# RADON-NIKODYM THEOREMS FOR VECTOR VALUED MEASURES[1]

BY

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ABSTRACT. Let $\mu$ be a nonnegative measure, and let $m$ be a measure having values in a real or complex vector space $V$. This paper presents a comprehensive treatment of the question: When is $m$ the indefinite integral with respect to $\mu$ of a $V$ valued function $f$? Previous results are generalized, and two new types of Radon-Nikodym derivative, the "type $\rho$" function and the "strongly $\Gamma$ integrable" function, are introduced. A derivative of type $\rho$ may be obtained in every previous Radon-Nikodym theorem known to the author, and a preliminary result is presented which gives necessary and sufficient conditions for the measure $m$ to be the indefinite integral of a type $\rho$ function. The treatment is elementary throughout, and in particular will include the first elementary proof of the Radon-Nikodym theorem of Phillips.

1. **Introduction.** Let $\mu$ be a nonnegative countably additive measure, and let $m$ be a set function which in some sense is countably additive, and which assumes values in a real or complex vector space $V$. We shall present a comprehensive treatment of the question: When is $m$ the indefinite integral with respect to $\mu$ of a $V$ valued function $f$ which in some sense is $\mu$-integrable?

The exposition will be organized as follows: In §2 we establish notation and basic terminology. In §3 we develop technical machinery which will permit an automatic generalization of subsequent Radon-Nikodym theorems from the case where $\mu$ is finite to a wide variety of measures $\mu$, including regular Borel measures on locally compact Hausdorff spaces. In Theorem 4.9 of §4 we present very general conditions which are sufficient for the measure $m$ to be an indefinite integral. These entail the relative compactness (in certain topologies) of sets of ratios of the form $m(E)/\mu(E)$, where $E$ is a measurable set such that $0 < \mu(E) < \infty$. A number of previous theorems have required this sort of hypothesis, and all turn out to be special cases of Theorem 4.9. The aim of §5 is to present, in special cases, conditions which are necessary as well as sufficient for $m$ to be an indefinite integral. Three types of Radon-Nikodym derivative are considered. So-called "type $\rho$" functions possess a certain compatibility with a lifting $\rho$. So-called "strongly $\Gamma$ integrable" functions are a little less than Bochner

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integrable and a little more than "weakly" integrable. The third type is the standard (locally) measurable function having values in a normed linear space. For this we shall also mention previous results ([14], [17], [23], [24]), each given the slight generalization implied by the material of §3. Finally, §6 will round out the presentation with some examples and open problems.

2. **Notation and basic terminology.** By a measure space we shall mean a triple $(X, S, \mu)$, where $X$ is a set, $S$ is a $\sigma$-ring of subsets of $X$, and $\mu$ is a countably additive set function on $S$ which assumes nonnegative or possibly infinite values. If we say that $(X, S, \mu)$ (or $\mu$) is finite or totally $\sigma$-finite, we shall imply that $X \in S$.

Let $E$ and $F$ be sets. We shall write $E \oplus F$ in place of $E \cup F$ when $E$ and $F$ are disjoint. By $E^c$ we shall denote the complement of $E$, by $E - F$ we shall denote the relative complement $E \cap F^c$, and by $\chi_E$ we shall denote the characteristic (or indicator) function of the set $E$. The symbol $\emptyset$ will denote the empty set. If $E, F \in S$, we shall say that $E \sim_\mu F$ if $E$ is $\mu$-equivalent to $F$, i.e. if $\chi_E = \chi_F$ almost everywhere. We shall employ the standard abbreviation "a.e." for "almost everywhere".

A set will be called measurable if it lies in $S$, and locally measurable if its intersection with any measurable set is measurable. A $\mu$-null or null set is a measurable set of measure 0, and a locally null set is a locally measurable set which intersects every measurable set in a null set. A function $f$ on $X$ having values in a normed linear space will be called strongly measurable, or just measurable, if it is the a.e. pointwise limit of a sequence of simple measurable functions, i.e. functions of the form $\sum_{i=1}^{n} v_i \chi_{E_i}$, where $E_i$ is measurable. We shall call $f$ locally measurable if $f\chi_E$ is measurable for every $E$ in $S$. If $f$ is locally measurable, and if $E \in S$, then $\mathrm{er}_E(f)$ will denote the essential range of $F$ on $E$ [24, Definition 1.2, p.469].

All vector spaces we consider will be either real or complex. The term "scalar" will refer indifferently to a real or complex number according as the space in question is real or complex. Let $V$ be a vector space, and let $\Gamma$ be a space of linear functionals on $V$. The net $\{v_\alpha\} \subset V$ will converge to the point $v \in V$ in the $\Gamma$ topology on $V$ if $\phi(v_\alpha) \to \phi(v)$ for every $\phi \in \Gamma$. The function $f: X \to V$ will be called (locally) $\Gamma$ measurable if $\phi f (= \phi \circ f)$ is (locally) measurable for every $\phi \in \Gamma$. In the original spirit of Pettis [20, Definition 2.1, p. 280] we shall say that $f$ is $\Gamma$ integrable on the set $E \in S$ if $\phi f$ is (Lebesgue) integrable on $E$ for all $\phi \in \Gamma$, and if there exists an element $v_E$ of $V$ such that $\phi(v_E) = \int_E \phi f\, d\mu$ for all $\phi \in \Gamma$. We shall write $\int_E f\, d\mu$ in place of $v_E$, and refer to $\int_E f\, d\mu$ as the $\Gamma$ integral of $f$ on $E$. Unless the $\Gamma$ topology is Hausdorff, or, equivalently, unless $\Gamma$ is total (i.e. $v = 0$ if and only if $\phi(v) = 0$ for all $\phi \in \Gamma$), the $\Gamma$ integral $\int_E f\, d\mu$ will not be

uniquely defined. However, when we write $\int_E f\,d\mu$, we shall imply an arbitrary selection among the suitable elements of $V$, and when we assert that $v = \int_E f\,d\mu$ we shall imply that $v$ is suitable and that we have chosen $\int_E f\,d\mu$ to equal $v$. If $M \subset S$, we shall say that $f$ is $\Gamma$ integrable on $M$ if $f$ is $\Gamma$ integrable on every set $E$ in $M$. The set function $m$ defined by $m(E) = \int_E f\,d\mu$ for all sets $E$ on which $f$ is $\Gamma$ integrable will be called the indefinite integral of $f$ (with respect to $\mu$).

Let $m$ be a set function defined on a subset of $S$ and having values in a normed linear space. Then $m$ is (norm) countably additive if, whenever $E = \bigoplus_{i=1}^{\infty} E_i$, and $m$ is defined for $E$ and for each $E_i$, then the series $\sum_{i=1}^{\infty} m(E_i)$ is (unconditionally) convergent to $m(E)$. By $|m|$ we shall denote the total variation of $m$ [4, pp.32 ff.]. If $m$ assumes values in the arbitrary space $V$, we shall say that $m$ is $\Gamma$ countably additive if $\phi m$ ( $= \phi \circ m$) is countably additive for every $\phi \in \Gamma$. We shall say that $m$ is $\mu$-continuous if $\mu(E) = 0$ implies that $m(E)$ is defined and equal to $0$.

Assume now that $V$ is a normed linear space. By $V^*$ we shall denote the dual space of $V$. We shall occasionally refer to the $V^*$ topology on $V$ as the "weak" topology.

Finally, the end of a proof will be signalled by the standard symbol ∎.

3. **Decomposable measure spaces.** The purpose of this section is to introduce machinery which will permit automatic generalization of our Radon-Nikodym theorems from finite to decomposable measure spaces, this latter comprising a wide variety of measure spaces which includes regular Borel measures on locally compact Hausdorff spaces [14]. We shall illustrate how the generalization proceeds with the classical Radon-Nikodym theorem, and then assume finiteness of the measure $\mu$ in the proofs of the next sections.

3.1. **Definition.** Let $(X, S, \mu)$ be a measure space. A subset $I$ of $S$ is called an *ideal* if $I$ is a ring, and if, given $E \in S$ and $F \in I$, we have $E \cap F \in I$. The subset $I$ is *dense (in $S$)* if every set $E$ in $S$ of positive measure contains a set $F$ in $I$ of positive measure. The subset $I$ will be called a *$\mu$-ideal* if $I$ is a dense ideal which contains the $\mu$-null sets.

