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HOMOLOGY IN VARIETIES OF GROUPS. IV

BY

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ABSTRACT. The study of homology groups $\mathfrak{V}_n(\Pi, A)$, $\mathfrak{V}$ a variety, Π a group in $\mathfrak{V}$, and A a suitable Π-module, is continued. A 'Tor' is constructed which gives a better (but imperfect) approximation to these groups than a Tor previously considered. $\mathfrak{V}_2(\Pi, Z)$ is calculated for various varieties $\mathfrak{V}$ and groups Π.

Introduction. We continue the study of homology groups $\mathfrak{V}_n(\Pi, A)$, where Π is a group in the variety $\mathfrak{V}$ and A is a suitable Π-module, as in [21], [25] and [31], here after referred to as [H I], [H II] and [H III]. These homology groups were compared with a certain Tor in [H I] and [H II], and it was shown that the two theories do not always agree in dimension 1. In §1 we introduce a different Tor which gives a better approximation to $\mathfrak{V}_n(\Pi, A)$; this theory arises from the consideration of two-sided modules as in the Hochschild theory, and was prompted by a remark of Barry Mitchell's. In §2, $\mathfrak{V}_2(\Pi, Z)$ is calculated approximately for $\mathfrak{V}$ the variety of metabelian groups that are nilpotent of class c and Π a finitely generated abelian group. The homological techniques are the same as those used in [H III] to calculate $\mathfrak{V}_2(\Pi, Z)$ for $\mathfrak{V}$ the variety of all nilpotent groups of class c and Π as above, though the group theory here is harder. Comparing the results in the two cases we deduce that neither Tor mentioned above always agrees with $\mathfrak{V}_n(\Pi, A)$ if $n = 2$, even when A is a module with trivial action. In §3 we repeat the above calculations, for $\mathfrak{V}$ the two varieties of all metabelian and all centre-by-metabelian groups. A refinement in the homology enables the calculations to be exact in these cases. The variety of metabelian groups is particularly interesting in that in this case it follows from a theorem of Grace Orzech that $\mathfrak{V}^2(\Pi, A)$, which can be calculated from $\mathfrak{V}_2(\Pi, Z)$ in the case of trivial action, may be interpreted as an obstruction group much as in the classical theory.

The principal conventions and notations of [H I], [H II] and [H III] will remain in force. A finitely generated abelian group will be said to be of rank r and type $(s; n_1, \cdots, n_t)$ if it is the direct product of s infinite cyclic groups, and t finite cyclic groups, one of order n_i for each i, where n_i divides n_{i+1} for $1 \leq i < t$, and $r = s + t$. A cyclic group generated by the element a will be written

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$C(a)$ if it is of infinite order and $C_n(a)$ if it is of order n. The category of abelian groups will be written as **Ab**, except when it is regarded as a variety of groups, in which case it will be written as $\mathfrak{A}$. If R is a ring, $\mathfrak{M}_R$ and ${}_R\mathfrak{M}$ denote the categories of right and left R-modules respectively.

