(x_a) : a \in A\}$ is $T$-Cauchy, uniformly on $d \in D$. Thus $T\text{-}\lim_a S_d(x_a)$ exists for each $d \in D$, and in fact uniformly on $d \in D$ [hence weakly uniformly on $d \in D$]. Let

$$y_d = T\text{-}\lim_{a} S_d(x_a).$$

We show that $\{y_d : d \in D\}$ [weakly] converges to $x$.

Let $K$ be a [weak] neighborhood of 0 in $(V, T)$. [Let $J$ be a weak neighborhood of 0 such that $J+J \subset K$.] Let $U \in L$ such that $U+U+U \subset K$ [such that $U+U \subset J$]. Let $N \in A$ such that if $a \geqq N$, $x-x_a \in U$. Choose $N' \in A$ such that if $a \geqq N'$, $S_d(x_a) - y_d \in U$ for all $d \in D$. Fix $a \in A$ such that $a$ follows both $N$ and $N'$. Choose $M \in D$ such that for $d \geqq M$, $x_a - S_d(x_a) \in U$ [such that $x_a - S_d(x_a) \in J$]. Then if $d \geqq M$, $x - y_d = (x-x_a)+(x_a - S_d(x_a))+(S_d(x_a)-y_d) \in K$. Thus $\{y_d\}$ [weakly] converges to $x$. It follows from I.1(3) that $S_d(x) = y_d$ for all $d \in D$. But $\{S_d(x_a) : a \in A\}$ $T$-converges to $y_d$ $(=S_d(x))$ uniformly on $d \in D$, so it follows from the definition of $T'$ that $\{x_a\}$ $T'$-converges to $x$. Q.E.D.

THEOREM II.3. *Let $\{S_d : d \in D\}$ be a weak S.O.B. for a locally convex space $V$. Suppose either that $\{S_d\}$ is u.b. and $V$ is evaluable, or that $V$ is barrelled. Then $\{S_d\}$ is an e-S.O.B. for $V$.*

**Proof.** Either hypothesis guarantees that $\{S_d\}$ is equicontinuous. Now $\{S_d(x) : d \in D\}$ is eventually $x$ if $x \in \bigcup_{d\in D} R(S_d)$, so $\{S_d : d \in D\}$ converges pointwise to the identity operator, $I$, on the subspace $\bigcup_{d \in D} R(S_d)$. Since $\{S_d\}$ is equicontinuous, it converges pointwise to $I$ on

$$\mathrm{cl}\left[\bigcup_{d\in D} R(S_d)\right] = \text{weak-cl}\left[\bigcup_{d\in D} R(S_d)\right] = V. \qquad \text{Q.E.D.}$$

THEOREM II.4. *An O.B. (resp. a D.O.B.) in a complete linear metric space is an e-S.O.B. (resp. an e-S.D.O.B.).*

**Proof.** Immediate from II.2 and the open mapping theorem. Q.E.D.

THEOREM II.5. *A weak O.B. for a Fréchet space is an e-S.O.B.*

**Proof.** II.2 and the open mapping theorem imply that the elements of the O.B. are continuous. The desired conclusion then follows from II.3. Q.E.D.

III. **Applications to duality theory.** In this section we let $V$ be a Hausdorff, locally convex, linear topological space. Endow $V^*$, the dual of $V$, with the strong topology $s(V^*, V)$. Recall that a local base for $V^*$ is $\{B^\circ : B$ is a bounded subset of $V\}$, where $B^\circ = \{f \in V^* : |f(b)| \leqq 1$, for all $b \in B\}$. Alternatively, $V^*$ has the topology of uniform convergence on bounded subsets of $V$.

If $P$ is a continuous linear operator on $V$ into $V$, define $P^*: V^* \to V^*$ by $P^*(f) = f \circ P$. $P^*$ is necessarily continuous on $V^*$ [9, p. 204, 21.6]. If $\{S_d : d \in D\}$ is a S.D.O.B. for $V$, let

$$Y = \bigcup_{d \in D} R(S_d^*),$$

and let $\bar{Y}$ be the strong closure of $Y$ in $V^*$. Note that $Y$, and hence $\bar{Y}$, is a linear subspace of $V^*$.

The main results of this section, Theorems III.11 and III.13, show that a locally convex space with a S.D.O.B. is semireflexive iff the S.D.O.B. is both shrinking and boundedly complete. Theorem III.16 then characterizes reflexivity of a space with a S.D.O.B. in terms of additional properties of the S.D.O.B.