Many vector valued measures (including signed measures) cannot meaningfully be defined on all of $S$, and we shall habitually assume that such measures are defined on $\mu$-ideals. Using an "exhaustion procedure" similar to that in [23, Lemma 1, p.72], or using a simple Zorn's lemma argument, we may quickly establish the following result.

3.2. **Proposition.** *Let $(X, S, \mu)$ be a measure space, and let $I$ be a dense subset of $S$. Then every $\sigma$-finite set $E$ in $S$ is $\mu$-equivalent to a countable*

*disjoint union of sets in I.*

  3.3. **Definition.** Let $(X, S, \mu)$ be a measure space. We shall say that
$(X, S, \mu)$ (or $\mu$) is *decomposable* if $\mu$ is $\sigma$-finite, and if there exists a family
$\{X_\alpha\}$ of disjoint sets of finite positive measure such that $\mu(E) = \sum_\alpha \mu(E \cap X_\alpha)$ for
all $E \in S$.

  In this definition we follow the terminology of Kelley and Srinivasan [14]. De-
composable measure spaces have also been called strictly localizable
[12, Definition 8, p.17], and essentially constitute direct sums of finite spaces
[25, Definition 3.1, p.282]. (Cf. also [4, Definition 5, p.179].) Note that the $\sigma$-
finiteness of $\mu$ will ensure that each set $E$ in $S$ intersects at most countably
many $X_\alpha$ in a nonnull set, and hence that the set $X - \bigoplus_\alpha X_\alpha$ is locally null.

  If we have established the classical Radon-Nikodym theorem for a finite mea-
sure $\mu$, then, in view of Proposition 3.2, the following generalization is immediate.

  3.4. **Theorem.** *Let $(X, S, \mu)$ be a nonnegative, decomposable measure space,
and let $m$ be a real valued measure defined on a $\mu$-ideal $M \subset S$. Then there exists
a real valued locally measurable function $f$ on $X$ such that $m(E) = \int_E f \, d\mu$ for all
$E \in M$ if and only if $m$ is $\mu$-continuous.*

  Decomposability is a convenient criterion for ensuring that a "compatible"
collection of measurable functions, each defined only on a member of a dense
subset of $S$, can be pieced together into a suitable globally defined and locally
measurable function. (This statement is made precise in Proposition 3.5.) Since
we shall need decomposability only for this purpose, and since we shall assume
it in all of the Radon-Nikodym theorems to follow, it is worth pointing out that it
need not be assumed in any specific instance where a piecing together procedure
can be carried out by some other means. An example occurs when the measure $m$,
such as in Theorem 3.4, is carried on a locally measurable set on which $\mu$ is de-
composable—in particular, if $m$ is carried on a $\sigma$-finite set. Another example may
occur when $m$ is assumed to be real valued. It is well known that the conclusion
of Theorem 3.4 is valid for all measures $m$, as described there, if and only if the
space $(X, S, \mu)$ is localizable ([25, Definition 2.6, p.279; Theorem 5.1, p.301],
[27, Theorem 9.5, p.182], [13, Theorem 3, p.91], [15, Theorem 2, p.9]; cf. also
[27, Theorem 9.4, p.181]). While decomposability implies localizability, it is un-
known whether the converse is true (however, cf. [8] and [15, pp.3ff.]).

  As we shall be dealing with vector valued measures and functions, it is worth
observing that decomposability is the weakest reasonable condition which can be
imposed in order to ensure an appropriate piecing together of locally defined mea-
surable functions.

3.5. **Proposition.** *Let* $(X, S, \mu)$ *be a measure space containing no atoms of infinite measure. Suppose that we are given a Banach space B, a dense subset M of S, and, for each set E in M, a measurable function* $f_E : E \to B$ *such that, for all E, F* $\in$ *M, we have* $f_E(x) = f_F(x)$ *a.e. on* $E \cap F$; *suppose, moreover, that these conditions will always imply the existence of a locally measurable function* $f : X \to B$ *such that, for all E* $\in$ *M, we have* $f(x) = f_E(x)$ *a.e. on E. Then* $(X, S, \mu)$ *is decomposable.*

**Proof.** Using Zorn's lemma, we let $\{X_\alpha\}_{\alpha \in A}$ be a maximal family of sets of finite positive measure such that $X_\alpha \cap X_\beta$ is $\mu$-null whenever $\alpha \neq \beta$. Because $\mu$ has no atoms of infinite measure, we would quickly contradict the maximality of $\{X_\alpha\}_{\alpha \in A}$ if we assumed that $\mu(E) \neq \sum_\alpha \mu(E \cap X_\alpha)$ for any $E \in S$.

We shall now show that there exists a family $\{X_\alpha'\}_{\alpha \in A}$ such that $X_\alpha' \sim_\mu X_\alpha$ for all $\alpha \in A$ and such that $X_\alpha' \cap X_\beta' = \emptyset$ whenever $\alpha \neq \beta$. Let $M = \{F \in S : F \subset X_\alpha$ for some $\alpha\}$. Then $M$ is dense in $S$. Let $\{e_\alpha\}_{\alpha \in A}$ be, for example an orthonormal subset of a suitably large Hilbert space. Given $E \in M$ such that $\mu(E) > 0$, there is exactly one $\alpha \in A$ such that $E \subset X_\alpha$. Define $f_E(x) = e_\alpha$ for all $x \in E$. If $\mu(E) = 0$, then $f_E$ may be defined arbitrarily. By hypothesis there exists a locally measurable function $f$ such that, for all $E \in M$, we have $f(x) = f_E(x)$ a.e. on $E$. Let $X_\alpha' = \{x \in X_\alpha : f(x) = e_\alpha\}$. Clearly $X_\alpha' \sim_\mu X_\alpha$, and, because $e_\alpha \neq e_\beta$ for $\alpha \neq \beta$, we have $X_\alpha' \cap X_\beta' = \emptyset$ whenever $\alpha \neq \beta$.

It remains to show that $\mu$ is $\sigma$-finite. Given $E \in S$, the function $f|_E$ is assumed to be measurable, so that, except possibly for a null subset of $E$, $f|_E$ will be separably valued. This clearly cannot be the case if $E \cap X_\alpha$ were nonnull for more than countably many $\alpha$. ∎

**4. Radon-Nikodym theorems: sufficient conditions.** This section will present (in Theorem 4.9) very general conditions which are sufficient for a vector valued measure $m$ to be the indefinite integral of a function $f$ with respect to a nonnegative measure $\mu$. The following material will be used in the proof of Theorem 4.9.

4.1. **Definition.** Let $(X, S, \mu)$ be a totally $\sigma$-finite measure space. A function $\rho : S \to S$ will be called a *lifting* (*for* $\mu$) if it satisfies the following conditions:

(4.1.1) $\rho(E) \sim_\mu E$.

(4.1.2) $E \sim_\mu F$ implies that $\rho(E) = \rho(F)$.

(4.1.3) $\rho(\emptyset) = \emptyset$; $\rho(X) = X$.

(4.1.4) $\rho(E \cap F) = \rho(E) \cap \rho(F)$.

(4.1.5) $\rho(E \cup F) = \rho(E) \cup \rho(F)$.

Note that the range of $\rho$ is a field.

Dieudonné [2, pp.78ff.] was the first to point out that it is often convenient to construct a Radon-Nikodym derivative $f$ with the aid of a lifting. A more recent result of the Ionescu Tulceas [12, Theorem 2, p.89] suggests that when $f$ is not

(locally) strongly measurable the use of a lifting may be unavoidable. It is well known that a lifting always exists when $(X, S, \mu)$ is complete. This fact, former- ly the major stumbling block to an elementary treatment of the present topic, has recently been given a totally elementary proof by Sion [26].

We may extend $\rho$ to $\mathcal{L}_\infty (\mu)$, the space of scalar valued, essentially bounded, measurable functions on $X$, by defining

$$\rho \left( \sum_{i=1}^{n} \lambda_i \chi_{E_i} \right) = \sum_{i=1}^{n} \lambda_i \chi_{\rho(E_i)},$$

and then defining $\rho(f)(x) = \lim_{n \to \infty} \rho(f_n)(x)$, where $\{f_n\}_{n=1}^{\infty}$ is any sequence of simple measurable functions converging uniformly a.e. to $f$. It is straightforward to verify that $\rho(f)$ is well defined and independent of the choice of the $f_n$, and as the occasion demands we shall use certain obvious properties of this extension without comment.

