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INTEGRAL DECOMPOSITION OF FUNCTIONALS ON C^*-ALGEBRAS

BY
HERBERT HALPERN[1]

Abstract. The spectrum of the center of the weak closure of a C^*-algebra with identity on a Hilbert space is mapped into a set of quasi-equivalence classes of representations of the C^*-algebra so that every positive σ-weakly continuous functional on the algebra can be written in a central decomposition as an integral over the spectrum of a field of states whose canonical representations are members of the respective quasi-equivalence classes except for a nowhere dense set. Various questions relating to disjointness of classes, factor classes, and uniformly continuous functionals are studied.

1. **Introduction.** It is now well known that *normal* (i.e. σ-weakly continuous positive) functionals on a von Neumann algebra can be written as integrals of fields of functionals over a base space. (It is not surprising then that many of the convergence theorems of the theory of integration have natural analogues in the theory of normal functionals.) Of significance are the structures of the base space, the measures and the fields of functionals involved in the integration. Since every C^*-algebra can be embedded as a weakly dense subalgebra of a von Neumann algebra so that the dual of the algebra is isometric isomorphic with the predual (i.e. the space of all σ-weakly continuous functionals) of the von Neumann algebra, every positive functional on a C^*-algebra can be written as an integral of a field of functionals on a base space. Here the field is obtained by restricting the field for the von Neumann algebra to the C^*-algebra.

For separable C^*-algebras a rather complete theory exists. Here the base spaces have been taken to be the state space of the algebra [25] or the set of quasi-equivalence classes of factor representations [16]. For the latter the measures which come from decompositions have been described [5]. The nonseparable case seems more difficult, however, probably since integration is essentially a countable process. An integration over the state space can be obtained, and Choquet's theorem, in a form proved by Bishop and de Leeuw, may be used to show that the measures involved vanish on Baire sets disjoint from the set of *primary* states (i.e. those states whose canonical representations give factor algebras) [29].

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However, such a condition still allows very pathological behavior [15] since Radon measures are not defined with respect to Baire sets only.

In this paper we also study integral decompositions. The viewpoint will be governed by global considerations developed in earlier work ([10]–[13]). Given a C^*-algebra $\mathscr{A}$ with identity represented on a Hilbert space H, it is shown that there is a map $\zeta \to (\zeta)$ of the spectrum Z of the center of the von Neumann algebra $\mathscr{A}''$ generated by $\mathscr{A}$ on H into the set of quasi-equivalence classes of representations of $\mathscr{A}$ such that every normal functional f (i.e. the restriction to $\mathscr{A}$ of a normal functional on $\mathscr{A}''$) can be written as an integral $f(A) = \int f_\zeta(A)\, d\nu(\zeta)$ and except on a nowhere dense set the canonical representation of $\mathscr{A}$ induced by the state f_ζ is in class (ζ). At present we cannot tell what the classes (ζ) are except in a type I algebra where the (ζ) are factor classes except for a nowhere dense set. For a von Neumann algebra $\mathscr{A}$ on H, the classes (ζ) and (ζ') are disjoint whenever $\zeta \neq \zeta'$. Further, it is shown that each class (ζ) may be taken to be a factor class if $\mathscr{A}$ is semifinite. This is an extension of a result of H. Takemoto, who considered a certain class of normal functionals on type I algebras [28].

Finally, some theorems concerning the separable case are stated without proof in order to place the foregoing material in proper perspective with the work of Mautner, Mackey, Dixmier, Sakai, Effros and Ernest (cf. Bibliography). Actually, central decompositions over the state space and over the set of quasi-equivalence classes of factor representations are the same. The decomposition obtained here is over the spectrum Z of commutative separable C^*-algebra on a Hilbert space. Every point $\zeta \in Z$ corresponds to a quasi-equivalence class (ζ) of representations. The measures which are important are spectral measures. There is a set X in Z such that $Z - X = N$ is a null set for every spectral measure, such that every (ζ) in X is a factor class and the classes in X are mutually disjoint. Further each positive functional on the algebra may be written as an integral of a field of $\{f_\zeta\}$ of positive functionals over Z to form a central decomposition so that f_ζ is in class (ζ) for all ζ in an F_σ-set whose complement has measure 0. This fits in with E. Effros' result [5] showing that those measures on the quasi-equivalence classes of factor representations with the Mackey-Borel structure which produce integral decompositions of the separable representations of the algebra are those measures for which there is a set M of measure 0 such that the Borel structure on the complement of M is induced by a Borel isomorphism of the complement and a complete metric space. In Effros' result M might vary from measure to measure. Here, however, the complete metric structure on Z is given and a set N of measure 0 with respect to all significant measures exists so that $Z - N$ consists of disjoint classes of factor representations.

2. **Decompositions.** Of primary importance in the following work are the following ideas. Let $\mathscr{A}$ be a type I von Neumann algebra with center $\mathscr{Z}$ on a Hilbert space H. Let E be an abelian projection in $\mathscr{A}$ with central support P. For each

$A \in \mathscr{A}$ there is a unique element $\tau_E(A)$ in $\mathscr{Z}P$ such that $EAE = \tau_E(A)E$. The function τ_E is a σ-weakly continuous $\mathscr{Z}$-module homomorphism of the $\mathscr{Z}$-module $\mathscr{A}$ into $\mathscr{Z}$-module $\mathscr{Z}P$. Now suppose E is a maximal abelian projection (i.e. an abelian projection whose central support is 1). For each maximal ideal ζ in the spectrum Z of $\mathscr{Z}$, the relation $f_\zeta(A) = \tau_E(A)^\wedge(\zeta)$ defines a state of $\mathscr{A}$. Here $C^\wedge$ denotes the Gelfand transform of C in $\mathscr{Z}$. Now for any positive functional f on a C^*-algebra $\mathscr{B}$ with identity, the set $L(f) = \{A \in B \mid f(B^*B) = 0\}$ defines a closed left ideal of $\mathscr{B}$. The factor $\mathscr{B}$-module $\mathscr{B} - L(f)$ has a natural inner product $\langle A - L(f), B - L(f)\rangle = f(B^*A)$. The completion of $\mathscr{B} - L(f)$ in this inner product is a Hilbert space $H(f)$. For each $A \in \mathscr{B}$, the function $B - L(f) \to AB - L(f)$ defines a bounded linear operator on the prehilbert space $\mathscr{B} - L(f)$. This operator may be extended uniquely to a bounded linear operator $\pi_f(A)$ on $H(f)$. The map $A \to \pi_f(A)$ is a representation of $\mathscr{B}$ on $H(f)$. It is called the *canonical representation* of $\mathscr{B}$ induced by f. If x_f is the vector $x_f = 1 - L(f)$ in $H(f)$, then $w_{x_f}\pi_f = f$. Here w_{x_f} denotes the functional $w_{x_f}(A) = (Ax_f, x_f)$. Now returning to the type I algebra $\mathscr{A}$, we let $[\zeta]$ be the smallest closed two-sided ideal in $\mathscr{A}$ containing the maximal ideal ζ. Then we have

$$[\zeta] = \text{uniform closure}\left\{\sum \{A_iB_i \mid 1 \leqq i \leqq n\} \,\middle|\, A_i \in \mathscr{A}, B_i \in \zeta, n = 1, 2, \ldots\right\}.$$