Singer [13] noted a duality between shrinking and boundedly complete Schauder bases: if $\{S_n\}_{n=1}^\infty$ are the partial sum operators associated with a Schauder basis for a Banach space $V$, then $\{S_n\}_{n=1}^\infty$ is shrinking iff $\{S_n^*\}_{n=1}^\infty$ is a boundedly complete basis for $V^*$; $\{S_n\}_{n=1}^\infty$ is boundedly complete iff $\{S_n^*\}_{n=1}^\infty$ is a shrinking basis for $\bar{Y}$. Dubinsky and Retherford [4] extended this result to Schauder bases in certain kinds of locally convex spaces. Theorems III.7, III.8, and III.10 verify that under reasonable conditions on $V$ (which are satisfied whenever $V$ is quasi-complete and evaluable), there is a duality between shrinking and boundedly complete S.O.B.'s and S.D.O.B.'s.

The following known lemma is useful for obtaining the results of this section.

LEMMA III.1. (a) *Let $\{P_i : i \in J\}$ be a uniformly bounded family of continuous linear operators on $V$ into $V$. Then $\{P_i^* : i \in J\}$ is equicontinuous.*

(b) *Let $V$ be sequentially complete and let $\{P_i : i \in J\}$ be a pointwise bounded family of continuous linear operators on $V$. Then $\{P_i : i \in J\}$ is uniformly bounded.*

(c) *A semireflexive space is sequentially complete.*

**Proof.** (a) Let $B^\circ$ be a basic neighborhood of 0 in $V^*$. Since $B$ is bounded and $\{P_i : i \in J\}$ is uniformly bounded, $C = \{P_i(b) : i \in J, b \in B\}$ is bounded in $V$. Hence $C^\circ$ is a neighborhood of 0 in $V^*$. We assert that $P_i^*[C^\circ] \subset B^\circ$, for all $i \in J$. Suppose that $f \in C^\circ$, $i \in J$, and $b \in B$. Then $|P_i^*(f)(b)| = |f(P_i(b))| \leqq 1$, since $P_i(b) \in C$. Hence $P_i^*(f) \in B^\circ$, and thus $\{P_i^* : i \in J\}$ is equicontinuous.

(b) and (c) are immediate from [9, p. 105, 12.4] and [9, p. 190, 20.2], respectively. Q.E.D.

DEFINITION III.2. Let $\{S_d : d \in D\}$ be a S.D.O.B. for $V$. $\{S_d\}$ is *boundedly complete* iff, for each bounded net $\{x_d : d \in D\}$ in $V$ satisfying $S_e(x_d) = x_e$ for all $e \leqq d$, there is $x \in V$ such that $S_d(x) = x_d$ for all $d \in D$.

Of course, the statement in the above definition that $S_d(x)=x_d$ for all $d\in D$ is equivalent to the statement that $\{x_d\}$ is convergent.

THEOREM III.3. *Let $\{S_d : d\in D\}$ be a boundedly complete e-S.D.O.B. for $V$. Then $V$ is complete.*

**Proof.** Let $V^{\sim}$ be the completion of $V$. For each $d$ in $D$, let $S_d^{\sim}$ be the continuous extension of $S_d$ to $V^{\sim}$. Note that $\{S_d^{\sim} : d\in D\}$ is equicontinuous (see [9, p. 38, 5.5 ff.]), so since it converges to $I$ pointwise on $V$, it converges to $I$ pointwise on $\mathrm{cl}\,(V)=V^{\sim}$. Now $R(S_d^{\sim})=R(S_d)$ because the latter is finite dimensional and hence complete. Thus if $y$ is in $V^{\sim}$, $\{S_d^{\sim}(y) : d\in D\}$ is a bounded net in $V$ which converges to $y$. Obviously if $e\leqq d$, $S_e(S_d^{\sim}(y))=S_e^{\sim}(y)$. But since $\{S_d : d\in D\}$ is boundedly complete, $\{S_d^{\sim}(y) : d\in D\}$ must converge in $V$. In other words, $y$ is in $V$. Q.E.D.

REMARK III.4. It follows from Theorem III.3 and [9, p. 192, 20.4] that if $V$ is evaluable and has a boundedly complete u.b.S.D.O.B., then $V$ is barrelled.

DEFINITION III.5. Let $\{S_d : d\in D\}$ be a S.O.B. (resp. a S.D.O.B.) for $V$. $\{S_d\}$ is *shrinking* iff $\{S_d^* : d\in D\}$ is a S.D.O.B. (resp. a S.O.B.) for $V^*$.

THEOREM III.6. *Let $\{S_d : d\in D\}$ be a shrinking S.O.B. or S.D.O.B. for $V$. Then $\{S_d\}$ is uniformly bounded and consequently $\{S_d^* : d\in D\}$ is an e-S.D.O.B. or e-S.O.B. for $V^*$.*

**Proof.** For each $f\in V^*$, $\{S_d^*(f) : d\in D\}$ is bounded, hence, for each $f\in V^*$, $\{f\circ S_d : d\in D\}$ is uniformly bounded, hence $\{S_d : d\in D\}$ is uniformly bounded. The equicontinuity of $\{S_d^* : d\in D\}$ follows from III.1(a).

THEOREM III.7. *Let $\{S_d : d\in D\}$ be a shrinking S.O.B. for $V$, and suppose that $V$ is evaluable. Then $\{S_d^* : d\in D\}$ is a boundedly complete e-S.D.O.B. for $V^*$.*

**Proof.** In view of III.6 and III.1(a), we need prove only that $\{S_d^* : d\in D\}$ is boundedly complete. Let $\{x_d : d\in D\}$ be a strongly bounded net in $V^*$ such that $S_e^*(x_d)=x_e$ for all $e\leqq d$. Since $V$ is evaluable, $\{x_d : d\in D\}$ is equicontinuous [9, p. 192, 20.4]. Let $f$ be in $R(S_e)$. Then

$$\lim_d x_d(f) = \lim_d x_d(S_e(f)) = \lim_d S_e^*(x_d)(f) = x_e(f),$$

so that $\{x_d : d\in D\}$ converges pointwise on $\bigcup_{e\in D} R(S_e)$. Since $\{x_d : d\in D\}$ is equicontinuous and $\bigcup_{e\in D} R(S_e)$ is dense in $V$, $\lim_d x_d(f)$ exists for each $f$ in $V$. Let $x$ be defined by

$$x(f) = \lim_d x_d(f),$$

for all $f$ in $V$. $x$ is in $V^*$ (i.e., $x$ is continuous on $V$) because it is the pointwise limit of an equicontinuous net. Clearly $S_d^*(x)=x_d$ for all $d$ in $D$. Q.E.D.

THEOREM III.8. *Let $\{S_d : d\in D\}$ be a boundedly complete S.D.O.B. for a barrelled space $V$. Then $\{S_d^* : d\in D\}$ is a shrinking e-S.O.B. for $\overline{Y}$.*

**Proof.** Note that $\{S_d\}$ is equicontinuous since $V$ is barrelled so that, by III.1, $\{S_d^*\}$ is equicontinuous. A standard argument (e.g., that used in II.3) shows that $\{S_d^* : d \in D\}$ is an e-S.O.B. for $\bar{Y}$. Let $f$ be in $\bar{Y}^*$. Since $S_d$ is a projection, $R(S_d)^*$ and $R(S_d^*)$ are naturally isomorphic. Since $R(S_d)$ is finite dimensional and thus reflexive, we can find $x_d$ in $R(S_d)$ such that for all $y$ in $R(S_d^*)$, $y(x_d)=f(y)$. Now if $e \leqq d$, $y(x_d)=f(y)=y(x_e)$ for all $y$ in $R(S_e^*)$. Thus the totality of $R(S_e^*)$ over $R(S_e)$ implies that $S_e(x_d)=S_e(x_e)=x_e$ for all $e \leqq d$. We show that $\{x_d : d \in D\}$ is bounded. Let $y$ be in $V^*$. Since $\{S_d^*\}$ is equicontinuous, $\{S_d^*(y) : d \in D\}$ is bounded and hence $\{f(S_d^*(y)) : d \in D\}$ is bounded. But

$$\{f(S_d^*(y)) : d \in D\} = \{S_d^*(y)(x_d) : d \in D\} = \{y(x_d) : d \in D\}.$$

Thus $\{x_d : d \in D\}$ is weakly bounded, hence, bounded. Since $\{S_d : d \in D\}$ is boundedly complete, there is $x$ in $V$ such that $S_d(x)=x_d$ for all $d$ in $D$. Clearly $y(x)=f(y)$ for all $y$ in $Y$, so $y(x)=f(y)$ for all $y$ in $\bar{Y}$. This argument and the totality of $Y$ over $V$ show that $V$ is canonically (algebraically) isomorphic to $\bar{Y}^*$. But since the topology of $\bar{Y}^*$ is the topology of uniform convergence on bounded subsets of $\bar{Y}$ and the topology of $V$ is the stronger topology of uniform convergence on weak* bounded subsets of $V^*$ [9, p. 171, 18.7 and p. 156, 17.7], $\{S_d^{**} : d \in D\}$ is a S.D.O.B. for $\bar{Y}^*$. Q.E.D.