4.2. **Lemma.** *Let $(X, S, \mu)$ be a finite, complete measure space, let $\rho$ be a lifting for $\mu$, and let $H$ be a family of bounded, real valued, measurable functions on $X$ such that $\rho(h) = h$ for all $h \in H$. Let $f(x) = \sup\{h(x): h \in H\}$. Then $f$ is measurable, and for any measurable function $g$ such that $g \geq h$ a.e. for all $h \in H$, we have $g \geq f$ a.e.*

**Proof.** Let $H'$ denote the set of functions of the form $h' = \sum_{i=1}^{n} h_i \chi_{F_i}$, where $h_i \in H$ and $\rho(F_i) = F_i$ for $i = 1, \cdots, n$, and where $X = \bigoplus_{i=1}^{n} F_i$. It is readily deduced that $\rho(h') = h'$ for all $h' \in H'$, that $f(x) = \sup\{h'(x): h' \in H'\}$ for all $x \in X$, and that $H'$ is directed for the relation $\leq$. The result now essentially follows from [4, Proposition 4, p.209] if it is observed that whenever $f \chi_E$ is integrable, it will be the $L_1$ supremum of the family $\{h' \chi_E: h' \in H'\}$. ∎

4.3. **Lemma.** *Let $(X, S, \mu)$ be a measure space, let $B$ be a Banach space, let $f$ be a $B$ valued, Bochner integrable function on $X$, let $X_f$ be the ($\sigma$-finite) support of $f$, and let $\rho$ be a lifting for $\mu$ restricted to the measurable subsets of $X_f$. Consider the collection $\Pi$ of partitions $\pi = \{F_1, \cdots, F_n\}$ such that $X_f = \bigoplus_{i=1}^{n} F_i$, and such that $\rho(F_i) = F_i \neq \emptyset$ for $i = 1, \cdots, n$. Define*

$$f_\pi = \sum_{i=1}^{n} \frac{m(F_i)}{\mu(F_i)} \chi_{F_i},$$

*where $m(E) = \int_E f d\mu$ for all $E \in S$, and where $m(E)/\mu(E) = 0$ when $\mu(E)$ is in- finite. Then $\Pi$ is directed under refinement (cf. [10, Example (5), p.31]). More- over, the net $\{f_\pi\}$ is uniformly Cauchy and converges to $f$ a.e. on $X_f$ if and only if $f(X_f - N)$ is relatively compact in $B$ for some $\mu$-null set $N$.*

**Proof.** In view of [7, Theorem 15, p.22] and of the (generalized) mean value theorem [24, Proposition 1.9, p.470], this result is essentially a straightforward generalization of a Dunford-Schwartz lemma [7, Lemma 3, p.500]. ∎

We have avoided the language of essential ranges [24, Definition 1.2, p.469]; however, the statement that $f(X_f - N)$ is relatively compact for some null set $N$ is equivalent to the statement that $\mathrm{er}_{X_f}(f)$ is compact. It follows from Egoroff's theorem and from [7, Theorem 15, p.22] (cf. [24, Proposition 1.1, p.469]) that if $f$ is an arbitrary Bochner integrable function, then the collection of measurable sets $E$ such that $\mathrm{er}_E(f)$ is compact constitutes a $\mu$-ideal in $S$. We therefore obtain from Proposition 3.2 that $f = \lim_{\pi} f_{\pi}$ a.e. in any event, although in general the net $\{f_{\pi}\}$ need not be uniformly Cauchy. Note that if $f$ is scalar valued and essentially bounded, then $\rho(f)(x) = \lim_{\pi} f_{\pi}(x)$ for all $x \in X$.

**4.4. Lemma.** *Let* $(X, S, \mu)$ *be a measure space, let* $B$ *be a Banach space, and let* $f$ *be a* $B$ *valued, Bochner integrable function on* $X$. *Then the set* $\{\int_E f\, d\mu: E \in S\}$ *is relatively compact in* $B$.

**Proof.** The result is evident for simple functions, and, in view of [7, Theorem 15, p.22], is easily obtained for arbitrary $f$ through approximation by simple functions. ∎

**4.5. Definition.** Let $V$ be a normed linear space. A subspace $\Gamma$ of $V^*$ will be called *norming* if $\|v\| = \sup\{|v^*(v)|: v^* \in \Gamma; \|v^*\| \leq 1\}$ for all $v \in V$.

Norming subspaces have also been called determining [11, Definition 2.8.2, p.34], and are clearly total. In fact the subspace $\Gamma$ is total if and only if it is $V$ dense in $V^*$ (i.e. dense in the $V$ topology on $V^*$), whereas $\Gamma$ is norming if and only if its intersection with the unit ball of $V^*$ is $V$ dense in the unit ball. (The proof of the latter statement is similar to that of Theorem 5 in [7, p.424].)

**4.6. Lemma.** *Let* $V$ *be a normed linear space, let* $\Gamma$ *be a norming subspace of* $V^*$, *and let* $m$ *be a* $V$ *valued,* $\Gamma$ *countably additive measure on an arbitrary measure space. Then the total variation* $|m|$ *is countably additive.*

**Proof.** The standard arguments for the case where $m$ is norm countably additive [4, pp.34–36] carry over to this setting with little change. ∎

**4.7. Lemma.** *Let* $(X, S, \mu)$ *be a measure space, let* $V$ *be a normed linear space, let* $\Gamma$ *be a norming subspace of* $V^*$, *and let* $f$ *be a locally* $\Gamma$ *measurable function on* $X$ *having values in a separable subspace of* $V$. *Then* $f$ *is locally (strongly) measurable. Moreover, if* $f$ *is* $\Gamma$ *measurable, then* $f$ *is measurable.*

**Proof.** This generalizes slightly the classical result of Pettis [20, Theorem 1.1, p.278], and no essentially new ideas are involved in the proof. ∎

**4.8. Lemma.** *Let $V$ be a normed linear space, and let $W$ be a subspace of $V$. Then $W$ is norm closed if and only if $W$ is $V^*$ closed.*

**Proof.** This follows readily from [7, Corollary 13, p.64], and is an easy special case of a well-known deeper result [7, Corollary 14, p.114]. ∎

**4.9. Theorem.** *Let $(X, S, \mu)$ be a complete, decomposable measure space, let $V$ be a vector space, let $\Gamma$ be a space of linear functionals on $V$, and let $m$ be a $V$ valued, $\Gamma$ countably additive, $\mu$-continuous measure defined on a $\mu$-ideal $M \subset S$. Assume that for each set $E$ in a dense subset of $M$ the set*

$$A_E(m) = \{m(F)/\mu(F): F \in S; F \subset E; 0 < \mu(F) < \infty\}$$

*is relatively compact in the $\Gamma$ topology on $V$. Then*

*(4.9.1) There exists a $V$ valued function $f$ on $X$ such that $f$ is $\Gamma$ integrable on $M$, and such that $m(E) = \int_E f d\mu$ for all $E \in M$.*

*(4.9.2) If, moreover, $V$ is a normed linear space, and if $\Gamma$ is a norming subspace of $V^*$, then $f$ may be chosen such that $\|f(\,\cdot\,)\|$ is locally measurable, and such that $|m|(E) = \int_E \|f(x)\| d\mu(x)$ for all $E \in M$.*

*(4.9.3) If, moreover, $V$ is a Banach space, and we have either that $V$ is separable or that $\Gamma = V^*$, then $f$ may be chosen to be locally strongly measurable and the $\Gamma$ integral $\int_E f d\mu$ realized as a Bochner integral for the $\mu$-ideal of sets $E$ in $M$ such that $|m|(E)$ is finite.*

**Proof.** In accordance with the results of §3, we may assume that $(X, S, \mu)$ is finite, that $M = S$, and that $A_X(m)$ is relatively compact in the $\Gamma$ topology on $V$. We shall preserve this assumption throughout the entire proof.

**Proof of (4.9.1).** Given $\phi \in \Gamma$, the measure $\phi m$ ( $= \phi \circ m$) is $\mu$-continuous. Thus, by the Radon-Nikodym theorem, there is a scalar valued function $f_\phi$ on $X$ such that $\phi m(E) = \int_E f_\phi d\mu$ for all $E \in S$.

Choose a lifting $\rho$ for $\mu$, and, in the manner of Lemma 4.3, let $\Pi$ denote the associated family of partitions of $X$. If $\pi = \{F_1, \cdots, F_n\} \in \Pi$, we define $f_\pi = \sum_{i=1}^n (m(F_i)/\mu(F_i))\chi_{F_i}$.

Since $A_X(m)$ is relatively compact in the $\Gamma$ topology, the set $A_X(\phi m) = \phi(A_X(m))$ will be relatively compact, and hence bounded. Therefore, by the mean value theorem, $f_\phi$ is essentially bounded, so that $\rho(f_\phi)$ is defined. We may assume that $f_\phi = \rho(f_\phi)$, and so may conclude from Lemma 4.3 that $f_\phi(x) = \lim_\pi \phi f_\pi(x)$ for all $x \in X$.

For each fixed $x \in X$, the net $\{f_\pi(x)\}$ lies in $A_X(m)$. There will therefore be a subnet converging in the $\Gamma$ topology to some point $f(x) \in V$ (not necessarily uniquely defined). Since $\phi f_\pi(x)$ already converges to $f_\phi(x)$, it is clear that $\phi f(x) = f_\phi(x)$ for all $x \in X$. Therefore $f$ is as desired.

**Proof of (4.9.2).** Because $\Gamma$ is norming, we have that $\|f(x)\| = \sup\{|v^* f(x)|: v^* \in \Gamma; \|v^*\| \leq 1\}$ for all $x \in X$. We recall that $\rho(v^* f) = v^* f$ in the construction above, and it is readily deduced from this that $\rho(|v^* f(\cdot)|) = |v^* f(\cdot)|$. The measurability of $\|f(\cdot)\|$ now follows from Lemma 4.2. Since the sets $E$ in $S$ on which $\|f(\cdot)\|$ is integrable constitute a $\mu$-ideal, we shall lose no generality by assuming in addition that $\|f(\cdot)\|$ is integrable.

Our goal now is to show that $|m|(E) = \int_E \|f(x)\| d\mu(x)$ for all $E \in S$. Evidently $\|m(E)\| \leq \int_E \|f(x)\| d\mu(x)$ for all $E \in S$, so that we have $|m|(E) \leq \int_E \|f(x)\| d\mu(x)$ for all $E \in S$ [4, p.35].

We shall now obtain the reverse inequality. By Lemma 4.6 the total variation $|m|$ is countably additive. It is also $\mu$-continuous and finite. Therefore, by the Radon-Nikodym theorem, there is a (nonnegative) function $b$ such that $|m|(E) = \int_E b \, d\mu$ for all $E \in S$. Let $v^* \in \Gamma$ have norm $\leq 1$. If we can show that $|v^* f(\cdot)| \leq b$ a.e., then an application of Lemma 4.2 will establish that $\|f(\cdot)\| \leq b$ a.e., and the proof will be complete. By [7, Theorem 20, p.114], we have $|v^* m|(E) = \int_E |v^* f(x)| d\mu(x)$ for all $E \in S$. Fix $E \in S$ and $\epsilon > 0$. Partition $E = \bigoplus_{i=1}^n E_i$ in such a way that

$$\int_E |v^* f(x)| d\mu(x) \leq \sum_{i=1}^n |v^* m(E_i)| + \epsilon \leq \sum_{i=1}^n \|m(E_i)\| + \epsilon \leq |m|(E) + \epsilon = \int_E b \, d\mu + \epsilon.$$

Since $E$ and $\epsilon$ were arbitrary, we may conclude that $|v^* f(\cdot)| \leq b$ a.e.

We remark that in order to extend this special case to the general setting, Lemma 4.6 is required in addition to Proposition 3.2.

**Proof of (4.9.3).** In view of (4.9.2), and because $V^*$ itself is always norming [7, Corollary 15, p.65], it will suffice to show that $f$ is strongly measurable. The condition $|m|(E) = \int_E \|f(x)\| d\mu(x)$ for all $E \in S$ will then imply that $f$ is Bochner integrable on every set $E$ such that $|m|(E)$ is finite. That the Bochner integral $\int_E f \, d\mu$ will coincide with the $\Gamma$ integral is obvious. We shall preserve the assumption that $\|f(\cdot)\|$ is integrable.

When $V$ is separable, strong measurability follows from Lemma 4.7. Thus (4.9.3) has been completely established when $V$ is separable, and we shall use this fact in what follows.

It remains to show that $f$ is strongly measurable for general $V$ when $\Gamma = V^*$. To this end it will suffice to show that the set $\{m(E): E \in S\}$ is relatively (norm) compact. Then $A_X(m)$, and hence the closed subspace $W$ generated by $A_X(m)$, will be separable. By Lemma 4.8, $W$ will also be $V^*$ closed, and so will contain $f(X)$. Lemma 4.7 will then give strong measurability of $f$.

Compactness is equivalent to sequential compactness for metric topologies, and so it will suffice to show that a sequence $\{m(E_n)\}_{n=1}^\infty$ has a convergent subsequence. To this end let $Y = \bigcup_{n=1}^\infty E_n$, let $T$ be the $\sigma$-algebra on $Y$ generated

by the $E_n$, let $\nu$ be $\mu$ restricted to $T$, let $n$ be $m$ restricted to $T$, and let $B$ be the closed subspace of $V$ generated by elements of the form $m(F)$, where $F$ is the finite intersection of some of the $E_n$. The elements $m(F)$ will be countable, and so $B$ is separable.

We shall show first that $n$ assumes values in $B$. Let $R = \{E \in S: m(E) \in B\}$. Then we have

(4.9.4) $E, F \in R$ and $E \cap F = \emptyset$ implies that $E \oplus F \in R$.

(4.9.5) $E, F \in R$ and $E \subset F$ implies that $F - E \in R$.

(4.9.6) If $\{F_n\}_{n=1}^{\infty}$ is an increasing sequence of sets in $R$, then $\bigcup_{n=1}^{\infty} F_n \in R$.

Note that (4.9.6) will hold because, as before, the subspace $B$ will be $V^*$ closed as well as norm closed. Since $R$ contains a generating set for $T$ which is closed under finite intersections, we may conclude that $R \supset T$ [18, Example 1.4.5, p.19].

The hypotheses of (4.9.3) are clearly satisfied by the $B$ valued measure $n$ with respect to $\nu$. Because $B$ is separable, and because we have $|n|(Y) \leq |m|(X) < \infty$, we have seen that we may obtain a Bochner integrable function $g$ such that $n(E) = \int_E g \, d\nu$ for all $E \in T$. Therefore, by Lemma 4.4, the set $\{\int_E g \, d\nu: E \in T\} = \{m(E): E \in T\}$ is relatively compact in $B$. It contains the original sequence $\{m(E_n)\}_{n=1}^{\infty}$, which therefore has a convergent subsequence. ∎

*Discussion of theorem.* The hypothesis that $A_E(m)$ be relatively $\Gamma$ compact is a convenient (and traditional) type of assumption to make. We note, however, that it is far stronger than what was actually needed in the proof of (4.9.1) (cf. Theorem 5.3).

The generality of Lemma 4.3 implies that we may replace the scalar field by an arbitrary Banach space $D$ and replace $\Gamma$ by a space of linear functions from $V$ to $D$. The generalization is spurious, however, for by composing elements of $\Gamma$ with linear functionals in $D^*$ we obtain in fact a special case of Theorem 4.9.

If $\Gamma$ is not assumed to be norming in (4.9.2), we may define, for all $v \in V$, a seminorm $\|v\|_{\Gamma} = \sup\{|v^*(v)|: v^* \in \Gamma; \|v^*\| \leq 1\}$, we may compute the total variation, $|m|_{\Gamma}$, of $m$ with respect to this seminorm, and we may conclude, by the arguments in the proof of (4.9.2), that $|m|_{\Gamma}(E) = \int_E \|f(x)\|_{\Gamma} \, d\mu(x)$ for all $E \in M$. Since each point $f(x)$ is not uniquely defined in general, there may be sufficient latitude in choosing $f(x)$ that we may obtain $\|f(x)\|_{\Gamma} = \|f(x)\|$ a.e. Such was the case in the classical (and first) proof of a special case of (4.9.1) given by Dunford and Pettis [6, Theorem 2.1.0, p.339].

If $V$ is only assumed to be a normed linear space in (4.9.3), we may pass to the completion $\overline{V}$, we may apply the arguments in the proof of (4.9.3), and we may conclude that $f$ is locally measurable in $V$ and Bochner integrable in $\overline{V}$ on every set $E$ such that $|m|(E)$ is finite. Of course the Bochner integral $\int_E f \, d\mu$ will also

lie in $V$.

Because the function $f$ in (4.9.3) is strongly (locally) measurable, it is not necessary to use the lifting theorem in order to derive it. Under the assumption that $\mu$ is finite, we may transfer both $\mu$ and $m$ to the Stone space of $(X, S, \mu)$ [10, Example (15), p.170; Example (7), p.223], in which the closed-open subsets constitute the range of a natural lifting. The strongly measurable Radon-Nikodym derivative for $m$ with respect to $\mu$ on the Stone space may then be transferred back to $X$.

*Discussion of previous results.* With the exception of certain Radon-Nikodym theorems for the Bochner integral, which will be summarized as part of Theorem 5.6, all previous theorems of this type which are known to us require, either explicitly or implicitly, an assumption about the relative compactness of sets of ratios of the form $m(E)/\mu(E)$. Each of these theorems constitutes a special case of Theorem 4.9. In addition to the Dunford-Pettis theorem mentioned earlier, various special cases of (4.9.1) have been proved by Dieudonné [3, Theorem 1, p.132], Bourbaki [1, Corollary 3, p.46], Dubins [5, Theorem 5, p.291], Métivier ([16, Theorem 6.6, p.334], [17, Theorem 7, p.199]), Dunford and Schwartz [7, Theorem 2, p.499], Dinculeanu [4, Theorem 5, p.269], the Ionescu Tulceas [12, Theorem 1, p.86], and Pellaumail [19, Theorem B2, p.361]. Some of these theorems, as well as some of the theorems cited in the next paragraph, require rewording in order to assume explicitly the form of a Radon-Nikodym theorem. A typical example is the Dunford-Schwartz theorem, a suitable rewording of which is given in [24, Theorem 5.1, p.481]. We remark that Dubins constructed only a "generalized random variable" [5, Definition 1, p.273] as a Radon-Nikodym derivative, and did not prove that it could always be taken to be an ordinary function. Dinculeanu gave the only other proof of (4.9.2) that we have seen. In the special case which he considered we have $V = \mathcal{L}(E, F)$, where $E$ and $F$ are Banach spaces, and where $\mathcal{L}(E, F)$ denotes the space of bounded linear operators from $E$ to $F$. The space $\Gamma$ is not explicitly mentioned, and we note that it would comprise the subspace of $\mathcal{L}(E, F)^*$ generated by pairs of the form $(e, z)$, where $e \in E$ and $z \in Z$, a norming subspace of $F^*$. The pair $(e, z)$ will map an operator $U \in \mathcal{L}(E, F)$ into the scalar $z(U(e))$.

Aside from the automatic generalization implied by the discussion in §3, (4.9.3) is a known result. It was first established by Phillips [21, Theorem 5-1, pp. 130 ff.], who gave two separate proofs. The second of these also appears in Grothendieck [9, Theorem 3, p.426] and in Bourbaki [1, Example 24, p.95]. The crux of the difficulty is to obtain strong measurability, and additional proofs of essentially this result have been given by Métivier [17, Theorem 11, p.203] and the Ionescu Tulceas [12, Propostion 1, p.91]. All four proofs depend upon results which

are both specialized and deep. The present proof, by contrast, exploits only the most basic feature of the weak topology which crucially distinguishes it from the other $\Gamma$ topologies: namely, that the weakly closed and the norm closed subspaces of a Banach space coincide. We remark that Rao [22, Theorem 3.3, p.114] has given a generalization of Phillips' theorem for the case where both $m$ and $\mu$ assume values in (possibly different) Banach spaces.

*Discussion of techniques of proof.* Our technique of using partitions consisting of sets in the range of a lifting was employed independently by Pellaumail [19]. In his first proof of (4.9.3) Phillips (erroneously) derived the function $f$ in a similar fashion, using partitions consisting of sets of strictly positive finite measure. These, however, are not directed under refinement, so that, given $x \in X$, the set $\{f_\pi(x)\}$ does not constitute a net.

Our argument in the proof of (4.9.2) that the function $\|f(\cdot)\|$ was locally measurable was adapted from ideas in Dinculeanu [4, Proposition 5, p.213]. However, his proof (when $V = \mathfrak{L}(E, F)$) that $|m|(E) = \int_E \|f(x)\| d\mu(x)$ for all $E \in M$ relies upon special properties of the $\Gamma$ topology which he considers, and does not readily extend to the more general setting.

## 5. Radon-Nikodym theorems: necessary and sufficient conditions.

In each of the three cases of Theorem 4.9 the Radon-Nikodym derivative possessed a "compatibility" with the lifting $\rho$ used to construct it. In this section we shall present necessary and sufficient conditions for the vector valued measure $m$ to be the indefinite integral of a function of this type, with respect to the nonnegative measure $\mu$. Since the definition (of a "type $\rho$" function) relies crucially upon the arbitrary selection of a lifting $\rho$, we have felt it desirable to lay stress upon special cases (such as Theorem 4.9 itself) which may require the involvement of a lifting in the proof, but which do not require it in the statement. Theorem 5.3 itself is more preliminary in nature.

In preparation for Definition 5.1, we shall extend the notion of a lifting to decomposable measure spaces. Let $(X, S, \mu)$ be a complete, decomposable measure space, and let $\{X_\alpha\}$ be a family of disjoint sets of positive finite measure such that $\mu(E) = \sum_\alpha \mu(E \cap X_\alpha)$ for all $E \in S$ (cf. Definition 3.3). For each $\alpha$, let $\rho_\alpha$ be a lifting for $\mu$ on $X_\alpha$, and then, given $E \in S$, define $\rho(E) = \bigoplus_\alpha \rho_\alpha(E \cap X_\alpha)$. We obtain a function $\rho: S \to S$ which satisfies (4.1.1)–(4.1.5), except that $\rho(X)$ is not defined. We shall call $\rho$ a lifting for $\mu$ in this setting, and we observe that a complete, $\sigma$-finite measure space is decomposable if and only if it admits a lifting of this sort. (The "if" argument involves a disjointization procedure, as in the proof of Propositiom 3.5.) Note that if $h$ is a scalar valued, essentially bounded, measurable function on $X$, then $\rho(h)$ may be defined in the manner described after Definition 4.1.

5.1. **Definition.** Let $(X, S, \mu)$ be a $\sigma$-finite measure space, let $\rho$ be a lifting for $\mu$, let $V$ be a vector space, let $\Gamma$ be a space of linear functionals on $V$, and let $f$ be a $V$ valued, locally $\Gamma$ measurable function on $X$. We shall say that $f$ is of *type* $\rho$ (with respect to $\Gamma$) if there exists a dense subset of $S$ such that for each set $E$ in this dense subset we have $\rho(\phi f \chi_E) = \phi f \chi_E$ for all $\phi \in \Gamma$, except possibly for a null set which is independent of the choice of $\phi$.

The motivation for the "null set which is independent of the choice of $\phi$" is to allow any function which is a.e. equal to a type $\rho$ function to be of type $\rho$ as well. We shall now formally label the type of function derived in (4.9.2).

5.2. **Definition.** Let $(X, S, \mu)$ be a ($\sigma$-finite) measure space, let $V$ be a normed linear space, let $\Gamma$ be a norming subspace of $V^*$, and let $f: X \to V$ be a function. Given $E \in S$, we shall say that $f$ is *strongly* $\Gamma$ *integrable on* $E$ if $f$ is $\Gamma$ integrable on all measurable subsets of $E$, if $\|f(\cdot)\|$ is measurable on $E$, and if we have $|m|(F) = \int_F \|f(x)\| d\mu(x)$ for all measurable subsets of $F$ of $E$, where $m(F)$ denotes the $\Gamma$ integral $\int_F f d\mu$. If $M \subset S$, we shall say that $f$ is *strongly* $\Gamma$ *integrable on* $M$ if $f$ is strongly $\Gamma$ integrable on each set $E$ in $M$.

We emphasize that the equality $|m|(F) = \int_F \|f(x)\| d\mu(x)$ does not imply that either expression is finite.

If $f$ is strongly $\Gamma$ integrable on a $\mu$-ideal $M \subset S$, it is straightforward to verify that whenever $\|f(\cdot)\| \chi_E(\cdot)$ is integrable, it will be the $L_1$ supremum of the family $\{|v^* f(\cdot)| \chi_E(\cdot) : v^* \in \Gamma; \|v^*\| \leq 1\}$. Therefore, if $g$ is $\Gamma$ integrable on $M$ and has the same indefinite integral as $f$ does, we may infer that $\|f(x)\| \leq \|g(x)\|$ a.e. We are unable, however, to ascertain whether this "minimal norm" property is sufficient to imply the strong $\Gamma$ integrability of $f$.

If $f$ is $\Gamma$ integrable on the $\mu$-ideal $M \subset S$, and if $f$ is of type $\rho$ for some lifting $\rho$ for $\mu$, then it follows from the arguments in the proof of (4.9.2) that $f$ is also strongly $\Gamma$ integrable. In particular, let $f$ be locally (strongly) measurable. Then the collection $M_f$ of sets $E$ in $S$ such that $\|f(\cdot)\| \chi_E(\cdot)$ is integrable, and hence such that $f$ is Bochner integrable on $E$ (in the completion, $\overline{V}$, of $V$), constitutes a $\mu$-ideal in $S$. It is now evident from Lemma 4.3 and from the remarks following it that $f$ is of type $\rho$ (with respect to $V^*$) for any lifting $\rho$ for $\mu$, and hence that $f$ is strongly $V^*$ ($= \overline{V}^*$) integrable in $V$ on $M_f$. Conversely, if the $V$ valued function $f$ is of type $\rho$ (with respect to $V^*$) for some lifting $\rho$ for $\mu$, and if $f$ is (strongly) $V^*$ integrable on a $\mu$-ideal $M \subset S$, then, as was essentially established in the proof of (4.9.3), $f$ will be locally measurable.

We remark that a necessary and sufficient condition for $f$ to be strongly $\Gamma$ integrable on the $\mu$-ideal $M$ is that: for each set $E$ in a dense subset of $M$ we have

$$\overline{m}_q(E) = \left( \int_E \|f(x)\|^q d\mu(x) \right)^{1/q} < \infty,$$

where $1 < q < \infty$, and where $\overline{m}_q$ denotes the "$q$-variation" of $m$. as described by Dinculeanu [4, pp.241 ff.]. (When $q = \infty$, an analogous condition can be given.) Both necessity and sufficiency follow readily from [4, Proposition 4, p.254]. The notion of $q$-variation occurs at least as early as Phillips' paper [21, pp.133 ff.] (cf. [4, Propostion 1, p.249]), and, in Corollary 5.6 [21, p.134], the above criterion is noted in connection with a Bochner integrable Radon-Nikodym derivative. This corollary thus amounts to a special case of Theorem 5.7 below.

For the next theorem we establish the following notation: If $(X, S, \mu)$ is a $\sigma$-finite measure space, if $\rho$ is a lifting for $\mu$, and if $x \in X$, we let $S(x)$ denote the "$\rho$ neighborhoods" of $x$, i.e.. the set $\{F \in \rho(S): x \in F\}$. Then $S(x)$ is nonvoid except possibly for a locally null set, and is directed by the standard relation $F_1 \leq F_2$ if $F_1 \supset F_2$.

5.3. **Theorem.** *Let* $(X, S, \mu)$, $\rho$, $V$, *and* $\Gamma$ *be as in Definition 5.1, and let* $m$ *be a* $V$ *valued,* $\Gamma$ *countably additive measure defined on a* $\mu$-*ideal* $M \subset S$. *Ther. there exists a* $V$ *valued function* $f$ *of type* $\rho$ *on* $X$ *such that* $f$ *is* $\Gamma$ *integrable on* $M$, *and such that* $m(E) = \int_E f d\mu$ *for all* $E \in M$ *if and only if*

(5.3.1) $m$ *is* $\mu$-*continuous for* $\Gamma$, *i.e.* $\mu(E) = 0$ *implies that* $\phi m(E) = 0$ *for all* $\phi \in \Gamma$;

(5.3.2) *for each set* $E$ *in a dense subset of* $M$, *the set* $A_E(m)$ *(as defined in Theorem 4.9) is* $\Gamma$ *bounded, i.e.* $\phi(A_E(m))$ *is bounded for all* $\phi \in \Gamma$;

(5.3.3) *for all* $x$ *not in a fixed locally null set, the net* $\{m(F)/\mu(F)\}_{F \in S(x)}$ *has a subnet which converges in the* $\Gamma$ *topology.*

**Remark.** If $V$ is a normed linear space, and if $\Gamma$ is a norming subspace of $V^*$, then $f$ will be strongly $\Gamma$ integrable, and it is readily deduced that the statement "$A_E(m) \cdots$ is $\Gamma$ bounded" in (5.3.2) may be replaced by the statement "$|m|(E)$ is finite".

If we have in addition that $V$ is separable or that $\Gamma = V^*$, then $f$ will also be locally measurable, and the $\Gamma$ integral $\int_E f d\mu$ can be computed as a Bochner integral if $|m|(E)$ is finite.

**Proof.** It is easy to see that the argument in the proof of (4.9.1) will yield a type $\rho$ Radon-Nikodym derivative $f$ for $m$ with respect to $\mu$ if $m$ satisfies the weaker conditions (5.3.1)–(5.3.3). Note that $f$ may be defined arbitrarily on the locally null set of points $x \in X$ for which no assumption is made about the net $\{m(F)/\mu(F)\}_{F \in S(x)}$ (or for which $S(x)$ is void).

Conversely, suppose that $m$ is the indefinite integral with respect to $\mu$ of a $\Gamma$ integrable function $f$ of type $\rho$. Then (5.3.1) is immediate. Let $E \in M$ be such that $\rho(\phi f \chi_E) = \phi f \chi_E$ a.e. for all $\phi \in \Gamma$, and such that the null set $N$ where the

two functions differ is independent of the choice of $\phi \in \Gamma$. Then, in particular, the function $\phi / \chi_E$ is essentially bounded for all $\phi \in \Gamma$, so that, by the mean value theorem, the set $A_E(m)$ is $\Gamma$ bounded.

To establish (5.3.3), we shall simplify notation by assuming that the set $E$ of the last paragraph equals $X$. Let $\{f_\pi\}$ be the net defined in the proof of (4.9.1). Then, because the function $\phi f$ is essentially bounded, Lemma 4.3 implies that $\rho(\phi f)(x) = \lim_\pi \phi f_\pi(x)$ for all $x \in X$. But for $x \notin N$, the fixed null set defined above, we have $\phi f(x) = \rho(\phi f)(x)$ for all $\phi \in \Gamma$. It is then clear from the definition of $f_\pi$ that $\phi f(x) = \lim_{F \in S(x)} \phi(m(F)/\mu(F))$ for all $\phi \in \Gamma$, and for each $x \notin N$. By Proposition 3.2, (5.3.3) is now immediate. ∎

We remark that hypotheses (5.3.1) and (5.3.3) alone are sufficient to yield a $\Gamma$ integrable Radon-Nikodym derivative for $m$ with respect to $\mu$; however, we are unable to characterize conveniently the type of $\Gamma$ integrable functions whose indefinite integrals satisfy (5.3.1) and (5.3.3) without necessarily being of type $\rho$, i.e. without necessarily satisfying (5.3.2) as well.

Without assuming hypothesis (5.3.3), we may still infer that, for nearly every $x \in X$, the net $\{\phi(m(F)/\mu(F))\}_{F \in S(x)}$ converges for all $\phi \in \Gamma$, and so determines a linear functional on $\Gamma$. This observation permits the conclusion that Theorem 5.3 will remain essentially valid in the absence of hypothesis (5.3.3) if the function $f$ is allowed to assume values in $\Gamma^+$, the space of linear functionals on $\Gamma$. We omit precise details, except for mention of the following distinguished special case.

5.4. **Theorem.** *Let* $(X, S, \mu)$ *be a complete, decomposable measure space, let* $\Gamma$ *be a vector space, let* $\Gamma^+$ *be the space of linear functionals on* $\Gamma$, *and let* $m$ *be a* $\Gamma^+$ *valued,* $\Gamma$ *countably additive measure defined on a* $\mu$-*ideal* $M \subset S$. *Then there exists a* $\Gamma^+$ *valued function* $f$ *on* $X$ *such that* $f$ *is of type* $\rho$ *for some lifting* $\rho$ *for* $\mu$, *such that* $f$ *is* $\Gamma$ *integrable on* $M$, *and such that* $m(E) = \int_E f d\mu$ *for all* $E \in M$ *if and only if* $m$ *is* $\mu$-*continuous, and, for each set* $E$ *in a dense subset of* $M$, *the set* $A_E(m)$ *is* $\Gamma$ *bounded (as defined in Theorem 5.3).*

We remark that by the Tychonoff theorem a set $K \subset \Gamma^+$ is $\Gamma$ bounded if and only if it is relatively $\Gamma$ compact.

Except for generalization to the decomposable setting, the following two theorems are essentially known. With the one exception noted in Theorem 5.6, all of the additional arguments which are needed to fully establish these theorems are entirely elementary.

The following theorem includes the familar weak[*] topologies as distinguished special cases.

5.5. **Theorem.** *Let $(X, S, \mu)$ be a complete, decomposable measure space, let $V$ be a normed linear space, let $\Gamma$ be a norming subspace of $V^*$ such that the unit ball of $V$ is relatively compact in the $\Gamma$ topology, and let $m$ be a $V$ valued, $\Gamma$ countably additive measure defined on a $\mu$-ideal $M \subset S$. Then there exists a $V$ valued function $f$ on $X$ such that $f$ is strongly $\Gamma$ integrable on $M$, and such that $m(E) = \int_E f d\mu$ for all $E \in M$ if and only if $m$ is $\mu$-continuous, and $|m|(E)$ is finite for each set $E$ in a dense subset of $M$.*

**Proof.** Cf. [3, Theorem 1, p.132]. ∎

When the unit ball of $V$ is not relatively $\Gamma$ compact, it is always possible to embed $V$ in a larger space $W$ such that $\Gamma$ may be regarded as a norming subspace of $W^*$, and such that the unit ball of $W$ is (relatively) $\Gamma$ compact. Theorem 5.5 carries over exactly to this setting, provided of course that we allow the function $f$ to assume values in $W$. We may always let $W = \Gamma^*$ (cf. Theorem 5.4); however, in specific instances there may arise a more natural choice for the space $W$. For example, in the theorem of Dinculeanu discussed earlier [4, Theorem 5, p.269], the space $V = \mathcal{L}(E, F)$ is embedded in the space $W = \mathcal{L}(E, Z^*)$ (where we recall that $Z$ was a norming subspace of $F^*$). Although Dinculeanu does not mention it explicitly, the space $\mathcal{L}(E, Z^*)$ may be seen to have a compact unit ball for the $\Gamma$ topology involved.

The following theorem states in particular that the conditions of (4.9.3) are both necessary and sufficient to obtain a locally measurable Radon-Nikodym derivative.

5.6. **Theorem.** *Let $(X, S, \mu)$ be a decomposable measure space, let $B$ be a Banach space, and let $m$ be a $B$ valued, (norm) countably additive measure defined on a $\mu$-ideal $M \subset S$. Then there exists a $B$ valued, locally measurable function $f$ on $X$ such that $f$ is strongly $B^*$ integrable on $M$, and such that $m(E) = \int_E f d\mu$ for all $E \in M$ if and only if $m$ is $\mu$-continuous, and any one of the following statements is true.*

*(5.6.1) For each set $E$ in a dense subset of $M$, the set $A_E(m)$ (as defined in Theorem 4.9) is relatively $B^*$ compact.*

*(5.6.2) For each set $E$ in a dense subset of $M$, we have both that $|m|(E)$ is finite, and that there exists a $B^*$ compact set $K \subset B - \{0\}$ (depending upon $E$) such that $m(F)$ is contained in the cone generated by $K$ for all measurable sets $F \subset E$.*

*(5.6.3) Given $\epsilon > 0$, the diameter of $A_E(m)$ is $< \epsilon$ for each set $E$ in a dense subset of $M$.*

*(5.6.4) For each set $E$ in a dense subset of $M$, the set $A_E(m)$ is dentable [23, Definition 1, p.71].*

**Remark.** Note that norm countable additivity is equivalent to $B^*$ countable additivity [20, Theorem 2.4, p.283].

**Proof.** (Cf. [24, Main Theorem, p.466], [17, Theorem 11, p.203], [14], [23, Theorem 1, p.71]. ) We remark that the proof in [24, p.479] that (5.6.2) implies (5.6.1) carries over from the norm compact to the $B^*$ compact setting by using the deeper Kreĭn-Smulian theorem [7, Theorem 4, p.434] in place of the elementary Mazur theorem [7, Theorem 6, p.416]. ∎

With note of the fact that reflexive Banach spaces are precisely those whose unit balls are (relatively) weakly compact [7, Theorem 7, p.425], we may combine Theorems 5.5 and 5.6 to produce the closest analogue of the classical Radon-Nikodym theorem for measures $m$ which are not scalar valued.

**5.7. Theorem.** *Let $(X, S, \mu)$ be a decomposable measure space, let $B$ be a reflexive Banach space, and let $m$ be a $B$ valued, countably additive measure defined on a $\mu$-ideal $M \subset S$. Then there exists a $B$ valued, locally measurable function $f$ on $X$ such that $f$ is strongly $B^*$ integrable on $M$, and such that $m(E) = \int_E f\,d\mu$ for all $E \in M$ if and only if $m$ is $\mu$-continuous, and $|m|(E)$ is finite for each set $E$ in a dense subset of $M$.*

6. **Examples and open problems.** The five examples to follow will complement those given by Rieffel [24, pp.484 ff.]. For each, let $X$ be the unit interval $[0, 1]$, let $S$ be the Lebesgue measurable subsets of $X$, and let $\mu$ be Lebesgue measure on $S$.

Our first example will present a function which has constant norm, and which is $\Gamma$ integrable, but not strongly $\Gamma$ integrable.

**6.1. Example.** Let $\{e_x\}_{x \in X}$ be an orthonormal set in a suitably large Hilbert space $H$. Define the function $f: X \to H$ by $f(x) = e_x$ for all $x \in X$. Then $f$ is $H^*$ ( $= H$) integrable on $S$, and we have $\int_E f\,d\mu = 0$ for all $E \in S$. However, $\|f(x)\| = 1$ for all $x \in X$. If we multiply $f$ by a nonmeasurable scalar valued function, we obtain a function with badly behaved norm which is still $H^*$ integrable on $S$.

Our next example will illustrate that the assumptions made about the finiteness of the total variation in Theorem 5.3, in Theorem 5.5, in (5.6.2), and in Theorem 5.7 cannot be eliminated.

**6.2. Example.** (Cf. [19, Example C.5. (b), p.365].) Let $B = L_p(\mu)$, where $1 < p < \infty$, and define the measure $m: S \to B$ by $m(E) = \chi_E$ for all $E \in S$. Then $|m|(E) = 0$ if $\mu(E) = 0$, and $|m|(E) = \infty$ if $\mu(E) > 0$. Let $\Gamma$ be a norming subspace of $B^*$. Since $B$ is separable, any $\Gamma$ measurable function on $X$ will be measurable, and therefore Bochner integrable on some $\mu$-ideal $M \subset S$. Thus $m$ cannot be the indefinite integral of any such function. Since $B$ is also reflexive, it may be seen that the results cited above will each fail if a finiteness assumption is not made about $|m|$.

Our next example will illustrate both that Theorem 5.7 fails in the nonreflexive setting, and that two strongly $\Gamma$ integrable functions with the same indefinite integral need not coincide at any $x$ in $X$.

6.3. **Example.** Now let $B = L_1(\mu)$, and let $m$ be defined as in Example 6.2. Then $|m|(E) = \mu(E)$ for all $E \in S$. By using (5.6.3), it is quickly checked that $m$ could not be the indefinite integral of a Bochner integrable function, and hence also not of a $\Gamma$ integrable function for any norming subspace $\Gamma$ of $B^*$.

However, by Theorem 5.5, $m$ will be the indefinite integral of a $B^{**}$ valued function $f$ on $X$ which is strongly $B^*$ integrable on $S$. We shall now describe $f$ explicity. We recall that $B^* = L_\infty(\mu)$ [7, Theorem 5, p.289], and that $B^{**} = ba(\mu)$ [7, Theorem 16, p.296], where $ba(\mu)$ denotes the space of finitely additive, $\mu$-continuous, scalar valued measures on $S$ with finite total variation. (Then $m(E)$, regarded as an element of $ba(\mu)$, is simply $\mu$ restricted to $E$.) Let $\rho$ be a lifting for $\mu$, and let $x \in X$. Using the construction in the proof of (4.9.1), we may easily establish that $f(x) = \rho_x \in ba(\mu)$, where $\rho_x(E) = 1$ if $x \in \rho(E)$, and $= 0$ otherwise, for all $E \in S$. (Note then that, given $\phi \in L_\infty(\mu)$, we have $\phi f(x) = \int \phi d\rho_x = \rho(\phi)(x)$, so that $\phi m(E) = \int_E \phi d\mu = \int_E \rho(\phi) d\mu = \int_E \phi f d\mu$ for all $E \in S$, and for all $\phi \in L_\infty(\mu)$.) It is now clear that two distinct liftings for $\mu$ will yield two $B^*$ integrable Radon-Nikodym derivatives for $m$ with respect to $\mu$ which are unequal at every point $x$ in $X$.

Our next example will present a strongly $\Gamma$ integrable function which is not of type $\rho$ for any lifting $\rho$, and whose indefinite integral is not the indefinite integral of any type $\rho$ function.

6.4. **Example.** We modify Example 6.3 as follows: Replace $ba(\mu)$ by $V$, the space of scalar valued, countably additive measures on $S$ under the total variation norm; replace $L_\infty(\mu)$ by $\Gamma$, the space of scalar valued, bounded measurable functions on $X$ under the supremum norm $\|\phi\|_\infty = \sup\{|\phi(x)| : x \in X\}$. Then $\Gamma$ may be regarded as a norming subspace of $V^*$, and the measure $m$ of Example 6.3 may be assumed to take values in $V$.

Given $x \in X$, let $g(x) = \delta_x$, where $\delta_x \in V$ denotes unit mass at the point $x$. Then it is readily checked that $g$ is strongly $\Gamma$ integrable on $S$, and that $m$ is its indefinite integral. Since $\phi g(x) = \int \phi d\delta_x = \phi(x)$ for all $\phi \in \Gamma$, it is clear that the function $g$ cannot be of type $\rho$ for any lifting $\rho$ for $\mu$. Moreover, if $f$ were a type $\rho$ Radon-Nikodym derivative for $m$ with respect to $\mu$, then, by Theorem 5.3, we would have $f(x) = \rho_x$ (as defined in Example 6.3) for almost all $x \in X$. However, the measure $\rho_x$ is not countably additive, and is therefore not in $V$.

Our final example will illustrate two intuitively obvious truisms:

(1) that a measure $m$ can be a $\Gamma$ indefinite integral of a Bochner integrable function without being its Bochner indefinite integral; and

(2) that the larger the space $V$ and the smaller the space $\Gamma$, the easier it is

to find well-behaved $\Gamma$ integrable Radon-Nikodym derivatives.

**6.5. Example.** In Example 6.4, let $\phi_0 \equiv 1$, and let $\Gamma_0$ be the one-dimensional subspace of $\Gamma$ generated by $\phi_0$. Define $f(x) = \mu \in V$ for all $x \in X$. Then $f$ is Bochner integrable, and we have $\int_E f \, d\mu = \mu(E) \cdot \mu \neq m(E)$ for every nonnull set $E \in S$; however, the $\Gamma_0$ integral $\int_E f \, d\mu$ equals $m(E)$ in the sense that $\phi m(E) = \int_E \phi f \, d\mu$ for all $E \in S$, and for all $\phi \in \Gamma_0$. Moreover, we have $|m|(E) = \int_E \|f(x)\| d\mu(x) = \mu(E)$, for all $E \in S$.

**6.6. Problem.** *Characterize the indefinite integrals of arbitrary $\Gamma$ integrable functions and of arbitrary strongly $\Gamma$ integrable functions.*

We might first try to characterize the class of "accessible" $\Gamma$ integrable functions, i.e. those whose indefinite integrals satisfy precisely (5.3.1) and (5.3.3), the latter with respect to a lifting $\rho$ for $\mu$. If such a function is not also of type $\rho$, then it will be wild (e.g. unbounded on every nonnull subset of a set of positive measure), and so we ask

**6.7. Problem.** *Can hypothesis (5.3.2) be eliminated from Theorem 5.3?*

The term "accessible" is intended to imply that the values of the function can be recovered in the limit from the values of the indefinite integral. While in this sense the function of Example 6.1 is highly inaccessible, there does exist an accessible function (namely, the function identically equal to zero) with the same indefinite integral. By contrast, the measure of Example 6.4 does not admit an accessible Radon-Nikodym derivative in any ordinary sense, and it appears that a radical departure from the present techniques will be needed to characterize such measures.

**6.8. Problem.** *If the function $f$ is $\Gamma$ integrable on a $\mu$-ideal, where $\Gamma$ is a norming subspace of the dual of a normed linear space, then does there exist a strongly $\Gamma$ integrable function with the same indefinite integral?*

**6.9. Problem.** *How far can the analogy between "strongly $\Gamma$ integrable" and "Bochner integrable" be carried?*

To the extent that "strongly $\Gamma$ integrable" can be combined with "type $\rho$", the answer appears to be: nearly as far as we like. For example, let $(X, S, \mu)$ be a complete, finite measure space, and let $V$ be the dual of a Banach space $\Gamma$. Fix a lifting $\rho$ for $\mu$, and consider the collection of functions $f: X \to V$ which are $\Gamma$ measurable, and which are of type $\rho$. Since $V$ is a dual space, it is evident that each such function $f$ is $\Gamma$ integrable (and hence strongly $\Gamma$ integrable) on the $\mu$-ideal of sets $E \in S$ such that $\|f(\cdot)\| \chi_E$ is integrable.

It is readily checked that this collection of functions is closed under algebraic operations and under a.e. pointwise (norm) convergence of sequences, and that any two such functions with the same indefinite integral will be equal a.e. In fact most of the standard results of measure theory (Egoroff's theorem, the Riesz-Weyl

theorem, the dominated convergence theorem, etc.) will carry over intact. The singular and unavoidable exception is the inability to approximate by sequences of simple functions.

We may also form an "$L_1$" space comprising equivalence classes of functions $f$ such that $\|f\|_1 = \int_X \|f(x)\| \, d\mu(x)$ is finite. From Example 6.3, we see that in general (unless $\Gamma$ is reflexive) this space will strictly contain the space $L_1(X, S, \mu, V)$ of Bochner integrable functions, and, by Theorem 5.5, that it will be isometrically isomorphic to the Banach space of $\Gamma$ countably additive measures $m$ on $S$ such that $|m|(X)$ is finite. In particular we obtain concrete representation for a greater portion of the dual space of $L_\infty(X, S, \mu, \Gamma)$ (the space of equivalence classes of $\Gamma$ valued, essentially bounded, measurable functions on $X$) than is given by $L_1(X, S, \mu, V)$. Similarly the other "$L_p$" spaces may be defined (for $1 < p \leq \infty$), and they constitute the respective dual spaces of the Bochner spaces $L_q(X, S, \mu, \Gamma)$, where $1 \leq q < \infty$, and where $1/p + 1/q = 1$.

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