1. **The Hochschild theory.** If Π is a group, Π^* will denote the opposite group of Π with elements $\{\pi^* \mid \pi \in \Pi\}$, and multiplication given by $\pi^*\rho^* = (\rho\pi)^*$. A two-sided Π-module A may be regarded as a left $Z\Pi^* \otimes_Z Z\Pi$-module by $(\pi^* \otimes \rho)a = \rho a \pi$, or as a right $Z\Pi^* \otimes_Z Z\Pi$-module by $a(\pi^* \otimes \rho) = \pi a \rho$, for all a in A and π, ρ in Π. Then the Hochschild homology groups $H_n^e(\Pi, A)$ are defined to be $\mathrm{Tor}_n^{Z\Pi^* \otimes Z\Pi}(Z\Pi, A)$; here $Z\Pi$ is a two-sided Π-module in the natural way, so that $Z\Pi$ and A may be regarded as right and left $Z\Pi^* \otimes_Z Z\Pi$-modules respectively. If A is a left Π-module then A may be regarded as a two-sided Π-module by $\rho a \pi = \rho a$. Then it is proved in [6] that $H_n^e(\Pi, A)$ and $H_n(\Pi, A) = \mathrm{Tor}_n^{Z\Pi}(Z, A)$ are isomorphic. Turning to the relative case, bear in mind that the varietal homology involves a dimension shift so that $\mathfrak{B}_n(\Pi, A)$ is to be compared with $\mathrm{Tor}_n^{\mathfrak{B}\Pi}(D\Pi, A)$, where $D\Pi = I\Pi \otimes_Z \mathfrak{B}\Pi$. The two-sided analogue of $\mathfrak{B}\Pi$ will be $\mathfrak{B}^e\Pi = \mathfrak{B}\Pi^* \otimes_Z \mathfrak{B}\Pi$. If Π_1, Π_2 are any groups in $\mathfrak{B}$, and $\alpha: \Pi_1 \to \Pi_2$ is a homomorphism, then α induces a homomorphism $\alpha^*: \Pi_1^* \to \Pi_2^*$ by $\pi^*\alpha^* = (\pi\alpha)^*$, and a homomorphism $\mathfrak{B}\alpha$: $\mathfrak{B}\Pi_1 \to \mathfrak{B}\Pi_2$, so putting $\mathfrak{B}^e\alpha = \mathfrak{B}\alpha^* \otimes \mathfrak{B}\alpha$: $\mathfrak{B}^e\Pi_1 \to \mathfrak{B}^e\Pi_2$, $\mathfrak{B}^e$ becomes a functor from $\mathfrak{B}$ to rings. We now construct the two-sided analogue of $D\Gamma$. For any group Γ define $\mu: Z\Gamma^* \otimes_Z Z\Gamma \to Z\Gamma$ by $(\gamma_1^* \otimes \gamma_2)\mu = \gamma_1\gamma_2$ and extend by linearity. Then putting $Z^e\Gamma = Z\Gamma^* \otimes_Z Z\Gamma$, μ becomes a homomorphism of right $Z^e\Gamma$-modules, where Γ acts naturally on $Z\Gamma$ on either side. Thus, putting $J\Gamma = \ker \mu$, $J\Gamma$ is a right $Z^e\Gamma$-module. If now $\Gamma \to \Pi \in |(\mathfrak{B}, \Pi)|$, there is a natural homomorphism of $Z^e\Gamma$ onto $\mathfrak{B}^e\Gamma$, and hence into $\mathfrak{B}^e\Pi$; and so we may form $J\Gamma \otimes_{Z^e\Gamma} \mathfrak{B}^e\Pi = D^e\Gamma$, say. D^e defines in a natural way a functor from $(\mathfrak{B}, \Pi)$ to right $\mathfrak{B}^e\Pi$-modules.

A two-sided $\mathfrak{B}\Pi$-module may equally be regarded as a left or right $\mathfrak{B}^e\Pi$-module in the same way that an arbitrary two-sided Π-module may be regarded as a left or right $Z^e\Pi$-module. In other words, the categories ${}_{\mathfrak{B}\Pi}\mathfrak{M}_{\mathfrak{B}\Pi}$ of two-sided $\mathfrak{B}\Pi$-modules, $\mathfrak{M}_{\mathfrak{B}^e\Pi}$ and ${}_{\mathfrak{B}^e\Pi}\mathfrak{M}$ are all isomorphic. If A is an abelian group then A can be made into a left or right $\mathfrak{B}\Pi$-module by making Π act trivially if and only if the exponent of Π divides the exponent of $\mathfrak{B}$. Thus, if A is a right $\mathfrak{B}\Pi$-module, A can be made into a two-sided $\mathfrak{B}\Pi$-module by making Π act trivially on the left; A with this additional structure will be denoted by ${}_\epsilon A$. Similarly, if A is a left $\mathfrak{B}\Pi$-module, A_ϵ may be defined.

It is the thesis of this paragraph that $\mathrm{Tor}_n^{\mathfrak{B}^e\Pi}(D^e\Pi, A_\epsilon)$ and $\mathrm{Ext}^n_{\mathfrak{B}^e\Pi}(D^e\Pi, {}_\epsilon A)$ are better approximations to $\mathfrak{B}_n(\Pi, A)$ and $\mathfrak{B}^n(\Pi, A)$ than are $\mathrm{Tor}_n^{\mathfrak{B}\Pi}(D\Pi, A)$ and $\mathrm{Ext}^n_{\mathfrak{B}\Pi}(D\Pi, A)$ respectively; but first a few details must be disposed of.

Lemma 1.1. *If $\Gamma \to \Pi \in |(\mathfrak{B}, \Pi)|$, then $D^e\Gamma$ is generated as a right $\mathfrak{B}^e\Pi$-module by $\{(\gamma^* \otimes 1 - 1^* \otimes \gamma) \otimes_{Z^e\Gamma} 1 \mid \gamma \in \Gamma\}$.*

Proof. By [6, