TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 167, May 1972

SOME CLASSES OF FLEXIBLE LIE-ADMISSIBLE ALGEBRAS(1)

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Abstract. Let $\mathfrak{A}$ be a finite-dimensional, flexible, Lie-admissible algebra over a field of characteristic $\neq 2$. Suppose that $\mathfrak{A}^-$ has a split abelian Cartan subalgebra $\mathfrak{H}$ which is nil in $\mathfrak{A}$. It is shown that if every nonzero root space of $\mathfrak{A}^-$ for $\mathfrak{H}$ is one-dimensional and the center of $\mathfrak{A}^-$ is 0, then $\mathfrak{A}$ is a Lie algebra isomorphic to $\mathfrak{A}^-$. This generalizes the known result obtained by Laufer and Tomber for the case that $\mathfrak{A}^-$ is simple over an algebraically closed field of characteristic 0 and $\mathfrak{A}$ is power-associative. We also give a condition that a Levi-factor of $\mathfrak{A}^-$ be an ideal of $\mathfrak{A}$ when the solvable radical of $\mathfrak{A}^-$ is nilpotent. These results yield some interesting applications to the case that $\mathfrak{A}^-$ is classical or reductive.

1. **Introduction.** Let $\mathfrak{A}$ be a finite-dimensional nonassociative algebra over a field. A. A. Albert [1] proved that if $\mathfrak{A}$ is a flexible algebra of characteristic $\neq 2, 3$ such that $\mathfrak{A}^+$ is a simple Jordan algebra of degree $\geqq 3$ then $\mathfrak{A}$ is either quasi-associative or a Jordan algebra. As an analog to this result, Laufer and Tomber [7] have proved that if $\mathfrak{A}$ is a flexible power-associative algebra over an algebraically closed field of characteristic 0 such that $\mathfrak{A}^-$ is a simple Lie algebra, then $\mathfrak{A}$ is a Lie algebra isomorphic to $\mathfrak{A}^-$. In this case it is shown that the simplicity of $\mathfrak{A}^-$ implies that $\mathfrak{A}$ is nil ([8], [10]). In the present paper, by assuming only that a split Cartan subalgebra of $\mathfrak{A}^-$ is nil in $\mathfrak{A}$, we extend the result of Laufer and Tomber to the case that the radical of $\mathfrak{A}^-$ is not 0 and the characteristic is not 2. We also obtain the same result for the algebra $\mathfrak{A}$ with $\mathfrak{A}^-$ classical.

If $\mathfrak{A}^-$ is a Lie algebra of characteristic 0, consider a Levi-factor $\mathfrak{S}$ of $\mathfrak{A}^-$. Although $\mathfrak{S}$ may be an ideal of $\mathfrak{A}^-$, $\mathfrak{S}$ need not be an ideal of $\mathfrak{A}$ even in case $\mathfrak{A}$ is a nilalgebra. Therefore, in terms of a Cartan subalgebra of $\mathfrak{S}$, we give a condition that $\mathfrak{S}$ be an ideal of $\mathfrak{A}$ when the solvable radical of $\mathfrak{A}^-$ is nilpotent. This provides some interesting applications to the case $\mathfrak{A}^-$ is reductive.

2. **Preliminaries.** For an algebra $\mathfrak{A}$, the algebra $\mathfrak{A}^-$ is defined as the same vector space as $\mathfrak{A}$ but with a multiplication given by $[x, y] = xy - yx$. Then $\mathfrak{A}$ is called Lie-

Presented to the Society, January 23, 1970 under the title *Flexible Lie-admissible algebras*; received by the editors February 22, 1971.

AMS 1969 *subject classifications*. Primary 1730, 1760, 1720.

Key words and phrases. Flexible algebra, power-associative algebra, Lie-admissible algebra, nilalgebra, Cartan subalgebra, root space, classical Lie algebra, Levi-factor, reductive Lie algebra.

(1) This paper is based on part of the author's doctoral thesis prepared at Michigan State University under the direction of Professor Marvin L. Tomber.

admissible if $\mathfrak{A}^-$ is a Lie algebra, that is, $\mathfrak{A}^-$ satisfies $[x, x]=0$ and the Jacobi identity $[x, [y, z]]+[y, [z, x]]+[z, [x, y]]=0$ for x, y, z in $\mathfrak{A}$. $\mathfrak{A}$ is said to be flexible if it satisfies the flexible law

$$(xy)x = x(yx) \tag{1}$$

and the linearized form of this is

$$(xy)z+(zy)x = x(yz)+z(yx), \tag{2}$$

or, equivalently,

$$L_{xy}-L_yL_x = R_{yx}-R_yR_x \tag{3}$$

where R_x and L_x are right and left multiplication by x in $\mathfrak{A}$.

If $\mathfrak{A}$ is an algebra over a field Φ of characteristic $\neq 2$, the algebra $\mathfrak{A}^+$ is defined as one with multiplication $x\cdot y=\frac{1}{2}(xy+yx)$ on the same vector space. If $\mathfrak{A}$ is flexible, by (1) we get $\frac{1}{2}[y^2, x]=y\cdot[y, x]$, and the linearization of this implies that $D_x\equiv R_x-L_x$ is a derivation of $\mathfrak{A}^+$, that is,

$$(y\cdot z)D_x = yD_x\cdot z+y\cdot zD_x \tag{4}$$

for all x, y, z in $\mathfrak{A}$. Hence if $\mathfrak{A}$ is flexible and Lie-admissible then D_x is a derivation of $\mathfrak{A}$ for every x in $\mathfrak{A}$, since D_x is a derivation of $\mathfrak{A}^-$ and $yz=y\cdot z+\frac{1}{2}[y, z]$. If we denote $T_x=\frac{1}{2}(R_x+L_x)$, from (4) we deduce

$$D_{y\cdot z} = D_yT_z+D_zT_y. \tag{5}$$

A power-associative algebra $\mathfrak{A}$ is said to be nil in case every element x of $\mathfrak{A}$ is nilpotent, that is, $x^n=0$ for some positive integer n. If $\mathfrak{A}$ is a flexible algebra of characteristic $\neq 2, 3, 5$, it is shown in [1] that $\mathfrak{A}$ is power-associative if and only if $x^2x^2=x^3x$ for all x in $\mathfrak{A}$.

Let $\mathfrak{A}$ now be a finite-dimensional, flexible, Lie-admissible algebra over a field Φ and $\mathfrak{H}$ be a split Cartan subalgebra of $\mathfrak{A}^-$. Then $\mathfrak{A}^-$ has a vector space decomposition; the Cartan decomposition relative to $\mathfrak{H}$:

$$\mathfrak{A}^- = \mathfrak{A}_\alpha+\mathfrak{A}_\beta+\cdots+\mathfrak{A}_\delta,$$

where $\mathfrak{A}_\alpha$, the root space corresponding to the root α, is the set of elements x_α in $\mathfrak{A}$ such that $x_\alpha(D_h-\alpha(h)I)^{r(h)}=0$ for all h in $\mathfrak{H}$. Then we have

$$\begin{aligned} [\mathfrak{A}_\alpha, \mathfrak{A}_\beta] &\subseteq 0 && \text{if } \alpha+\beta \text{ is not a root,} \\ &\subseteq \mathfrak{A}_{\alpha+\beta} && \text{if } \alpha+\beta \text{ is a root.} \end{aligned} \tag{6}$$

Let $\mathfrak{A}_{\alpha(h)}=\{x\in\mathfrak{A} \mid x(D_h-\alpha(h)I)^k=0 \text{ for some } k\}$ for h in $\mathfrak{H}$. Since D_h is a derivation of $\mathfrak{A}$, it is shown that

$$\begin{aligned} \mathfrak{A}_{\alpha(h)}\mathfrak{A}_{\beta(h)} &\subseteq 0 && \text{if } \alpha(h)+\beta(h) \text{ is not a characteristic root,} \\ &\subseteq \mathfrak{A}_{\alpha(h)+\beta(h)} && \text{if } \alpha(h)+\beta(h) \text{ is a characteristic root} \end{aligned}$$

(see [6, p. 54]). Recalling $\mathfrak{A}_\alpha=\bigcap_{h\in H}\mathfrak{A}_{\alpha(h)}$, we see that

$$\begin{aligned} \mathfrak{A}_\alpha\mathfrak{A}_\beta &\subseteq 0 && \text{if } \alpha+\beta \text{ is not a root,} \\ &\subseteq \mathfrak{A}_{\alpha+\beta} && \text{if } \alpha+\beta \text{ is a root.} \end{aligned} \tag{7}$$

Since $\mathfrak{H}$ is the root space corresponding to the root 0, (7) implies that any split Cartan subalgebra of $\mathfrak{A}^-$ is a subalgebra of $\mathfrak{A}$. If $\mathfrak{M}$ is a subset of $\mathfrak{A}$, then we note that the centeralizer of $\mathfrak{M}$ in $\mathfrak{A}^-$ is a subalgebra of $\mathfrak{A}$ and so is the center of $\mathfrak{A}^-$. For the basic results on Lie algebras we refer the reader to [3], [6], and [11].

Throughout this paper we assume, unless otherwise stated, that $\mathfrak{A}$ denotes a finite-dimensional, flexible, Lie-admissible algebra over a field Φ of characteristic $\neq 2$.

3. **General results in case** rad $\mathfrak{A}^- \neq 0$. This section is devoted to the proof of the theorem:

THEOREM 3.1. *Let $\mathfrak{A}$ be a finite-dimensional, flexible, Lie-admissible algebra over a field Φ of characteristic $\neq 2$. Suppose that $\mathfrak{A}^-$ has a split abelian Cartan subalgebra $\mathfrak{H}$ which is nil in $\mathfrak{A}$. If* $\dim \mathfrak{A}_\alpha = 1$ *for* $\alpha \neq 0$ *and $\mathfrak{A}^-$ has the center* 0, *then $\mathfrak{A}$ is a Lie algebra isomorphic to $\mathfrak{A}^-$.*

For the proof we first deduce some basic identities for flexible algebras and prove lemmas which are useful in this paper.

Let $\mathfrak{A}$ be a flexible algebra of characteristic $\neq 2$. Then from the flexible law (5) we get

$$D_{h^2} = 2D_hT_h = 2T_hD_h \tag{8}$$

for all h in $\mathfrak{A}$. If h is a power-associative element (that is, the subalgebra generated by h in $\mathfrak{A}$ is associative), then by (5) we see $D_{h^3} = D_hT_{h^2} + D_{h^2}T_h$ and so from (8) we have

$$D_{h^3} = D_h(2T_h^2 + T_{h^2}). \tag{9}$$

From (8) and (5) we also obtain

$$D_{h^4} = 2D_{h^2}T_{h^2} = 4D_hT_hT_{h^2} = 4T_{h^2}T_hD_h = D_hT_{h^3} + D_{h^3}T_h, \tag{10}$$

$$D_{h^5} = D_{h^2}T_{h^3} + D_{h^3}T_{h^2}. \tag{11}$$

LEMMA 3.2. *Let $\mathfrak{A}$ be a flexible algebra over a field Φ of characteristic $\neq 2$. Let h be a power-associative element of $\mathfrak{A}$. Then*

(i) *if an element x of $\mathfrak{A}$ is a characteristic vector of D_h and D_{h^2}, then $xD_{h^3} = xD_{h^4} = 0$ implies $xD_{h^2} = 0$, and furthermore*

(ii) *if x is a characteristic vector of D_h, D_{h^2}, R_h, and R_{h^2}, then $xD_{h^4} = xD_{h^5} = 0$ implies $xD_{h^3} = 0$.*

Proof. (i) Let $xD_h = \lambda x$ and $xD_{h^2} = \mu x$ for λ, μ in Φ. If $\lambda = 0$, we use (8) to obtain $\mu = 0$. If $\lambda \neq 0$, we get $xT_{h^2} = -2xT_h^2$ from (9). Hence, by (10), $0 = xD_{h^4} = 4xT_{h^2}T_hD_h = -8xT_h^3D_h = -8\lambda xT_h^3$ since $T_hD_h = D_hT_h$, thus $xT_h^3 = 0$. But then, by (8), $xT_h = \mu(2\lambda)^{-1}x$ and so $\mu = 0$; $xD_{h^2} = 0$.

(ii) Let $xD_h = \lambda x$, $xD_{h^2} = \mu x$, $xR_h = \nu x$, and $xR_{h^2} = \omega x$ for λ, μ, ν, ω in Φ. If $\lambda = 0$, we use (9) to conclude $xD_{h^3} = 0$. Now suppose $\lambda \neq 0$. If $\mu = 0$ then, by (8), $xT_h = 0$, and we use this together with (9) to obtain $xD_{h^3}=xD_hT_{h^2}=\lambda\omega x$. Hence by (11) this implies that $0=xD_{h^3}T_{h^2}=\lambda\omega xT_{h^2}=\lambda\omega^2x$ (recall $\mu=0$), thus $\omega=0$ and $xD_{h^3}=0$. If $\mu\neq 0$, then, by (10), $0=xD_{h^4}=2xD_{h^2}T_{h^2}=2\mu xT_{h^2}$ and so $xT_{h^2}=0$, and since $\lambda\neq 0$, this implies $xD_hT_{h^2}=0$. Hence, by (9), we have $xD_{h^3}=2xD_hT_h^2=2\lambda xT_h^2=2\lambda\bar{\nu}^2x$ where $\bar{\nu}=\nu-\frac{1}{2}\lambda$. Recalling $xT_{h^2}=0$, this and (11) imply $0=xD_{h^5}=xD_{h^2}T_{h^3}+xD_{h^3}T_{h^2}=\mu xT_{h^3}+2\lambda\bar{\nu}^2xT_{h^2}=\mu xT_{h^3}$ and so $xT_{h^3}=0$. Therefore from (10) it follows that $0=xD_{h^4}=xD_hT_{h^3}+xD_{h^3}T_h=\lambda xT_{h^3}+2\lambda\bar{\nu}^2xT_h=2\lambda\bar{\nu}^3x$, thus $\bar{\nu}=0$ and $xD_{h^3}=0$. This completes the proof.

LEMMA 3.3. *Let $\mathfrak{H}$ be a split Cartan subalgebra of $\mathfrak{A}^-$ with $h^2=0$ for all h in $\mathfrak{H}$. If α is a nonzero root of $\mathfrak{H}$ such that D_h is a scalar on $\mathfrak{A}_\alpha$ for all h in $\mathfrak{H}$, then $\mathfrak{A}_\alpha T_h=0$ for all h in $\mathfrak{H}$.*

Proof. Let h be any element of $\mathfrak{H}$ and let $D_h=\alpha(h)$ on $\mathfrak{A}_\alpha$. By (8), $D_hT_h=0$ and hence if $\alpha(h)\neq 0$, then $0=\mathfrak{A}_\alpha D_hT_h=\alpha(h)\mathfrak{A}_\alpha T_h$ implies $\mathfrak{A}_\alpha T_h=0$. If $\alpha(h)=0$, then $\alpha(h')\neq 0$ for some h' in $\mathfrak{H}$ since α is a nonzero root. We linearize (8) to get $0=\mathfrak{A}_\alpha D_hT_{h'}+\mathfrak{A}_\alpha D_{h'}T_h=\mathfrak{A}_\alpha D_{h'}T_h=\alpha(h')\mathfrak{A}_\alpha T_h$, and thus $\mathfrak{A}_\alpha T_h=0$. The proof is complete.

We are now ready to prove the theorem.

Proof of Theorem 3.1. Since $\mathfrak{H}$ is finite dimensional and is nil in $\mathfrak{A}$, there exists a positive integer $t>1$ such that $h^t=0$ for all h in $\mathfrak{H}$. We first show that $h^2=0$ for all h in $\mathfrak{H}$. Suppose $t\geqq 3$ and let n be the least integer such that $3n\geqq t$. For any element $h\in\mathfrak{H}$, let $g=h^n$. Then $g^3=0$. Since $\dim\mathfrak{A}_\alpha=1$ for $\alpha\neq 0$ and $\mathfrak{H}$ is abelian, by (7) and Lemma 3.2(i), g^2 belongs to the center of $\mathfrak{A}^-$ and so $g^2=0$, or $h^{2n}=0$. If $2n>4$, let m be the least positive integer with $3m\geqq 2n$. Then we see $m<n$. The above argument implies $h^{2m}=0$ and hence by repeated applications of this, we have either $h^4=0$ or $h^2=0$. But if $h^4=0$ then, by Lemma 3.2 (ii), $h^3=0$ and so $h^2=0$. Since $\mathfrak{H}$ is abelian, this implies that $\mathfrak{H}$ is a zero algebra i.e., $\mathfrak{H}^2=0$. Therefore if $\mathfrak{A}_\alpha=\Phi x$ and $[x,h]=\alpha(h)x$, then by Lemma 3.3 we have $xh=-hx=\frac{1}{2}\alpha(h)x$.

Let α,β be any nonzero roots. If $\alpha+\beta$ is not a root, by (7), $\mathfrak{A}_\alpha\mathfrak{A}_\beta=\mathfrak{A}_\beta\mathfrak{A}_\alpha=0$. Now suppose $\alpha+\beta$ is a root. If $\alpha+\beta=0$, choose an h in $\mathfrak{H}$ with $\alpha=\alpha(h)\neq 0$ and let $xh=\sigma x$ and $\mathfrak{A}_\beta=\Phi y$, so $hx=(\sigma-\alpha)x$. Since xy and yx are in $\mathfrak{H}$, from the flexible law $(hx)y-h(xy)+(yx)h-y(xh)=0$, we have $xy=\sigma\alpha^{-1}[x,y]$. If $\alpha+\beta$ is a nonzero root, then by (7) $\mathfrak{A}_\alpha\mathfrak{A}_\beta\subseteq\mathfrak{A}_{\alpha+\beta}=\Phi z$. Therefore, for any roots α and $\beta\neq 0$, we have

$$xy-yx=\lambda z,\qquad xy=\mu z,\qquad yx=(\mu-\lambda)z \tag{12}$$

for x in $\mathfrak{A}_\alpha$ and some $z\neq 0$ in $\mathfrak{A}_{\alpha+\beta}$. We now choose an h in $\mathfrak{H}$ with $\beta(h)\neq 0$, and let $\alpha=\alpha(h)$ and $\beta=\beta(h)$. Then by Lemma 3.3

$$xh=-hx=\tfrac{1}{2}\alpha x,\qquad zh=-hz=\tfrac{1}{2}(\alpha+\beta)z \tag{13}$$

for x in $\mathfrak{A}_\alpha$ and $z\neq 0$ in $\mathfrak{A}_{\alpha+\beta}$. We use $(hx)y-h(xy)+(yx)h-y(xh)=0$ together with (12) and (13) to obtain

$$\tfrac{1}{2}[-\alpha\mu+(\alpha+\beta)\mu+(\mu-\lambda)(\alpha+\beta)-\alpha(\mu-\lambda)]z=0.$$

Since $z \neq 0$, this gives $\beta(2\mu-\lambda)=0$ and so $\lambda=2\mu$. Therefore, from (12), $xy=-yx=\frac{1}{2}[x, y]$ and this holds for all x, y in $\mathfrak{A}$. This completes the proof.

It is shown in [7] that if $\mathfrak{A}^-$ is semisimple over an algebraically closed field of characteristic 0, then $\mathfrak{A}$ is a direct sum of ideals $\mathfrak{A}_i$ of $\mathfrak{A}$ such that the $\mathfrak{A}_i^-$'s are simple Lie algebras. In this case it follows that if $\mathfrak{A}$ is power-associative then $\mathfrak{A}$ is a nilalgebra ([8] and [10]) and hence $\mathfrak{A}^-$ being semisimple satisfies the conditions in Theorem 3.1. Therefore Theorem 3.1 generalizes the result of Laufer and Tomber.

A finite-dimensional Lie algebra $\mathfrak{L}$ over a field of characteristic $\neq 2, 3$ is called classical if (1) the center of $\mathfrak{L}$ is 0; (2) $\mathfrak{L}=[\mathfrak{L}, \mathfrak{L}]$; (3) $\mathfrak{L}$ has an abelian Cartan subalgebra $\mathfrak{H}$ (called a classical Cartan subalgebra), relative to which (a) $\mathfrak{L}=\sum_\alpha \mathfrak{L}_\alpha$ where the adjoint map ad h is a scalar on $\mathfrak{L}_\alpha$ for any $h \in \mathfrak{H}$ and α; (b) if $\alpha \neq 0$ is a root, dim $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]=1$; (c) if α and β are roots and $\beta \neq 0$, not all $\alpha+k\beta$ are roots. In any classical Lie algebra, dim $\mathfrak{L}_\alpha=1$ for $\alpha \neq 0$ (see [10, p. 30]). We also recall that classical Lie algebras need not be semisimple. As an immediate consequence of Theorem 3.1 we have

COROLLARY 3.4. *Suppose that $\mathfrak{A}^-$ is a classical Lie algebra having a classical Cartan subalgebra which is nil in $\mathfrak{A}$. Then $\mathfrak{A}$ is a Lie algebra isomorphic to $\mathfrak{A}^-$.*

In Theorem 3.1, the conditions that $\mathfrak{H}$ is abelian, dim $\mathfrak{A}_\alpha=1$ for $\alpha \neq 0$ and the center of $\mathfrak{A}^-$ is 0 are not strong enough to imply that $\mathfrak{H}$ is nil in $\mathfrak{A}$. The condition of center $\mathfrak{A}^-=0$ is also essential in the theorem. We now give examples for these two facts in the following:

EXAMPLE 3.5. Let $\mathfrak{A}$ be a 3-dimensional algebra over Φ with the multiplication table given by

$$xh = x, \qquad yh = \tfrac{1}{2}(\alpha+1)y, \qquad hy = \tfrac{1}{2}(1-\alpha)y, \qquad h^2 = h,$$

and all other products are 0, where $\alpha \neq 0$, 1 in Φ. We easily check that $\mathfrak{A}$ is flexible, and $\mathfrak{A}^-$ is given by

$$[x, y] = 0, \qquad [x, h] = x, \qquad [y, h] = \alpha y,$$

and so is a solvable Lie algebra. We also note that $\Phi h+\Phi x+\Phi y$ is the Cartan decomposition of $\mathfrak{A}^-$ for the Cartan subalgebra $\mathfrak{H}=\Phi h$ and $\mathfrak{A}_1=\Phi x$, $\mathfrak{A}_\alpha=\Phi y$ for the roots 1 and α. The center of $\mathfrak{A}^-$ is 0 but $\mathfrak{H}$ is not nil in $\mathfrak{A}$.

EXAMPLE 3.6. Let $\mathfrak{A}$ be a 4-dimensional algebra over the field Φ with the multiplication given by

$$xy = z+\tfrac{1}{2}h, \qquad yx = z-\tfrac{1}{2}h, \qquad xh = -hx = \tfrac{1}{2}x,$$
$$yh = -hy = -\tfrac{1}{2}y, \qquad h^2 = -z,$$

and all other products are 0. It can be shown that $\mathfrak{A}$ is flexible Lie-admissible but not associative since $(yh)h \neq yh^2=0$. We also see that $\mathfrak{H}=\Phi h+\Phi z$ is an abelian Cartan subalgebra of $\mathfrak{A}^-$ and $\mathfrak{A}^-=\mathfrak{H}+\Phi x+\Phi y$ is the Cartan decomposition for $\mathfrak{H}$ such that $\mathfrak{A}_1=\Phi x$ and $\mathfrak{A}_{-1}=\Phi y$ for the roots 1 and -1. It follows that $\mathfrak{H}$ is a nil subalgebra of $\mathfrak{A}$ such that $u^3=0$ for all u in $\mathfrak{H}$ and the center of $\mathfrak{A}^-$ is Φz, but $\mathfrak{A}$ is not a Lie algebra.

4. **Levi-factors of** $\mathfrak{A}^-$. Let $\mathfrak{A}$ be a finite-dimensional, flexible, Lie-admissible algebra over a field Φ of characteristic 0. A Levi-factor $\mathfrak{S}$ of $\mathfrak{A}^-$ is a semisimple subalgebra of $\mathfrak{A}^-$ such that $\mathfrak{A}^-=\mathfrak{R}+\mathfrak{S}$ (vector space direct sum) where $\mathfrak{R}$ is the solvable radical of $\mathfrak{A}^-$. Though $\mathfrak{S}$ may be an ideal of $\mathfrak{A}^-$, $\mathfrak{S}$ is not in general an ideal of $\mathfrak{A}$ even in case $\mathfrak{A}$ is nil. In fact, if $\mathfrak{A}$ is the algebra in Example 3.6, we see that $\mathfrak{A}$ is a nilalgebra with $u^3=0$ for all u in $\mathfrak{A}$ and $\mathfrak{S}=\Phi x+\Phi y+\Phi h$ is a Levi-factor of $\mathfrak{A}^-$ which is an ideal of $\mathfrak{A}^-$ but not of $\mathfrak{A}$. Here we notice that the Cartan subalgebra Φh of $\mathfrak{S}$, being nilpotent in $\mathfrak{A}$, is not a subalgebra of $\mathfrak{A}$. Therefore we wish to give a condition that the Levi-factor $\mathfrak{S}$ is an ideal of $\mathfrak{A}$ in terms of a Cartan subalgebra of $\mathfrak{S}$.

THEOREM 4.1. *Let $\mathfrak{A}$ be a finite-dimensional, flexible, power-associative, Lie-admissible algebra over an algebraically closed field Φ of characteristic* 0 *such that the radical $\mathfrak{R}$ of $\mathfrak{A}^-$ is nilpotent. Then a Levi-factor $\mathfrak{S}$ of $\mathfrak{A}^-$ is an ideal of $\mathfrak{A}$ if and only if $\mathfrak{S}$ has a Cartan subalgebra $\mathfrak{H}$ that is a nil subalgebra of $\mathfrak{A}$ and* $[\mathfrak{R}, \mathfrak{H}]=0$. *In this case $\mathfrak{R}$ is a subalgebra of $\mathfrak{A}$, and furthermore if $\mathfrak{A}$ is simple, then either $\mathfrak{A}$ is a Lie algebra or $\mathfrak{A}^-$ is nilpotent.*

Proof. Suppose that $\mathfrak{S}$ is an ideal of $\mathfrak{A}$. Then since $\mathfrak{S}$ is power-associative under the multiplication in $\mathfrak{A}$ and semisimple under Lie multiplication, by the remark following the proof of Theorem 3.1, $\mathfrak{S}$ is a Lie algebra under the multiplication in $\mathfrak{A}$. Hence any Cartan subalgebra of $\mathfrak{S}$ satisfies the conditions in the theorem.

Now suppose that $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{S}$ satisfying the conditions. We show $[\mathfrak{S}, \mathfrak{R}]=0$ and hence $\mathfrak{S}$ is an ideal of $\mathfrak{A}^-$. Let $\mathfrak{S}_{\alpha_0}$ be the root space of $\mathfrak{S}$ for the root $\alpha_0\neq 0$ of $\mathfrak{H}$. Then $\mathfrak{S}_{\alpha_0}=\Phi x$ and $\alpha_0=\alpha_0(h)\neq 0$ for some h in $\mathfrak{H}$. By the Jacobi identity $\alpha_0[\mathfrak{R}, x]=[\mathfrak{R}, \alpha_0 x]=[\mathfrak{R}, [x, h]]\subseteq[[\mathfrak{R}, h], x]+[[x, \mathfrak{R}], h]=0$, and hence $[\mathfrak{R}, x]=0$ and $[\mathfrak{R}, \mathfrak{S}]=0$. It now follows from [4, p. 20] that there exists a Cartan subalgebra $\mathfrak{H}'$ of $\mathfrak{A}^-$ such that $\mathfrak{H}'=\mathfrak{H}+\mathfrak{H}'\cap\mathfrak{R}$ and $\mathfrak{A}_\alpha=\mathfrak{A}_\alpha\cap\mathfrak{S}+\mathfrak{A}_\alpha\cap\mathfrak{R}$ where $\mathfrak{A}_\alpha$ is the root space of $\mathfrak{A}^-$ for $\mathfrak{H}'$ corresponding to the root α. Since $[\mathfrak{H}, \mathfrak{R}]=0$ and $\mathfrak{H}$ and $\mathfrak{R}$ are nilpotent, so is $\mathfrak{H}+\mathfrak{R}$ in $\mathfrak{A}^-$. Since $\mathfrak{H}'$ is maximal [3, p. 380], we see that $\mathfrak{H}'=\mathfrak{H}+\mathfrak{R}$. Therefore $\mathfrak{A}_\alpha\subseteq\mathfrak{S}$ for $\alpha\neq 0$ and $\mathfrak{S}\supseteq\mathfrak{H}+\sum_{\alpha\neq 0}\mathfrak{A}_\alpha$, that is,

$$\mathfrak{S} = \mathfrak{H}+\sum_{\alpha\neq 0}\mathfrak{A}_\alpha.$$

Since $\alpha(z)=0$ for $z\in\mathfrak{R}$, this is the Cartan decomposition of $\mathfrak{S}$ for $\mathfrak{H}$ and so $\dim\mathfrak{A}_\alpha=1$ for $\alpha\neq 0$. Since $\mathfrak{H}$ is a commutative nil subalgebra of $\mathfrak{A}$ and the center of $\mathfrak{S}$ is 0, by the same method as in the proof of Theorem 3.1, we obtain $\mathfrak{H}^2=0$, and hence, by Lemma 3.3, $L_h=-R_h$ on $\mathfrak{S}$ for all h in $\mathfrak{H}$. Therefore

$$xh = -hx = \tfrac{1}{2}[x, h] = \tfrac{1}{2}\alpha(h)x \tag{14}$$

for x in $\mathfrak{A}$.

We first prove that $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$. If α and β are nonzero roots and

$\alpha+\beta\neq 0$, then $\mathfrak{A}_\alpha\mathfrak{A}_\beta\subseteq\mathfrak{S}$. Let $\alpha\neq 0$ and let x and y be nonzero elements of $\mathfrak{A}_\alpha$ and $\mathfrak{A}_{-\alpha}$, respectively. Then, by (7), $xy\in\mathfrak{H}'$ and so $xy=h+z$ for $h\in\mathfrak{H}$ and $z\in\mathfrak{R}$. Since $\alpha\neq 0$, by (14), we may choose $h'\in\mathfrak{H}$ such that $yh'=-h'y=\lambda y$ for $\lambda\neq 0$ in Φ. From $(xy)h'+(h'y)x=x(yh')+h'(yx)$, we have $(h+z)h'-\lambda yx=\lambda xy+h'(h''+z)$ where $yx=h''+z$ for $h''\in\mathfrak{H}$. Since $\mathfrak{H}^2=0$ and $[\mathfrak{R},\mathfrak{H}]=0$, this implies $\lambda(xy+yx)=0$ and $xy+yx=0$ since $\lambda\neq 0$. Hence $h+h''+2z=0$ and so $h+h''=z=0$. Therefore $xy=-yx$ belongs to $\mathfrak{H}$ and by (14) this proves that $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$.

We now prove that $\mathfrak{S}$ is an ideal of $\mathfrak{A}$. Since $\mathfrak{S}$ is a subalgebra of $\mathfrak{A}$, as before, we see that $\mathfrak{S}$ is a Lie algebra under the multiplication in $\mathfrak{A}$. Hence for a root $\alpha\neq 0$ of $\mathfrak{H}$ there exists a 3-dimensional simple subalgebra $\mathfrak{S}^{(\alpha)}$ of $\mathfrak{S}$ with a basis x, y, h such that

$$xh = x, \qquad yh = -y, \qquad xy = h, \tag{15}$$

where $\mathfrak{A}_\alpha=\Phi x$ and $\mathfrak{A}_{-\alpha}=\Phi y$. If z is any element of $\mathfrak{R}$, then we have

$$xz = zx = \lambda x, \qquad yz = zy = \mu y \tag{16}$$

for λ, μ in Φ. Writing $\mathfrak{H}'=\Phi h+\mathfrak{B}$ (vector space direct sum), we see that $hz=zh=\nu h+b$ for $b\in\mathfrak{B}$. Equation (2) applied to x, y, z together with (15) and (16) implies $2(\nu h+b)=2\mu h$ and so $\nu=\mu$ and $b=0$. Similarly, (2) applied to y, x, z, (15), and (16) give $\nu=\lambda$. Since there exists a basis $h_1, h_2, \ldots, h_r$ of $\mathfrak{H}$ such that each h_i is embedded in the canonical basis of $\mathfrak{S}^{(\alpha)}$ as in (15) for a root $\alpha\neq 0$ (see [6]), we also see that $\mathfrak{H}\mathfrak{R}=\mathfrak{R}\mathfrak{H}\subseteq\mathfrak{H}$. Therefore $\mathfrak{S}$ is an ideal of $\mathfrak{A}$ and moreover

$$xz = zx = \lambda x, \qquad yz = zy = \lambda y, \qquad hz = zh = \lambda h \tag{17}$$

for $z\in\mathfrak{R}$ and $\alpha\neq 0$, where $\lambda\in\Phi$ depends on z and α.

Since $[\mathfrak{S},\mathfrak{R}]=0$ and the center of $\mathfrak{S}$ is 0, $\mathfrak{R}=\{x\in\mathfrak{A}\mid[x,\mathfrak{S}]=0\}$ and so $\mathfrak{R}$ is a subalgebra of $\mathfrak{A}$. If $\mathfrak{A}^-$ is not nilpotent and $\mathfrak{A}$ is simple, then $\mathfrak{S}\neq 0$ is an ideal of $\mathfrak{A}$ and $\mathfrak{A}=\mathfrak{S}$. Hence $\mathfrak{A}$ is a Lie algebra. This completes the proof of Theorem 4.1.

REMARKS. (1) Under the conditions in Theorem 4.1, $\mathfrak{S}$ is the unique Levi-factor of $\mathfrak{A}^-$. Indeed, let $\mathfrak{S}'$ be any Levi-factor of $\mathfrak{A}^-$. Then by the theorem of Malcev-Harish-Chandra there exists an invariant automorphism A such that $\mathfrak{S}^A=\mathfrak{S}'$, but since $[\mathfrak{S},\mathfrak{R}]=0$, A is the identity map on $\mathfrak{S}$ and $\mathfrak{S}=\mathfrak{S}'$.

(2) Power-associativity for $\mathfrak{A}$ is needed for the "only if" part of the proof, while, in view of Theorem 3.1, the "if" part of the proof requires only that the Cartan subalgebra $\mathfrak{H}$ is power-associative in $\mathfrak{A}$.

COROLLARY 4.2. *Let* $\mathfrak{A}$, $\mathfrak{R}$, $\mathfrak{S}$, *and* $\mathfrak{H}$ *be the same as in Theorem* 4.1. *For a root* $\alpha\neq 0$ *of* $\mathfrak{H}$, *let* $\mathfrak{S}^{(\alpha)}$ *be the* 3-*dimensional simple Lie algebra given by* (15). *Then for any root* $\alpha\neq 0$ *of* $\mathfrak{H}$, *there exists a subalgebra* $\mathfrak{B}_\alpha$ *of* $\mathfrak{A}$ *such that*

(i) *the Lie algebra* $\mathfrak{B}_\alpha^-=\mathfrak{R}+\mathfrak{S}^{(\alpha)}$ *is a Levi decomposition of* $\mathfrak{B}_\alpha^-$, *and*

(ii) *the multiplications between* $\mathfrak{R}$ *and* $\mathfrak{S}^{(\alpha)}$ *are given by* $sz=zs=f_\alpha(z)s$ *for* $s\in\mathfrak{S}^{(\alpha)}$ *and* $z\in\mathfrak{R}$, *where* f_α *is a linear function on* $\mathfrak{R}$.

Proof. Since $\mathfrak{N}$ and $\mathfrak{S}^{(\alpha)}$ are subalgebras of $\mathfrak{A}$ and $\mathfrak{N}\mathfrak{S}^{(\alpha)}=\mathfrak{S}^{(\alpha)}\mathfrak{N}\subseteq\mathfrak{S}^{(\alpha)}$, $\mathfrak{B}_\alpha=\mathfrak{N}+\mathfrak{S}^{(\alpha)}$. Clearly $\mathfrak{N}+\mathfrak{S}^{(\alpha)}$ is a Levi decomposition for $\mathfrak{B}_\alpha^-$ and the second part follows from (17).

COROLLARY 4.3. *Let $\mathfrak{A}$ and $\mathfrak{N}$ be the same as in Theorem* 4.1. *If $\mathfrak{N}$ is nil in $\mathfrak{A}$, then the linear function f_α is* 0 *for $\alpha\neq 0$ and $\mathfrak{N}$ is an ideal of $\mathfrak{A}$.*

Proof. Let z be an element of $\mathfrak{N}$ and α a nonzero root. Suppose that R_z is nilpotent. Then, by Corollary 4.2(ii), $f_\alpha(z)=0$. Hence $\mathfrak{N}$ is an ideal of $\mathfrak{A}$. Since D_z is nilpotent, it suffices to prove the following:

LEMMA 4.4. *Suppose that $\mathfrak{A}$ is a flexible, power-associative algebra over Φ of arbitrary dimension. If x is a nilpotent element of $\mathfrak{A}$ such that D_x is nilpotent, then so are R_x and L_x.*

Proof. If x is nilpotent in $\mathfrak{A}$, so is x in the algebra $\mathfrak{A}^+$. Hence, by [5], $T_x=\frac{1}{2}(R_x+L_x)$ is nilpotent, and by the flexible law $L_xR_x=R_xL_x$, $T_x+\frac{1}{2}D_x=R_x$ and $T_x-\frac{1}{2}D_x=L_x$ are also nilpotent.

5. **The reductive case.** A finite-dimensional Lie algebra $\mathfrak{L}$ over a field of characteristic 0 is called reductive if ad $\mathfrak{L}$ is completely reducible. It is known that $\mathfrak{L}$ is reductive if and only if the quotient algebra $\mathfrak{L}/\mathfrak{Z}$ by the center $\mathfrak{Z}$ of $\mathfrak{L}$ is semisimple [3, p. 255]. One can easily see that if $\mathfrak{L}$ is reductive, the radical of $\mathfrak{L}$ coincides with $\mathfrak{Z}$. Let $\mathfrak{S}$ be a Levi-factor of $\mathfrak{L}$. Then $[\mathfrak{L},\mathfrak{L}]=[\mathfrak{S},\mathfrak{S}]=\mathfrak{S}$ since $\mathfrak{S}$ is semisimple. Hence if $\mathfrak{L}$ is reductive, $\mathfrak{L}$ has a unique Levi-factor $[\mathfrak{L},\mathfrak{L}]$. Therefore by Theorem 4.1 (recall the remark to Theorem 4.1) we obtain

THEOREM 5.1. *Let $\mathfrak{A}$ be a finite-dimensional, flexible algebra over a field of characteristic* 0 *such that $\mathfrak{A}^-$ is a reductive Lie algebra. If $[\mathfrak{A},\mathfrak{A}]$ has a split Cartan subalgebra that is a power-associative, nil subalgebra of $\mathfrak{A}$, then $[\mathfrak{A},\mathfrak{A}]$ is an ideal of $\mathfrak{A}$ and so is a Lie algebra. Moreover, if $\mathfrak{A}$ is simple, then $\mathfrak{A}$ is either commutative or a Lie algebra.*

In Theorem 5.1 the algebra $\mathfrak{A}$ need not be power-associative.

EXAMPLE 5.2. Let $\mathfrak{S}$ be the 3-dimensional Lie algebra over a field Φ of characteristic $\neq 2, 3, 5$ such that $xh=x$, $yh=-y$, $xy=h$. Let $\mathfrak{A}(\alpha)=\mathfrak{S}+\Phi z$ (vector space direct sum) be the algebra defined by $zs=sz=\alpha s$ for all s in $\mathfrak{S}$ and $z^2=z$, $\alpha\in\Phi$. Then $\mathfrak{A}(\alpha)$ is flexible Lie-admissible and $[\mathfrak{A}(\alpha),\mathfrak{A}(\alpha)]=\mathfrak{S}$ has a Cartan subalgebra Φh with $h^2=0$. We see that $u^3u=u^2u^2$ for all $u\in\mathfrak{A}(\alpha)$ if and only if $2\alpha^3-3\alpha^2+\alpha=0$; that is, $\mathfrak{A}(\alpha)$ is power-associative if and only if $\alpha=0,\frac{1}{2}$, or 1. We also note that $\mathfrak{A}(\alpha)$ is isomorphic to $\mathfrak{A}(\beta)$ if and only if $\alpha=\beta$.

THEOREM 5.3. *Let $\mathfrak{A}$ be a finite-dimensional, flexible, power-associative algebra over a field of characteristic* 0 *such that $\mathfrak{A}^-$ is a reductive Lie algebra. If there exists a split Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{A}^-$ with $h^3=0$ for $h\in\mathfrak{H}$, then the center $\mathfrak{Z}$ of $\mathfrak{A}^-$ is an ideal of $\mathfrak{A}$. Moreover if $\mathfrak{A}$ is simple then it is a Lie algebra.*

Proof. We recall that dim $\mathfrak{A}_\alpha = 1$ for $\alpha \neq 0$ and $\mathfrak{H}$ abelian. Hence, by Lemma 3.2(i), h^2 belongs to $\mathfrak{Z}$ for all $h \in \mathfrak{H}$. Let h_1, h_2 be elements of $\mathfrak{H}$. Then $(h_1+h_2)^2 = h_1^2 + 2h_1h_2 + h_2^2$ is in $\mathfrak{Z}$ and hence $\mathfrak{H}^2 \subseteq \mathfrak{Z}$. Since $\mathfrak{Z}$ is contained in $\mathfrak{H}$, by Lemma 4.4, $\mathfrak{Z}\mathfrak{A}_\alpha = 0$ for $\alpha \neq 0$ and $\mathfrak{Z}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{A}$ is simple, either $\mathfrak{Z} = 0$ or $\mathfrak{Z} = \mathfrak{A}$. Suppose $\mathfrak{Z} = \mathfrak{A}$. Then $\mathfrak{A}$ is a commutative algebra with $x^3 = 0$ for $x \in \mathfrak{A}$, and by [1, p. 557] $\mathfrak{A}$ is a Jordan algebra. But since there is no simple Jordan nilalgebra of finite dimension, we must have $\mathfrak{Z} = 0$ and therefore, by Theorem 3.1, $\mathfrak{A}$ is a Lie algebra, and the proof is complete.

Remarks. In case $\mathfrak{A}$ is nil and simple, Theorem 4.1 allows $\mathfrak{A}^-$ nilpotent. At the present time, it is not known whether there exist simple, flexible, Lie-admissible nilalgebras $\mathfrak{A}$ with $\mathfrak{A}^-$ nilpotent even in commutative case of dimension > 3. For instance, it is not known that if $\mathfrak{A}$ is a flexible Lie-admissible nilalgebra with $\mathfrak{A}^-$ nilpotent, then $\mathfrak{A}$ is nilpotent. In the characteristic 0 and commutative case, Gerstenhaber [5] has shown that if $\mathfrak{A}$ is a nilalgebra of dimension $\leqq 3$ then $\mathfrak{A}$ is nilpotent. If there is no simple, commutative nilalgebra, by a theorem of Block [2], it is shown that any finite-dimensional, simple, flexible nilalgebra of characteristic $\neq 2$ is anticommutative. From the result of Gerstenhaber and the theorem of Block it also follows that any simple flexible nilalgebra of dimension $\leqq 3$ is anticommutative if the characteristic is 0.

In connection with these questions and results, for the characteristic 0 and non-commutative Lie-admissible case, the author has however been able to prove in [9] that (i) if $\mathfrak{A}$ is a finite-dimensional flexible nilalgebra such that $\mathfrak{A}^-$ is nilpotent and has an abelian ideal $\mathfrak{B}$ of codimension 1, then the center of $\mathfrak{A}^-$ is an ideal of $\mathfrak{A}$ as well as $\mathfrak{B}$, (ii) if $\mathfrak{A}$ is a flexible nilalgebra of dimension $\leqq 4$ such that $\mathfrak{A}^-$ is nilpotent, then $\mathfrak{A}$ is also nilpotent, and (iii) if $\mathfrak{A}$ is a simple flexible nilalgebra of dimension $\leqq 4$ then $\mathfrak{A}$ is a 3-dimensional simple Lie algebra. The first two results together with Theorem 4.1 may be used to determine some special classes of simple flexible nilalgebras. The results (ii) and (iii) for an arbitrary dimension remain, as far as the author knows, unsolved.

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