REMARK III.9. Under the hypotheses of III.8, $V$ and $\bar{Y}^*$ are isomorphic. Let $T$ be the barrelled topology on $V$ and let $T'$ be the topology on $V$ of uniform convergence on bounded subsets of $\bar{Y}$. We are asserting that $T=T'$. III.8 shows that $T'$ is weaker than $T$. Let $\{x_j\}$ be a net in $V$ which $T'$-converges to 0. Let $C$ be a weak* bounded subset of $V^*$. $C$ is strongly bounded [9, p. 171, 18.7], so that $J=\bigcup \{S_d^*[C] : d \in D\}$ is a strongly bounded subset of $\bar{Y}$, and thus $\{f(x_j)\}$ converges to 0 uniformly on $f$ in $J$. Let $\bar{J}$ be the weak* closure of $J$. Clearly $C$ is a subset of $\bar{J}$. Now $\{\hat{x}_j\}$ is a net of weak* continuous functions on $V^*$ and $\{\hat{x}_j(f)\}$ converges to 0 uniformly for $f$ in the weak* dense subset $J$ of $\bar{J}$, hence $\{\hat{x}_j(f)\}$ converges to 0 uniformly for $f$ in $\bar{J}$. Thus $T \subset T'$.

One obvious, but interesting, application of III.8 and III.9 is that a Banach space with a boundedly complete S.D.O.B. is isomorphic to a conjugate Banach space.

THEOREM III.10. *Let $\{S_d : d \in D\}$ be a S.D.O.B. for a quasi-complete space $(V, T)$ (i.e., bounded T-Cauchy nets are T-convergent). Suppose that $\{S_d^* : d \in D\}$ is a shrinking S.O.B. for $\bar{Y}$. Then $\{S_d : d \in D\}$ is boundedly complete.*

**Proof.** Note that by III.1(b), $\{S_d : d \in D\}$ is uniformly bounded, so that, by III.1(a), $\{S_d^* : d \in D\}$ is equicontinuous. Since $\bar{Y}$ is total over $V$, we can identify $V$ with a subset of $\bar{Y}^*$. The relativised topology, $T'$, induced on $V$ by the strong topology on $\bar{Y}^*$, is the topology of uniform convergence on (strongly) bounded subsets of $\bar{Y}$. Now, the weak* bounded and strongly-bounded subsets of $V^*$ agree because $V$ is quasi-complete [9, p. 170, 18.5], so that the proof of III.9 shows that $T$ is weaker than $T'$. Let $\{x_d : d \in D\}$ be a bounded net in $V$ such that $S_e(x_d)$

$=x_e$ for all $e \leqq d$. To show that $\{x_d : d \in D\}$ is $T$-convergent, it is sufficient to show that $\{x_d : d \in D\}$ is $T$-Cauchy. Now $\{x_d : d \in D\}$ is $T$-bounded, hence is equicontinuous on $V^*$. If $f$ is in $R(S_e^*)$,

$$\lim_d f(x_d) = \lim_d S_e^*(f)(x_d) = \lim_d f(S_e(x_d)) = f(x_e).$$

Since $\{x_d : d \in D\}$ is equicontinuous on $V^*$, $\lim_d f(x_d)$ exists for all $f$ in $\bar{Y}$. Define $F$ on $\bar{Y}$ by

$$F(f) = \lim_d f(x_d).$$

$F$ is continuous on $\bar{Y}$ because it is the pointwise limit of the equicontinuous net $\{x_d : d \in D\}$. Since $\{S_d^* : d \in D\}$ is shrinking, it follows that if $B$ is a bounded subset of $\bar{Y}$, then

$$\lim_d f(x_d) = F(f),$$

uniformly on $f$ in $B$. Thus $\{x_d : d \in D\}$ is $T'$-Cauchy, hence $T$-Cauchy. Q.E.D.

THEOREM III.11. *Let $\{S_d : d \in D\}$ be a S.D.O.B. for a semireflexive space $V$. Then $\{S_d\}$ is both shrinking and boundedly complete.*

**Proof.** Since $V$ is semireflexive it follows from III(c) and III(b) that $\{S_d\}$ is uniformly bounded, and thus from III(a) that $\{S_d^* : d \in D\}$ is equicontinuous. Since $V$ is semireflexive, the weak* and weak topologies on $V^*$ agree, so that our usual argument shows that $\{S_d^* : d \in D\}$ is a S.O.B. for $V^*$. Thus $\{S_d : d \in D\}$ is shrinking.

To show that $\{S_d\}$ is boundedly complete, we let $\{x_d : d \in D\}$ be a bounded net in $V$ such that $S_e(x_d) = x_e$ whenever $e \leqq d$. We show that $\{x_d : d \in D\}$ is weakly Cauchy (hence weakly convergent by [9, p. 190, 20.2]). Let $f$ be in $V^*$ and let $\varepsilon > 0$. Since $\{S_d : d \in D\}$ is shrinking and $\{x_d : d \in D\}$ is bounded, there is $\bar{d}$ in $D$ such that, if $d \geqq \bar{d}$, $|S_d^*(f)(x_i) - f(x_i)| < \varepsilon/2$ for all $i$ in $D$. Now suppose that $d$ and $e$ both follow $\bar{d}$. Pick $j$ in $D$ so that $j$ follows both $d$ and $e$. Then

$$\begin{aligned} |f(x_d) - f(x_e)| &\leqq |f(x_d) - f(x_j)| + |f(x_j) - f(x_e)| \\ &= |f(S_d(x_j)) - f(x_j)| + |f(x_j) - f(S_e(x_j))| \\ &= |S_d^*(f)(x_j) - f(x_j)| + |f(x_j) - S_e^*(f)(x_j)| < \varepsilon. \end{aligned}$$

Thus $\{x_d : d \in D\}$ is weakly Cauchy and thus weakly converges to, say, $x$. Clearly $S_d(x) = x_d$ for all $d$ in $D$, so $\{S_d : d \in D\}$ is boundedly complete. Q.E.D.

An easy modification of the above proof yields the following corollary.

COROLLARY III.12. *If $\{S_n\}_{n=1}^{\infty}$ is a shrinking S.D.O.B. for a weakly sequentially complete space, then $\{S_n\}_{n=1}^{\infty}$ is boundedly complete. In particular, a shrinking finite dimensional Schauder decomposition for a weakly sequentially complete space is boundedly complete (and hence the space is semireflexive by* III.13).

THEOREM III.13. *Let $\{S_d : d \in D\}$ be a boundedly complete, shrinking S.D.O.B. for $V$. Then $V$ is semireflexive.*

**Proof.** Let $F$ be in $V^{**}$. As in the proof of III.8, for each $d$ in $D$ there is $x_d$ in $R(S_d)$ such that for all $f$ in $R(S_d^*)$, $f(x_d)=F(f)$ and $S_e(x_d)=x_e$ whenever $e\leqq d$. We show that $\{x_d : d\in D\}$ is bounded. It is sufficient to show that $\{x_d\}$ is weakly bounded. Let $f$ be in $V^*$. Then $f(x_d)=f(S_d(x_d))=F(S_d^*(f))$. But $\{S_d^*(f) : d\in D\}$ is bounded and $F$ is continuous, so that $\{F(S_d^*(f)) : d\in D\}$ is bounded. This shows that $\{x_d : d\in D\}$ is bounded. Thus there is $x$ in $V$ such that $S_d(x)=x_d$ for all $d$ in $D$. Clearly $f(x)=F(f)$ for all $f$ in $Y$. Since $x$ and $F$ are both continuous, $f(x)=F(f)$ for all $f$ in $\bar{Y}$. But $\bar{Y}=V^*$, since $\{S_d : d\in D\}$ is shrinking. Thus the canonical embedding of $V$ into $V^{**}$ is onto, and $V$ is semireflexive. Q.E.D.

A semireflexive space is reflexive iff it is evaluable (or, equivalently, barrelled) [9, p. 194, 20.6 and 20.7]. Thus to characterize reflexive spaces which admit a S.D.O.B. it is natural to ask what properties a S.D.O.B. in a evaluable or barrelled space must have.

Suppose that $V$ is barrelled (resp. evaluable) and $\{S_d : d\in D\}$ is a S.O.B. or S.D.O.B. for $V$ (resp. a u.b.S.O.B. or u.b.S.D.O.B. for $V$). Let $f$ be a bounded linear functional on $V$ satisfying the condition that

$$\lim_d f(S_d(x)) = f(x)$$

for all $x$ in $V$. For each $d$ in $D$, $R(S_d)$ is finite dimensional, so that $f$ is continuous on $R(S_d)$. Thus $f\circ S_d$ is continuous for each $d$ in $D$. But $\{S_d : d\in D\}$ is uniformly bounded and $f$ is bounded, so that $\{f\circ S_d : d\in D\}$ is a uniformly bounded net of continuous linear functionals on $V$. It follows from [9, p. 191, 20.3] that $\{f\circ S_d : d\in D\}$ is equicontinuous. Thus $f$, the pointwise limit of $\{f\circ S_d : d\in D\}$, is continuous.

The preceding observation motivates Definition III.14.

DEFINITION III.14. Let $\{S_d : d\in D\}$ be a S.O.B. or S.D.O.B. for $V$. $\{S_d : d\in D\}$ is full iff every bounded linear functional, $f$, satisfying

$$\lim_d f(S_d(x)) = f(x)$$

for all $x$ in $V$, is continuous.

Obviously every S.D.O.B. or S.O.B. for a bound space is full. The remarks preceding Definition III.14 prove that a S.O.B. or S.D.O.B. for a barrelled space is full and that a u.b.S.O.B. or u.b.S.D.O.B. for an evaluable space is full.

The definition of full bases is similar to Jones' definition in [8] of $A'$ Schauder bases. One of Jones' results is that if $V$ admits an $A'$ Schauder basis, then $V^*$ is complete. Theorem III.15 extends this result to full S.O.B.'s and full S.D.O.B.'s. ($L(V, X)$ is the space of all continuous linear maps from $V$ to $X$ endowed with the topology of uniform convergence on bounded subsets of $V$.)

THEOREM III.15. *Let $\{S_d : d\in D\}$ be a full S.O.B. or full S.D.O.B. for $V$.* (a) *$V^*$ is complete.* (b) *If, in addition, $V$ is Mackey, then $L(V, X)$ is complete for every complete locally convex space $X$.*

**Proof.** (a) In view of [9, p. 169, 18.4], it is sufficient to show that every linear functional which is continuous on bounded sets is continuous. But this follows immediately from the definition of fullness. Q.E.D.

(b) As in (a), it is sufficient to show that every linear map $K$ on $V$ into X which is continuous on bounded sets is continuous. (a) guarantees that such a $K$ is continuous considered as a mapping from $V$ into $(X, w(X, X^*))$. But then by [9, pp. 203–204, 21.5 and 21.4] $K$ is Mackey continuous, hence, continuous. Q.E.D.

Let $V$ be reflexive and let $\{S_d : d \in D\}$ be a S.D.O.B. for $V$. $\{S_d : d \in D\}$ is shrinking and boundedly complete by III.11. $V$ is barrelled [9, p. 194, 20.6], so that $\{S_d : d \in D\}$ is full. Since $V$ is evaluable, every bound absorbing barrel is a neighborhood of 0. These observations lead us to consider the following theorem.

THEOREM III.16. *Let $(V, T)$ be a Mackey space and let $\{S_d : d \in D\}$ be a S.D.O.B. for $(V, T)$. Let $T'$ be the topology on $V$ which has for a local base the collection of all bound absorbing barrels in $(V, T)$. Then $(V, T)$ is reflexive iff $\{S_d : d \in D\}$ is shrinking, boundedly complete, and full, and $\{S_d : d \in D\}$ is a S.D.O.B. for $(V, T')$.*

**Proof.** The "only if" part follows from the remarks preceding the theorem. To go the other way, note that $(V, T)$ is semireflexive by III.13. Now suppose $f$ is a $T'$-continuous linear functional on $V$. Note that $(V, T)$ and $(V, T')$ have the same bounded sets, so that $f$ is a $T$-bounded linear functional. Since $\{S_d : d \in D\}$ is a S.D.O.B. for $(V, T')$ and $f$ is $T'$-continuous,

$$\lim_d f(S_d(x)) = f(x)$$

for all $x$ in $V$. It follows from the fullness of $\{S_d : d \in D\}$ that $f$ is $T$-continuous. Thus $T'$ is compatible with the duality $(V, V^*)$. Since $T$ is Mackey, $T = T'$, so that $T$ is evaluable. Q.E.D.

IV. **Further applications to duality theory.** In this section we continue to let $\{S_d : d \in D\}$ be a u.b.S.D.O.B. for the locally convex space $V$ and $\bar{Y}$ be the strong closure of $\bigcup_{d \in D} R(S_d^*)$.

It is well known that $V^*$ need not be evaluable even if $V$ is a Fréchet space with a Schauder basis (cf., e.g., [9, p. 221, G]). On the other hand we have the fundamental

THEOREM IV.1. *Let $\{S_d : d \in D\}$ be a u.b.S.D.O.B. for the locally convex space $V$ and let $T$ be the relativisation of $s(V^*, V)$ to $\bar{Y}$. Then $(\bar{Y}, T)$ is evaluable.*

**Proof.** Let $B$ be a bound absorbing barrel in $(\bar{Y}, T)$. Let $K = \bigcap_{d \in D} S_d^{*-1}[B]$. $K$ is a barrel in $V^*$ because, for each $f \in V^*$, $\{S_d^*(f) : d \in D\}$ is strongly bounded and is thus absorbed by $B$. Hence $\bar{K}$, the weak* closure of $K$, is a weak* barrel in $V^*$ and is thus a strong neighborhood of 0 in $V^*$. We intend to show that $\bar{K} \cap \bar{Y} \subset B$, from which it follows that $B$ is a neighborhood of 0 in $(\bar{Y}, T)$.

Let $y \in \bar{K} \cap \bar{Y}$ and let $\{y_j\}$ be a net in $K$ which weak* converges to $y$. By the definition of $K$, $S_d^*(y_j) \in B$ for each $d \in D$ and for each $j$. For each $d \in D$,

$$S_d^*(y) = \text{weak*-}\lim_j S_d^*(y_j)$$

because $S_d^*$ is weak* continuous. Now the strong topology must agree with the weak* topology on the finite dimensional space $R(S_d^*)$, hence also,

$$S_d^*(y) = \text{strong-}\lim_j S_d^*(y_j).$$

Thus $S_d^*(y) \in B$ because $B$ is strongly closed. Finally,

$$\text{strong-}\lim_j S_d^*(y) = y$$

because $y \in \bar{Y}$, so that by again using the strong closedness of $B$ we conclude that $y \in B$. Q.E.D.

COROLLARY IV.2. *If $V$ admits a shrinking S.D.O.B. or a shrinking S.O.B. then $V^*$ is evaluable. If, in addition, $V$ admits a full S.D.O.B. or a full S.O.B., then $V^*$ is barrelled.*

**Proof.** A shrinking S.D.O.B. or S.O.B. is uniformly bounded by III.6, so the proof of IV.1 shows that $V^*$ is evaluable. Since a complete evaluable space is barrelled [9, p. 192, 20.4], the second assertion follows from III.15. Q.E.D.

COROLLARY IV.3. *Let $V$ be a barrelled space which admits a boundedly complete S.D.O.B. $\{S_d : d \in D\}$. Then $V$ is isomorphic to the strong dual of a complete barrelled space.*

**Proof.** By III.9, $V$ is isomorphic to $\bar{Y}^*$. $\bar{Y}$ is evaluable by IV.1 and is complete by III.15, hence $\bar{Y}$ is barrelled by [9, p. 192, 20.4]. Q.E.D.

In case $V$ is not barrelled, there is a weaker form of IV.3 which is true:

THEOREM IV.4. *Let $\{S_d : d \in D\}$ be a u.b.S.D.O.B. for the locally convex space $(V, T)$. Consider the following properties:*

(1) *$V$ is semireflexive in the duality $(V, \bar{Y})$;*

(2) *$\{S_d : d \in D\}$ is boundedly complete;*

(3) *the canonical embedding of $V$ into $\bar{Y}^*$ is onto;*

(4) *$s(\bar{Y}, V)$ is the relativisation of $s(V^*, V)$ to $\bar{Y}$.*

*Then the following implications hold:*

$$\begin{array}{l}(1) \Rightarrow (2) \Leftrightarrow (3);\ (2)\ \textit{and}\ (4) \Rightarrow (1).\\ \Downarrow \\ (4)\end{array}$$

**Proof.** (1) $\Rightarrow$ (2). Note that $w(V, \bar{Y})$ is weaker than $T$ and each $S_d$ is $w(V, \bar{Y})$-continuous, so that $\{S_d : d \in D\}$ is a S.D.O.B. for $(V, w(V, \bar{Y}))$. Thus, by III.11, $\{S_d : d \in D\}$ is $w(V, \bar{Y})$-boundedly complete, from which it easily follows that (2) is true.

(2) $\Rightarrow$ (3). This implication follows from the proof of Theorem III.8.

(3) $\Rightarrow$ (2). Let $\{x_d : d \in D\}$ be a bounded net in $(V, T)$ such that $S_e(x_d) = x_e$

whenever $e \leqq d$. Note that $\{x_d : d \in D\}^\circ$ is a (strong) neighborhood of 0 in $V^*$, so that $\{\hat{x}_d : d \in D\}$ is equicontinuous on $V^*$. Now if $y \in R(S_d^*)$, then $\lim_e \hat{x}_e(y) = y(x_d)$. Thus $\lim_e \hat{x}_e(y)$ exists for each $y \in Y$. Since $\{\hat{x}_d : d \in D\}$ is equicontinuous, $\lim_e \hat{x}_e(y)$ exists for each $y$ in cl $Y = \bar{Y}$. Define $f$ on $\bar{Y}$ by $f(y) = \lim_e \hat{x}_e(y)$. $f$ is the pointwise limit of an equicontinuous net of functionals on $\bar{Y}$ and is thus continuous. That is, $f$ is in $\bar{Y}^*$. By (3), there is $x \in V$ such that $f(y) = y(x)$ for all $y \in \bar{Y}$. Clearly $S_d(x) = x_d$ for all $d \in D$, hence (2) holds.

(1) $\Rightarrow$ (4). $s(\bar{Y}, V)$ is obviously stronger than the relativisation of $s(V^*, V)$ to $\bar{Y}$. However, the latter topology is evaluable (and hence Mackey) by IV.1, and by (1) the dual of $\bar{Y}$ is $V$ when $\bar{Y}$ is endowed with either topology, so that the two topologies on $\bar{Y}$ are the same.

(2) and (4) $\Rightarrow$ (1). Note that (4) is just another way of saying that every $w(V, \bar{Y})$-bounded subset of $V$ is $T$-bounded. From this observation and (2) it follows that $\{S_d : d \in D\}$ is a shrinking, boundedly complete S.D.O.B. for $(V, w(V, \bar{Y}))$, so that III.13 applies. Q.E.D.

Remark IV.5. We continue to use the notation of IV.4. If $V^*$, or even $\bar{Y}$, is sequentially complete (as it will be when $\{S_d : d \in D\}$ is full), the implication (2) $\Rightarrow$ (1) is true. To see this, note that $\bar{Y}$ is barrelled because it is evaluable and sequentially complete, hence, $\bar{Y}^*$ is $w(\bar{Y}^*, \bar{Y})$-quasi-complete [9, p. 171, 18.7]. Since (3) holds, this last statement says that $V$ is $w(V, \bar{Y})$-quasi-complete, which is equivalent to (1).

Example IV.6. In the notation of IV.4, there is a space in which (2) holds but (1) fails. Let $V = l_1$ and let $V^* = \text{sp}\,(K \cup \varphi)$, where $K$ is the set of all real sequences $\{x_i\}_{i=1}^\infty$ such that $x_i = \pm 1$ and $\varphi$ is the set of all real sequences which have only finitely many nonzero terms. Note that $V^* \cap c_0 = \varphi$. It is not hard to see that a subset of $V$ is bounded in the $l_1$ norm iff it is $w(V, V^*)$ bounded. Thus the usual Schauder basis for $l_1$ is a u.b. boundedly complete Schauder basis for $(V, w(V, V^*))$ and the strong topology $s(V^*, V)$ is the sup-norm topology on $V^*$. Hence $\bar{Y} = c_0 \cap V^* = \varphi$. But a subset of $V$ is $w(V, \varphi)$ bounded iff it is coordinatewise bounded. Thus (4) fails and hence (1) fails. Note that this example also shows that $\bar{Y}$ need not be barrelled.

A slight modification of this example shows that (1) does not imply that $\bar{Y}$ is sequentially complete. Indeed, replace $\varphi$ by any proper barrelled subspace $L$ of $c_0$ such that $\varphi \subset L$. By combining the proof of IV.5 with Example IV.6, we conclude that (1) holds, but $L$ $(= \bar{Y})$ is not sequentially complete.

**Added in proof.** Independently, N. J. Kalton has discovered analogous results to many of the theorems of §III and §IV in the context of Schauder bases of subspaces. His work is titled *Schauder decompositions in locally convex spaces*, Proc. Cambridge Philos. Soc. **68** (1970), 377–392.

Prior to the discovery of our results, T. A. Cook (*Schauder decompositions and semi-reflexive spaces*, Math. Ann. **182** (1969), 232–235) had proved the analogues of III.11 and III.13 for Schauder bases of subspaces.

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