For each $A \in \mathscr{A}$, let $A(\zeta)$ denote the image of A under the canonical homomorphism of $\mathscr{A}$ onto $\mathscr{A}/[\zeta]$ and, for a subset S of $\mathscr{A}$, let $S(\zeta) = \{A(\zeta) \mid A \in S\}$. The algebra $\mathscr{A}(\zeta)$ is a C^*-algebra under the norm $\|A(\zeta)\| = \text{glb}\,\{\|A+B\| \mid B \in [\zeta]\}$. For future reference we notice that

$$\|A(\zeta)\| = \text{glb}\,\{\|AP\| \mid P \text{ is a projection in } \mathscr{Z} \text{ with } P^\wedge(\zeta) = 1\}.$$

Now the linear space $\mathscr{A}E(\zeta)$ is a Hilbert space under the inner product $(AE(\zeta), BE(\zeta)) = \tau_E(B^*A)^\wedge(\zeta)$, and $A \to A(\zeta)$ is a representation ψ_ζ of $\mathscr{A}$ on $H(\zeta)$. Here $A(\zeta)(BE(\zeta)) = ABE(\zeta)$. The representation ψ_ζ has kernel $[\zeta]$. Furthermore, there is an isometry U of $H(f_\zeta)$ onto $H(\zeta)$ such that $U(1 - L(f_\zeta)) = E(\zeta)$ and $U\pi_{f_\zeta}(A) = \pi_\zeta(A)U$ for every $A \in \mathscr{A}$.

The following theorem on metric convergence is needed.

LEMMA 1. *Let E be an abelian projection in a type* I *von Neumann algebra $\mathscr{A}$ on the Hilbert space H. Let $A \in \mathscr{A}$ and let $\{A_n\}$ be a sequence in $\mathscr{A}$ such that* $\lim A_nE = AE$ (*strongly*). *For every $x \in H$ and $\varepsilon > 0$, there is a projection P in the center $\mathscr{Z}$ of $\mathscr{A}$ and a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $\|x - Px\| < \varepsilon$ and* $\lim A_{n_i}EP = AEP$ (*uniformly*).

Proof. Let $B_n = \tau_E((A_n - A)^*(A_n - A))^{1/2}$ for $n = 1, 2, \ldots$, and let $\mathscr{B}$ be the von Neumann algebra on H generated by the B_n $(n = 1, 2, \ldots)$ and 1. Because $\mathscr{B}$ is contained in $\mathscr{Z}$, the functional w_x is a gage on $\mathscr{B}$ in the sense of I. E. Segal [27]. Thus by employing his Corollary 13.1, we may find a projection P in $\mathscr{B}$ and a subsequence $\{B_{n_i}\}$ of $\{B_n\}$ such that $\|x - Px\| < \varepsilon$ and $\lim B_{n_i}P = 0$ (uniformly) since $\{B_n\}$ converges strongly to 0. This means that $\lim A_{n_i}EP = AEP$ (uniformly).

REMARK. Lemma 1 may be derived from an earlier result on algebraic convergence of J. von Neumann [20].

Two representations π_1 and π_2 of a C^*-algebra $\mathscr{A}$ on the Hilbert spaces H_1 and H_2 respectively are said to be *quasi-equivalent* (denoted by $\pi_1 \sim \pi_2$) if there is an isomorphism ϕ of the von Neumann algebra generated by $\pi_1(\mathscr{A})$ on H_1 onto the von Neumann algebra generated by $\pi_2(\mathscr{A})$ on H_2 such that $\phi(\pi_1(A)) = \pi_2(A)$ for every $A \in \mathscr{A}$. A representation π_1 of $\mathscr{A}$ on H_1 is said to be a *subrepresentation* of π on H if there is a projection E in the commutant $\pi(\mathscr{A})'$ of $\pi(\mathscr{A})$ such that $E(H) = H_1$ and $\pi(A)E = \pi_1(A)$ for every A in $\mathscr{A}$. If π_1 and π_2 are subrepresentations of π corresponding respectively to the projections E_1 and E_2 in $\pi(\mathscr{A})'$, then $\pi_1 \sim \pi_2$ if and only if E_1 and E_2 have the same central supports in $\pi(\mathscr{A})'$. Then it is clear that π_1 and π_2 are quasi-equivalent whenever π_1 is quasi-equivalent to a subrepresentation of π_2 (in symbols, $\pi_1 \prec \pi_2$) and π_2 is quasi-equivalent to a subrepresentation of π_1. If there is no nonzero subrepresentation of π_1 which is quasi-equivalent to a subrepresentation of π_2, then π_1 and π_2 are said to be disjoint. Since the notion of quasi-equivalence is an equivalence relation, the set of representations of $\mathscr{A}$ may be partitioned into quasi-equivalence classes of representations. If π is a factor representation of $\mathscr{A}$, then every element in the quasi-equivalence class determined by π is a factor representation. Since every factor representation is quasi-equivalent to each of its nonzero subrepresentations, two quasi-equivalence classes of factor representations of $\mathscr{A}$ are either equal or disjoint ([16], [17], [3, §5]).

LEMMA 2. *Let $\mathscr{A}$ be a C^*-algebra with identity, let π be a representation of $\mathscr{A}$ on the Hilbert space $H(\pi)$, and let f be a normal functional on the von Neumann algebra $\pi(\mathscr{A})''$ generated by $\pi(\mathscr{A})$. If π_f is the canonical representation of $\pi(\mathscr{A})''$ on $H(f)$ induced by f, then $\pi_f \cdot \pi$ is quasi-equivalent to a subrepresentation of π.*

Proof. Let $\pi_1 = \pi_f \cdot \pi$ and let π_0 be the direct sum $\pi_0 = \pi_1 \oplus \pi$ of π_1 and π on the Hilbert space $H = H(f) \oplus H(\pi)$. Let x be a cyclic vector in $H(f)$ under $\pi_f(\pi(\mathscr{A})'')$ such that $w_x \cdot \pi_f = f$. Since $\pi(\mathscr{A})$ is weakly dense in $\pi(\mathscr{A})''$ and since π_f is σ-weakly continuous on $\pi(\mathscr{A})''$, the vector x is a cyclic vector under $\pi_1(\mathscr{A})$. Let $\{x_i\}$ be a sequence in $H(\pi)$ such that $\sum \|x_i\|^2 < +\infty$ and $f = \sum w_{x_i}$. Now the projections E and F of H on $H(f)$ and $H(\pi)$ respectively are in the commutant of $\pi_0(\mathscr{A})$ on H. Let P and Q be their respective central supports.

Suppose π_1 is not quasi-equivalent to a subrepresentation of π. Then the projection $R = P - PQ$ is a nonzero central projection of $\pi_0(\mathscr{A})''$. But $\pi_0(\mathscr{A})$ is strongly dense in $\pi_0(\mathscr{A})''$ and so there is a net $\{A_n\}$ in $\mathscr{A}$ such that $\lim \pi_0(A_n) = R$ (strongly). Because $RE \neq 0$ and because

$$E(H) = \text{closure}\,\{\pi_0(A)x \mid A \in \mathscr{A}\} = \text{closure}\,\{\pi_1(A)x \mid A \in \mathscr{A}\},$$

we have that $Rx \neq 0$. However, we have

$$\begin{aligned}(Rx, x) &= \lim w_x(\pi_0(A_n)) = \lim w_x \cdot \pi_f(\pi(A_n)) \\ &= \lim f(\pi(A_n)) = \lim \sum w_{x_i}(F\pi_0(A_n)) = \sum w_{x_i}(FR) = 0.\end{aligned}$$

This is a contradiction. Thus, we must have that $\pi_1 \prec \pi$. Q.E.D.

REMARK. The representation $\pi_f \cdot \pi = \pi_1$ on the Hilbert space $H(f)$ is unitarily equivalent to the canonical representation π_g of $g = f \cdot \pi$ on the Hilbert space $H(g)$. Indeed, for A, B, C in $\mathscr{A}$, we have that

$$\begin{aligned}(\pi_1(A)(\pi(B)-L(f)), \pi(C)-L(f))\\ = f(\pi(C^*AB)) = g(C^*AB) = (\pi_g(A)(B-L(g)), C-L(g)).\end{aligned}$$

Thus the relation $U(B-L(g)) = \pi(B)-L(f)$ defines a linear isometry U of the linear manifold $\mathscr{A}-L(g)$ in $H(g)$ onto the linear manifold $\pi(\mathscr{A})-L(f)$ in $H(f)$. We show that $\pi(\mathscr{A})-L(f)$ is dense in $H(f)$. Since $\pi(\mathscr{A})''-L(f)$ is dense in $H(f)$, it is sufficient to show that $\pi(\mathscr{A})-L(f)$ is dense in $\pi(\mathscr{A})''-L(f)$. If $A \in \pi(\mathscr{A})''$ there is a bounded net $\{A_n\}$ in $\pi(\mathscr{A})$ which converges strongly to A by the Kaplansky density theorem [14]. Then we have that

$$\lim \|(A-L(f))-(A_n-L(f))\|^2 = \lim f((A-A_n)^*(A-A_n)) = 0$$

since $\lim (A-A_n)^*(A-A_n) = 0$ (strongly) and f is strongly continuous. So $\pi(\mathscr{A}) -L(f)$ is dense in $\pi(\mathscr{A})''-L(f)$ and hence dense in $H(f)$. Because $\mathscr{A}-L(g)$ is dense in $H(g)$, the map U may be extended to a linear isometry, which we also call U, of $H(g)$ onto $H(f)$. By the extension of equalities, we obtain that $U\pi_g(A) = \pi_1(A)U$ for every A in $\mathscr{A}$.

THEOREM 3. *Let $\mathscr{A}$ be a C^*-algebra wih identity on a Hilbert space H, and let $\mathscr{A}''$ be the von Neumann algebra generated by $\mathscr{A}$. Let E_1 and E_2 be maximal abelian projections in the commutant $\mathscr{Z}'$ of the center $\mathscr{Z}$ of $\mathscr{A}$. For each ζ in the spectrum Z of $\mathscr{Z}$ let $f_{i\zeta}(A)$ be the state of $\mathscr{A}$ given by $f_{i\zeta}(A) = \tau_{E_i}(A)^\frown(\zeta)$ for every $A \in \mathscr{A}$ $(1 \leqq i \leqq 2)$. Let $\pi_{i\zeta}$ be the canonical representation induced by $f_{i\zeta}$ $(1 \leqq i \leqq 2)$. Then there is a nowhere dense set N such that $\pi_{1\zeta} \sim \pi_{2\zeta}$ for every $\zeta \notin N$.*

Proof. Let $E_1 = E$ and $E_2 = F$ and let U be a partial isometry in $\mathscr{Z}'$ with domain support E and range support F. Let P be a nonzero projection in $\mathscr{Z}$. First we show that there is a nonzero projection Q in $\mathscr{Z}P$ such that $\pi_{1\zeta} \prec \pi_{2\zeta}$ for every ζ in $\{\zeta \in Z \mid Q^\frown(\zeta) = 1\}$. There is no loss of generality in assuming that PE is a cyclic projection corresponding to the subspace closure $\{Ax \mid A \in \mathscr{Z}\}$ for some unit vector x in $P(H)$. There is a bounded sequence $\{B_n\}$ in the *-algebra generated by $\mathscr{A}P$ and $\mathscr{A}'P$ such that $\{B_n x\}$ converges to Ux since this *-algebra is strongly dense in $\mathscr{Z}'P$ [14]. Therefore, the sequence $\{B_n E\}$ converges strongly to UP. Using Lemma 1 and taking a nonzero projection in $\mathscr{Z}P$ and a subsequence of $\{B_n\}$ if necessary, we may assume that $\lim B_n EP = UP$ (uniformly). Thus, the sequence $\{\tau_E(B_n^* A B_n)\}$ of $\mathscr{Z}$-module homomorphisms converges uniformly to $\tau_F(A)$ on the unit sphere of $\mathscr{A}''P$.

Let n be a fixed positive integer and let $B = B_n$. Let $\{A_i \mid 1 \leqq i \leqq m\}$ and $\{C_i \mid 1 \leqq i \leqq m\}$ be finite subsets of $\mathscr{A}$ and $\mathscr{A}'P$ respectively with $\sum A_i C_i = B$. We have that

$$\tau_E(B^*AB) = \sum \{\tau_E(A_i^* A A_j C_i^* C_j) \mid 1 \leqq i, j \leqq m\},$$

for every A in $\mathscr{A}''$. Since each $C_i^*C_j$ can be written as a linear combination of positive elements in $\mathscr{A}'P$ and since $\tau_E(AC) \leqq \tau_E(A)\|C\|$ for every A in $\mathscr{A}''^+$ whenever C is in $\mathscr{A}'^+$, by the Radon-Nikodym theorem for module homomorphisms [12, §2] there is no loss of generality in assuming that there is a finite subset $\{A_i \mid 1 \leqq i \leqq 2m\}$ of elements of $\mathscr{A}''P$ such that $\tau_E(B^*AB) = \sum \{\tau_E(A_i^*AA_{i+m}) \mid 1 \leqq i \leqq m\}$ for every A in $\mathscr{A}''$. For each A_i there is a bounded sequence $\{A_{ij}\}$ of elements of $\mathscr{A}$ such that $\lim A_{ij}x = A_ix$. By choosing subsequences of each sequence $\{A_{ij}\}_j$ if necessary, we may find projections P_{nj} in $\mathscr{Z}P$ such that $\|P_{nj}x - x\| \leqq 4^{-n}(2m)^{-1}$ and such that $\lim {}_j A_{ij}EP_{ni} = A_iEP_{ni}$ (uniformly). Let $P_n = P_{n1} \cdots P_{n2m}$. We have that $\|P_nx - x\| \leqq 4^{-n}$ and $\lim A_{ij}EP_n = A_iEP_n$ (uniformly) for $i = 1, 2, \ldots, 2m$. Now letting n vary, we set $Q = \operatorname{glb}\{P_1P_2 \cdots P_n \mid n = 1, 2, \ldots\}$. Since

$$\begin{aligned} \|Qx - x\| &= \lim \|P_1 \cdots P_nx - x\| \\ &\leqq \limsup \sum \{\|P_0 \cdots P_jx - P_0 \cdots P_{j-1}x\| \mid 1 \leqq j \leqq n\} \\ &\leqq \sum 4^{-n} = 3^{-1}, \end{aligned}$$

where $P_0 = P$, we have that $Q \neq 0$.

Now let $X = \{\zeta \in Z \mid Q\hat{\ }(\zeta) = 1\}$. In the preceding paragraph we showed that given any $n = 1, 2, \ldots$ then there are $\{A_i \mid 1 \leqq i \leqq 2m\}$ in $\mathscr{A}P$ such that

$$\left\|\tau_F(A) - \sum \{\tau_E(A_i^*AA_{i+m}) \mid 1 \leqq i \leqq m\}\right\| \leqq n^{-1}\|A\|$$

for every A in $\mathscr{A}''Q$. Thus for every A in $\mathscr{A}''$ and every projection R in $\mathscr{Z}Q$, we have that

$$\left\|\left(\tau_F(A) - \sum \tau_E(A_i^*AA_{i+m})\right)R\right\| \leqq n^{-1}\|AR\|.$$

Thus for every ζ in X we have that

$$\left|\left(\tau_F(A) - \sum \tau_E(A_i^*AA_{i+m})\right)\hat{\ }(\zeta)\right| \leqq n^{-1}\|A(\zeta)\|$$

for every A in $\mathscr{A}''$ (cf. remarks at beginning of §2).

Now let ζ be fixed in X. Let $\psi = \psi_\zeta$ be the representation of $\mathscr{Z}'$ with kernel $[\zeta]$ on the Hilbert space $H(\zeta) = \mathscr{Z}'E(\zeta)$ given by $\psi(A)BE(\zeta) = ABE(\zeta)$. Let $K(\zeta)$ be the subspace of $H(\zeta)$ given by $K(\zeta) = \text{closure}\{\psi(A)E(\zeta) \mid A \in \mathscr{A}\}$. Then the projection G of $H(\zeta)$ on $K(\zeta)$ is in the commutant $\mathscr{A}(\zeta)'$ of $\mathscr{A}(\zeta)$. Let $x_i = x_i(\zeta) = A_iE(\zeta)$ $(1 \leqq i \leqq m)$, let $y_i = A_{i+m}E(\zeta)$ $(1 \leqq i \leqq m)$, and let h_n be the functional

$$\sum \{w_{x_i, y_i} \mid 1 \leqq i \leqq m\}$$

on the algebra $\mathscr{L}$ of all bounded linear operators on $H(\zeta)$. Let h be the functional on $\mathscr{L}$ given by $h(A) = (AU(\zeta), U(\zeta))$. It is clear that $h_n \cdot \psi(A) = \sum \tau_E(A_i^*AA_{i+m})\hat{\ }(\zeta)$ and $h \cdot \psi(A) = \tau_F(A)\hat{\ }(\zeta)$ for all $A \in \mathscr{Z}'$. But for every A in $\mathscr{A}$, we have that $|h(\psi(A)) - h_n(\psi(A))| \leqq n^{-1}\|A(\zeta)\| = n^{-1}\|\psi(A)\|$. By the Kaplansky density theorem [14], $|h(A) - h_n(A)| \leqq n^{-1}\|A\|$ for every $A \in \psi(\mathscr{A})''$ since h and h_n are σ-weakly continuous on $\psi(\mathscr{A})''$. Because the x_i and y_i are in $K(\zeta)$, we have that $h_n(A) = h_n(GA)$

$=h_n(AG)$ for every A in $\psi(\mathscr{A})''$. Let R be the central support of G in $\psi(\mathscr{A})'$. We have that $|h_n(AG)-h_m(AG)| = |h_n(AR)-h_m(AR)| = |h_n(AR)-h(AR)| + |h(AR) - h_m(AR)| \leqq (n^{-1}+m^{-1})\|AR\| = (n^{-1}+m^{-1})\|AG\|$. Setting g_n equal to the restriction of h_n to $\psi(\mathscr{A})''G$ for every $n=1, 2, \ldots$, we obtain a Cauchy sequence $\{g_n\}$ in the predual of $\psi(\mathscr{A})''G$. Since the predual of $\psi(\mathscr{A})''G$ is a complete normed linear space, we may find two square summable sequences $\{z_i\}$ and $\{t_i\}$ in $K(\zeta)$ such that $\{g_n\}$ converges uniformly to $\sum w_{z_i,t_i}$ on $\psi(\mathscr{A})''G$. Hence, for every A in $\mathscr{A}$ we have that

$$\tau_F(A)\hat{}(\zeta) = h(\psi(A)) = \lim h_n(\psi(A)) = \lim h_n(\psi(A)G) = \sum w_{z_i,t_i}(\psi(A)G).$$

This means, first of all, that $g=\sum w_{z_i,t_i}$ is positive on $\psi(\mathscr{A})G$, and since every positive element in $\psi(\mathscr{A})''G$ is the σ-weak limit of positive elements of $\psi(\mathscr{A})G$, that g is positive on $\psi(\mathscr{A})''G$. Hence, there is a square summable sequence $\{u_i\}$ in $K(\zeta)$ such that $g=\sum w_{u_i}$ on $\psi(\mathscr{A})''G$. Now, let π be the representation of $\mathscr{A}$ on $K(\zeta)$ given by $\pi(A)=\psi(A)G$ for every A in $\mathscr{A}$. We have that $f=g|\pi(\mathscr{A})''=\sum w_{u_i}$ is normal on $\pi(\mathscr{A})''$ and by Lemma 2 we have that $\pi_f\cdot\pi \prec \pi$. But $\pi_f\cdot\pi$ is unitarily equivalent to the canonical representation $\pi_{2\zeta}$ of $\mathscr{A}$ induced by $f\cdot\pi=f_{2\zeta}$ [remarks and Lemma 2]. Also π is unitarily equivalent with $\pi_{1\zeta}$. Indeed, for every A, B, C in $\mathscr{A}$ we have that

$$\begin{aligned}(\pi(C)AE(\zeta), BE(\zeta)) &= \tau_E(B^*CA)\hat{}(\zeta) = f_{1\zeta}(B^*CA)\\ &= (\pi_{1\zeta}(C)(A-L(f_{1\zeta})), B-L(f_{1\zeta})).\end{aligned}$$

Since $\{AE(\zeta) \mid A\in\mathscr{A}\}$ and $\{A-L(f_{1\zeta}) \mid A\in\mathscr{A}\}$ are uniformly dense in $K(\zeta)$ and $H(\pi_{1\zeta})$, respectively, we have that π and $\pi_{1\zeta}$ are unitarily equivalent. This proves that $\pi_{2\zeta}\prec\pi_{1\zeta}$. Because ζ is arbitrary in X, we have that $\pi_{2\zeta}\prec\pi_{1\zeta}$ for every $\zeta\in X$. This completes the first part of the proof.

The rest of the proof requires a maximality argument. Let $\{P_i\}$ be a maximal set of nonzero orthogonal projections in $\mathscr{Z}$ such that for each $\zeta\in\bigcup\{\zeta\in Z \mid P_i\hat{}(\zeta)=1\}$ the relation $\pi_{2\zeta}\prec\pi_{1\zeta}$ holds. If $1-\sum P_i=P$ is not zero, then there is a nonzero projection Q in $\mathscr{Z}P$ such that $\pi_{2\zeta}\prec\pi_{1\zeta}$ for every $\zeta\in\{\zeta\in Z \mid Q\hat{}(\zeta)=1\}$. This comes from the first part of the proof. However, such a situation contradicts the maximality of the set $\{P_i\}$. Therefore $\sum P_i=1$ and $Y_1=\bigcup\{\zeta\in Z \mid P_i\hat{}(\zeta)=1\}$ is an open dense set of Z.

By a similar argument, there is an open subset Y_2 of Z such that $\pi_{1\zeta}\prec\pi_{2\zeta}$ for every $\zeta\in Y_2$. Therefore for every ζ in the open dense subset $Y_1\cap Y_2$ we have that $\pi_{1\zeta}\sim\pi_{2\zeta}$. Q.E.D.

The following lemma is known for primary functionals [16].

LEMMA 4. *Let $\mathscr{A}$ be a C^*-algebra with identity and let $\{f_i\}$ be a sequence of positive functionals such that the partial sums $\{\sum\{f_i \mid 1\leqq i\leqq n\}\}$ converge to the positive functional f. If each canonical representation π_{f_i} induced by f_i lies in the same quasi-equivalence class, then the canonical representation π_f induced by f lies in this same class.*

Proof. Let $\pi=\sum\oplus\pi_{f_i}$ on $H=\sum\oplus H(f_i)$ and let x_i be a vector in $H(f_i)$ cyclic under $\pi_{f_i}(\mathscr{A})$ such that $w_{x_i}\cdot\pi_{f_i}=f_i$. Then $\sum\|x_i\|^2=\sum f_i(1)<+\infty$. Because the projection E_i of H on $H(f_i)$ is in the commutant of $\pi(\mathscr{A})$, we have that $w_x\cdot\pi(A)=\sum w_{x_i}(\pi(A))=\sum f_i(A)=f(A)$ for every $A\in\mathscr{A}$. Here x is the vector in H given by $x=\sum x_i$. Let E be the projection of H on the subspace $K=$closure $\{\pi(A)x \mid A\in\mathscr{A}\}$ of H. Then E is in the commutant of $\pi(\mathscr{A})$ on H and the central support P of E majorizes the central support of each E_i since $Px_i=x_i$ for each i. Let π_1 be the representation of $\mathscr{A}$ on K given by $\pi_1(A)=\pi(A)E$. Then π_1 is unitarily equivalent to the representation π_f of $\mathscr{A}$ on $H(f)$ because $f=w_x\cdot\pi_1$ and x is cyclic for K under $\pi_1(\mathscr{A})$. So it is sufficient to show that $\pi_1\sim\pi_{f_i}$. However, the fact that all the π_{f_i} $(i=1,2,\ldots)$ are quasi-equivalent implies that all the projections E_i have the same central support. The common central support must be 1 since $\sum E_i=1$. Hence, we must also have that $P=1$. This means that $\pi\sim\pi_1$. Q.E.D.

Let $\mathscr{A}$ be a von Neumann algebra with center $\mathscr{Z}$ on a Hilbert space H. If f is a normal functional on $\mathscr{A}$, then there is normal module homomorphism ϕ of the $\mathscr{Z}$-module $\mathscr{A}$ onto $\mathscr{Z}$ such that $\phi\,(1)=1$ (i.e. ϕ is a *state*) and $f=(f|\mathscr{Z})\cdot\phi$. If P is the support of $f|\mathscr{Z}$, then $P\phi$ is uniquely determined [12].

Now let $\mathscr{A}$ be a type I von Neumann algebra with center $\mathscr{Z}$ and let $\mathscr{A}_\sim$ be the Banach $\mathscr{Z}$-module of all σ-weakly continuous $\mathscr{Z}$-module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$. If ϕ is a positive element of $\mathscr{A}_\sim$, there is a decreasing sequence $\{A_i\}$ in $\mathscr{Z}^+$ which converges uniformly to 0 and whose partial sums are bounded above and a sequence $\{E_i\}$ of orthogonal abelian projections whose central supports P_i are equal to the supports of the A_i respectively such that $\phi(A)=\sum A_i\tau_{E_i}(A)$ (strongly) ([11], [12]).

Now let $\mathscr{A}$ be an arbitrary von Neumann algebra with center $\mathscr{Z}$ on H. Let $\mathscr{A}_\sim$ be the set of all σ-weakly continuous $\mathscr{Z}$-module homomorphisms of $\mathscr{A}$ into $\mathscr{Z}$. If ϕ is a positive element of $\mathscr{A}_\sim$, then ϕ may be extended to a positive σ-weakly continuous $\mathscr{Z}$-module homomorphism of the commutant $\mathscr{Z}'$ of $\mathscr{Z}$ into $\mathscr{Z}$. Since $\mathscr{Z}'$ is type I, the preceding representation of the extension induces a representation of the homomorphism itself.

In the sequel, a functional on a C^*-algebra $\mathscr{A}$ with identity on a Hilbert space H will be called *normal* if it is the restriction to $\mathscr{A}$ of a (necessarily unique) normal functional on the von Neumann algebra generated by $\mathscr{A}$ on H.

The following lemma extends Theorem 3.

LEMMA 5. *Let $\mathscr{A}$ be a C^*-algebra with identity on the Hilbert space H, let $\mathscr{A}''$ be the von Neumann algebra generated by $\mathscr{A}$ and H and let $\mathscr{Z}$ be the center of $\mathscr{A}$. Let ϕ_1 and ϕ_2 be states in the space $\mathscr{A}^{\pm}_{\sim}$ of σ-weakly continuous positive module homomorphisms of $\mathscr{A}''$ into $\mathscr{Z}$. For each ζ in the spectrum Z of $\mathscr{Z}$ let $f_{i\zeta}$ be the state of $\mathscr{A}$ given by $f_{i\zeta}(A)=\phi_i(A)^\wedge(\zeta)$ for every $A\in\mathscr{A}$ $(1\leqq i\leqq 2)$. Let $\pi_{i\zeta}$ be the canonical representation of $\mathscr{A}$ induced by $f_{i\zeta}$ $(1\leqq i\leqq 2)$. Then there is a nowhere dense subset N of Z such that $\pi_{1\zeta}\sim\pi_{2\zeta}$ for every $\zeta\notin N$.*

Proof. It is sufficient to find a maximal abelian projection E in the commutant $\mathscr{Z}'$ of $\mathscr{Z}$ and a nowhere dense set N of Z such that for every $\zeta \notin N$ the representation $\pi_{1\zeta}$ is quasi-equivalent to the canonical representation induced by the state f_ζ of $\mathscr{A}$ defined by $\tau_E(A)^\frown(\zeta)$ (Theorem 3). We proceed to do this. By the remarks preceding this lemma the function ϕ_1 may be written

$$\phi_1(A) = \sum \{A_i \tau_{E_i}(A) \mid i = 1, 2, \ldots\}$$

(strong limit), where $\{A_i\}$ is a summable sequence in $\mathscr{Z}^+$ (in the strong topology) that converges monotonically to 0 and $\{E_i\}$ is a sequence of mutually orthogonal abelian projections such that the central support P_i of each E_i is related to the support of A_i by $\{\zeta \in Z \mid P_i^\frown(\zeta) = 1\} = \text{closure}\,\{\zeta \in Z \mid A_i^\frown(\zeta) > 0\}$. There is a nowhere dense subset N_0 of Z such that $\sum A_i^\frown(\zeta) = (\sum A_i)^\frown(\zeta)$ whenever $\zeta \notin N_0$. If $g_{n\zeta}(A) = A_n^\frown(\zeta)\tau_{E_n}(A)^\frown(\zeta)$ $(A \in \mathscr{A})$, because $\|g_{n\zeta}\| = g_{n\zeta}(1) = A_n^\frown(\zeta)$ for every ζ and n, it is clear that the partial sums $\{\sum \{g_{n\zeta} \mid 1 \leqq n \leqq m\}\}$ converge in the uniform topology of the dual of $\mathscr{A}$ to the functional $f_{1\zeta}$ for all $\zeta \notin N_0$. But $g_{1\zeta}(1)$ is nonzero for $\zeta \notin N_0$ since $g_{1\zeta}(1) \geqq g_{n\zeta}(1)$ and $f_{1\zeta}(1) = 1$ for all $\zeta \in Z$. This means that the central support P_1 of E_1 is 1. We show that $E = E_1$ has the desired properties. Let π_ζ be the canonical representation of $\mathscr{A}$ induced by the state $f_\zeta(A) = \tau_E(A)^\frown(\zeta)$ of $\mathscr{A}$. For every $i = 2, 3, \ldots$ there is a maximal abelian projection F_i such that $F_i \geqq E_i$. By Theorem 3, for every $i = 2, 3, \ldots$ there is a nowhere dense set N_i of Z such that the canonical representation induced by the state $\tau_{F_i}(A)^\frown(\zeta)$ is in the same class as π_ζ whenever $\zeta \notin N_i$. However, the set $N = N_0 \cup (\bigcup N_i)$ is nowhere dense in Z [1]. For every $\zeta \notin N$ and every $i = 1, 2, \ldots$ either $g_{i\zeta} = 0$ or the canonical representation of $\mathscr{A}$ induced by $g_{i\zeta}$ is in the same quasi-equivalence class as π_ζ. However, the functional $g_{1\zeta}$ is nonzero for every $\zeta \notin N$. By Lemma 4 we may conclude, therefore, that the canonical representation $\pi_{1\zeta}$ of $\mathscr{A}$ induced by $f_{1\zeta}$ is in the same quasi-equivalence class as the canonical representation π_ζ of $\mathscr{A}$ induced by f_ζ. Q.E.D.

The next theorem is the main decomposition theorem.

THEOREM 6. *Let $\mathscr{A}$ be a C^*-algebra on a Hilbert space H. Suppose $\mathscr{A}$ contains the center $\mathscr{Z}$ of the von Neumann algebra $\mathscr{A}''$ generated by $\mathscr{A}$ on H. Then there is a one-one map $\zeta \to (\zeta)$ of the spectrum Z of $\mathscr{Z}$ into a set of mutually disjoint quasi-equivalent classes of representations of $\mathscr{A}$ with the following properties: If f is a normal functional on $\mathscr{A}$, then there is a field $\{f_\zeta \mid \zeta \in Z\}$ of states of $\mathscr{A}$ indexed by Z and a Radon measure ν on Z such that*

(1) *ν is the spectral measure on Z obtained by restricting f to $\mathscr{Z}$;*

(2) *the canonical representation π_{f_ζ} of $\mathscr{A}$ induced by f_ζ is in class (ζ) except perhaps on a nowhere dense set of Z;*

(3) *$f_\zeta(A) = A^\frown(\zeta)$ for every $A \in \mathscr{Z}$ and $\zeta \in Z$;*

(4) *$\zeta \to f_\zeta(A)$ is continuous on Z for fixed A in $\mathscr{A}$; and*

(5) *$f(A) = \int f_\zeta(A)\, d\nu(\zeta)$ for every $A \in \mathscr{A}$.*

REMARK 1. In particular, a von Neumann algebra $\mathscr{A}$ on H satisfies the hypotheses of the theorem.

REMARK 2. If $\{f'_\zeta \mid \zeta \in Z\}$ is a second field of states indexed by Z, and if ν' is a second Radon measure on Z which satisfy properties (1)–(5), then $\nu'=\nu$ and $f'_\zeta=f_\zeta$ for all ζ in the support of ν. Indeed, ν' is then the spectral measure determined by $f|\mathscr{Z}$ while $\{f'_\zeta \mid \zeta \in \text{support } \nu\}$ determines the element ϕ of $\mathscr{A}_\sim$ with $f=(f|Z)\cdot\phi$.

REMARK 3. This theorem also gives a decomposition of the Hilbert space $H(f)$ into the direct integral of Hilbert spaces $H(f_\zeta)$.

Proof. Let E be a maximal abelian projection in the commutant $\mathscr{Z}'$ of $\mathscr{Z}$. For each $\zeta \in Z$, let (ζ) be the quasi-equivalence class of representations of $\mathscr{A}$ determined by the canonical representation of the positive functional $\tau_E(A)^\frown(\zeta)$ on $\mathscr{A}$. Then (ζ) and (ζ') are disjoint whenever ζ and ζ' are distinct points of Z. In fact, if π and π' are nonzero subrepresentations of the canonical representations induced by $\tau_E(A)^\frown(\zeta)$ and $\tau_E(A)^\frown(\zeta')$, respectively, then there exists no isometry U of the underlying Hilbert spaces such that $U\pi(A)=\pi'(A)U$ for every $A \in \mathscr{A}$ since $\pi(A) = A^\frown(\zeta)\pi(1)$ and $\pi(A)=A^\frown(\zeta')\pi'(1)$ for every $A \in \mathscr{Z}$.

Now there is a state $\phi \in \mathscr{A}_\sim$ such that $f=(f|\mathscr{Z})\cdot\phi$. Since $f|\mathscr{Z}$ is normal on $\mathscr{Z}$, there is a vector $x \in H$ such that $f|\mathscr{Z}=w_x$. Let ν be the spectral measure on Z which corresponds to w_x by the relation $w_x(A)=\int A^\frown(\zeta)\, d\nu(\zeta)$ for every $A \in \mathscr{Z}$. Then setting $f_\zeta(A)=\phi(A)^\frown(\zeta)$ for every $\zeta \in Z$, we obtain $f(A)=\int f_\zeta(A)\, d\nu(\zeta)$ for every A in $\mathscr{A}$. Since $\phi(A) \in \mathscr{Z}$ for every $A \in \mathscr{A}$, the function $\zeta \to f_\zeta(A)$ of Z into the complex field is continuous for each fixed A in $\mathscr{A}$. Thus $\{f_\zeta(A) \mid \zeta \in Z\}$ satisfies property (4). Also $f_\zeta(A)=\phi(A)^\frown(\zeta)=A^\frown(\zeta)\phi(1)^\frown(\zeta)=A^\frown(\zeta)$ for every $A \in \mathscr{Z}$. Thus, f_ζ satisfies property (3). Finally, property (2) follows from Lemma 5. Q.E.D.

To apply Theorem 6 to a C^*-algebra $\mathscr{A}$ with identity which does not contain the center $\mathscr{Z}$ of the von Neumann algebra which it generates, one might work with the C^*-algebra $\mathscr{A}+\mathscr{Z}$ generated by $\mathscr{A}$ and $\mathscr{Z}$. The objects in the representation for $\mathscr{A}+\mathscr{Z}$ may then be restricted to $\mathscr{A}$. One would then obtain a function $\zeta \to (\zeta)$ of Z into the set of quasi-equivalence classes of representations of $\mathscr{A}$ such that to each normal functional f there corresponds a field $\{f_\zeta \mid \zeta \in Z\}$ of functionals and a measure ν on Z satisfying (1)–(5). At present we cannot tell whether $\zeta \to (\zeta)$ is one-one or whether even the functionals f_ζ are distinct.

We now obtain some information on the quasi-equivalence classes.

THEOREM 7. *Let $\mathscr{A}$ be a type* I *C^*-algebra with identity, let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$, and let ϕ be a state of $\mathscr{B}_\sim$. For each ζ in the spectrum Z of the center $\mathscr{Z}$ of $\mathscr{B}$ let f_ζ be the state of $\mathscr{A}$ defined by $f_\zeta(A)=\phi(A)^\frown(\zeta)$ for every A in $\mathscr{A}$. Then the set $\{\zeta \in Z \mid f_\zeta$ is not a primary functional$\}$ is nowhere dense in Z.*

REMARK. Since Theorem 7 will not be applied to von Neumann algebras, it is stated in terms of the enveloping algebra.

Proof. By Lemma 5 it is sufficient to find a particular maximal abelian projection F in $\mathscr{B}$ such that $\{\zeta \in Z \mid \tau_F(A)^\frown(\zeta)$ is not primary$\}$ is nowhere dense in Z. We actually construct a maximal abelian projection F in $\mathscr{B}$ such that $\{\zeta \in Z \mid \tau_F(A)^\frown(\zeta)$ is not a pure state$\}$ is nowhere dense. Using the same proof as [13, Theorem 6 (part a1 implies a2)] extended to the GCR case from the CCR case, we may find an increasing net $\{P_i \mid 0 \leqq i \leqq i_0\}$ of projections in $\mathscr{Z}$ indexed by the ordinals with the following properties: (1) $P_0=0$; (2) $P_{i_0}=1$; (3) if j is a limit ordinal, then lub $\{P_i \mid i<j\}=P_j$; and (4) given i there is an abelian projection F_i of central support $P_{i+1}-P_i$ and a positive element A_i in $\mathscr{Z}(P_{i+1}-P_i)$ with closure $\{\zeta \in Z \mid A_i^\frown(\zeta)>0\}$ $=\{\zeta \in Z \mid P_i^\frown(\zeta)=1\}$ such that A_iF_i is in $\mathscr{A}(P_{i+1}-P_i)$.

Now using Lemma 3 of [13], we may find by a maximality argument a net $\{Q_i\}$ of orthogonal central projections of sum 1 such that each algebra $\mathscr{A}Q_i$ contains an abelian projection F_i of $\mathscr{B}$ of central support Q_i. For each $\zeta \in Z$ with $Q_i^\frown(\zeta)=1$, we show $f_\zeta(A)=\tau_{F_i}(A)^\frown(\zeta)$ is a pure state of $\mathscr{A}$. Indeed, let A_i be an element of $\mathscr{A}$ with $A_iQ_i=F_i$. Then by taking the appropriate function of A_i, we may assume that $0 \leqq A_i \leqq 1$ and thus $f_\zeta(1-A_i)=0$. Now let f be a positive functional on $\mathscr{A}$ which is majorized by f_ζ. This means that $f(A)=f(A_iAA_i)$ for every $A \in \mathscr{A}$. By the Schwarz inequality we obtain that $|f(A)| \leqq f_\zeta(A^*A)^{1/2}$ for every A in $\mathscr{A}$. However, the function $p(A)=(\tau_E(A^*A)^\frown(\zeta))^{1/2}$ is a seminorm on $\mathscr{B}$, and by the Hahn-Banach Theorem the functional f has an extension to a functional g on $\mathscr{B}$ such that $|g(A)| \leqq p(A)$ for every $A \in \mathscr{B}$. However, the seminorm p vanishes on the smallest closed two-sided ideal $[\zeta]$ in $\mathscr{B}$ which contains ζ. This means that g also vanishes on $[\zeta]$. But A_iAA_i $-f_\zeta(A)A_i$ is an element of $[\zeta]$. So we obtain that

$$f(A) = f(A_iAA_i) = g(A_iAA_i) = f_\zeta(A)g(A_i)$$

for every A in $\mathscr{A}$. Thus f_ζ is a pure state. Now let F be the maximal abelian projection of $\mathscr{B}$ given by $F=\sum F_i$. Then for $\zeta \in Z$ with $Q_i^\frown(\zeta)=1$ for some Q_i we have that $\tau_F(A)^\frown(\zeta)=(\tau_F(A)Q_i)^\frown(\zeta)=\tau_{F_i}(A)^\frown(\zeta)$. So $\tau_F(A)^\frown(\zeta)$ is a pure state except perhaps on the nowhere dense set $\bigcap \{\zeta \in Z \mid Q_i^\frown(\zeta)=0\}$. Q.E.D.

COROLLARY. *Let $\mathscr{A}$ be a C^*-algebra with identity, let $\mathscr{B}$ be its enveloping von Neumann algebra, let $\mathscr{Z}$ be the center of $\mathscr{B}$, and let Z be the spectrum of $\mathscr{Z}$. For every maximal abelian projection E in the commutant $\mathscr{Z}'$ of $\mathscr{Z}$ and for every ζ in Z let $(\zeta)_E$ be the quasi-equivalence class of representations of $\mathscr{A}$ determined by the canonical representation of $\mathscr{A}$ induced by the functional $\tau_E(A)^\frown(\zeta)$. Then $\mathscr{A}$ is a type* I *C^*-algebra if and only if the set $X_E=\{\zeta \in Z \mid (\zeta)_E$ does not contain an irreducible representation$\}$ is nowhere dense in Z for every maximal abelian projection E in $\mathscr{Z}'$.*

Proof. If $\mathscr{A}$ is type I, then the corollary follows from the proof of Theorem 7 and from Theorem 3.

If $\mathscr{A}$ is not type I, then there is a state f of $\mathscr{A}$ such that π_f is a factor type III on $H(f)$ [31]. There is a minimal projection P in $\mathscr{Z}$ and a maximal abelian projection F in $\mathscr{Z}'$ such that $f(A)P=\tau_F(A)P$ for every A in $\mathscr{A}$. Thus X_F has a nonvoid interior and, by Theorem 3, X_E has a nonvoid interior for every maximal abelian projection in $\mathscr{Z}'$. Q.E.D.

REMARK. P. Fillmore and D. Topping [8, §3] state without proof that the functionals f_ζ are irreducible provided there is an abelian projection E of central support 1 in $\mathscr{A}$ and $\mathscr{Z}$ is contained in $\mathscr{A}$. Here $\mathscr{Z}$ is not assumed to be in $\mathscr{A}$.

The next theorem might be considered as a generalization of a result of H. Takemoto [28]. He showed that the field of functionals $\{f_\zeta\}$ in a representation given by the present author was in fact a field of primary functionals.

THEOREM 8. *Let $\mathscr{A}$ be a semifinite von Neumann algebra on the Hilbert space H. Then there is a one-one map $\zeta \rightarrow (\zeta)$ of the spectrum Z of the center $\mathscr{Z}$ of $\mathscr{A}$ onto a set of mutually disjoint quasi-equivalence classes of factor representations of $\mathscr{A}$ with the following properties: If f is a normal functional on $\mathscr{A}$, then there are states $\{f_\zeta \mid \zeta \in Z\}$ of $\mathscr{A}$ and a measure ν on Z which satisfy properties* (1)–(5) *of Theorem* 6.

Proof. There is no loss in generality in considering the type I and the type II cases separately.

If $\mathscr{A}$ is type I, then let E be a maximal abelian projection of $\mathscr{A}$. For each ζ in Z the functional $f_\zeta(A)=\tau_E(A)\hat{\ }(\zeta)$ is a pure state of $\mathscr{A}$ [10, Proposition 2.1]. Thus, the result follows from Theorem 6 for this case.

Now let $\mathscr{A}$ be a type II algebra. Let E be a finite projection in $\mathscr{A}$ of central support 1 and let $A^{\#}$ denote the $\mathscr{Z}$-valued canonical trace of an element A in $E\mathscr{A}E$ [2, III, §5]. The function on $\mathscr{A}$ defined by $\phi(A)=(EAE)^{\#}$ is a state in $\mathscr{A}_\sim$ because the map $A \rightarrow EAE$ is a positive σ-weakly continuous linear function of $\mathscr{A}$ onto $E\mathscr{A}E$. We show that $f_\zeta(A)=\phi(A)\hat{\ }(\zeta)$ is a primary functional for every ζ in Z. Indeed, let π_ζ be the canonical representation of $\mathscr{A}$ on $H(\zeta)$ induced by f_ζ. Let x_ζ be a cyclic unit vector for $H(\zeta)$ under $\pi_\zeta(\mathscr{A})$ such that $w_{x_\zeta}\cdot\pi_\zeta=f_\zeta$. Then $\|\pi_\zeta(E)x_\zeta\|=1$ implies $\pi_\zeta(E)x_\zeta=x_\zeta$. This means that the central support of $\pi_\zeta(E)$ in the von Neumann algebra $\pi_\zeta(\mathscr{A})''$ generated by $\pi_\zeta(\mathscr{A})$ on $H(\zeta)$ is 1. Now let A be an element in the center of $\pi_\zeta(\mathscr{A})''$. Then $A\pi_\zeta(E)$ is in the center of $\pi_\zeta(E)\pi_\zeta(\mathscr{A})''\pi_\zeta(E)$ on $\pi_\zeta(E)H(\zeta)=K(\zeta)$. But $\pi_\zeta(E)\pi_\zeta(\mathscr{A})''\pi_\zeta(E)$ is the von Neumann algebra on $K(\zeta)$ which is generated by $\pi_\zeta(E\mathscr{A}E)$ on $K(\zeta)$. Also the vector x_ζ is in $K(\zeta)$ and x_ζ is cyclic under $\pi_\zeta(E\mathscr{A}E)$. This means that the representation $\pi_\zeta|E\mathscr{A}E$ on $K(\zeta)$ is unitarily equivalent to the canonical representation induced by $f_\zeta|E\mathscr{A}E$. However, the functional $f_\zeta|E\mathscr{A}E$ is a primary functional ([30], cf. [22, II, §7]). This implies that $\pi_\zeta(E)\pi_\zeta(\mathscr{A})''\pi_\zeta(E)$ is a factor and hence $A\pi_\zeta(E)$ is a scalar multiple of the identity. Since $\pi_\zeta(E)$ has central support 1, the element A itself is a scalar multiple of the identity. Hence $\pi_\zeta(\mathscr{A})''$ is a factor and so the functional f_ζ is primary. If (ζ) denotes the quasi-equivalence class of representations which contains the canonical representation induced by f_ζ, then $\zeta \rightarrow (\zeta)$ is a one-one map of Z onto a set of mutually disjoint quasi-equivalent factor representations of $\mathscr{A}$. Then Theorem 8 follows from Lemma 5 and Theorem 6. Q.E.D.

3. **Separable algebras.** We now state some results for separable C^*-algebras to indicate the relation between the present decomposition and the usual central decomposition.

PROPOSITION A. *Let $\mathscr{A}$ be a separable C^*-algebra with identity on a Hilbert space H and let E be a maximal abelian projection in the commutant of the center $\mathscr{Z}$ of the* *von Neumann algebra generated by $\mathscr{A}$. For each ζ in the spectrum Z of $\mathscr{Z}$ let f_ζ be the state of $\mathscr{A}$ given by $f_\zeta(A) = \tau_E(A)^\wedge(\zeta)$. Then the set $\{\zeta \in Z \mid f_\zeta$ is not a primary state of $\mathscr{A}\}$ is nowhere dense in Z.*

Passing to the enveloping von Neumann algebra, we obtain the following theorem:

THEOREM B. *Let $\mathscr{A}$ be a separable C^*-algebra with identity, let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$ and let $\mathscr{Z}$ be the center of $\mathscr{B}$. Let E be a maximal abelian projection of the commutant of $\mathscr{Z}$ and let $\mathscr{C}$ be the separable abelian C^*-algebra generated by $\{\tau_E(A) \mid A \in \mathscr{A}\}$. Let Z be the spectrum of $\mathscr{C}$. For each $\zeta \in Z$ let (ζ) be the quasi-equivalence class of representations of $\mathscr{A}$ determined by the canonical representation induced by the state $\tau_E(A)^\wedge(\zeta)$. Then there is a set X in Z such that $Z - X$ has measure 0 for every spectral measure on Z with respect to H and such that if ζ_1 and ζ_2 are distinct points of X, then (ζ_1) and (ζ_2) are disjoint classes of factor representations of $\mathscr{A}$. Now let f be a positive functional on $\mathscr{A}$. Then there is a field $\{f_\zeta \mid \zeta \in Z\}$ of states on $\mathscr{A}$ indexed by Z, a Radon measure ν on Z, and an F_σ-set Y on Z with $\nu(Z) = \nu(Y)$ such that*

(1) *ν is a spectral measure Z obtained by restricting to $\mathscr{C}$ the unique normal extension of f to $\mathscr{B}$;*

(2) *the canonical representation induced by f_ζ is in class (ζ) for all $\zeta \in Y$;*

(3) *for each fixed A in $\mathscr{A}$, the function $\zeta \to f_\zeta(A)$ is an essentially bounded ν-measurable function of Z;*

(4) *the set $\{\zeta \to f_\zeta(A) \mid A \in \mathscr{A}\}$ is dense in the set of essentially bounded ν-measurable functions of Z in the w^*-topology; and*

(5) *there is a linear function ϕ of $\mathscr{Z}$ into the essentially bounded ν-measurable functions on Z such that*

$$f(AB) = \int \phi(A)(\zeta) f_\zeta(B)\, d\nu(\zeta)$$

for every A in $\mathscr{Z}$ and B in $\mathscr{A}$. Here H is the Hilbert space of $\mathscr{B}$.

REMARK. The properties listed for the measures and fields of Theorem B are enough to determine an essentially unique representation over Z.

The last proposition corresponds to the theorem concerning smooth duals.

PROPOSITION C. *Let the hypotheses be the same as Theorem* B. *Then the algebra $\mathscr{A}$ is of type* I *if and only if X, in addition to its other properties, may be chosen so that each class (ζ) for $\zeta \in X$ contains an irreducible representation.*

ADDED IN PROOF ON JANUARY 27, 1972. S. Strătilă and L. Zsidó (C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A1452-A1456) have announced a result similar to Theorem 8.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221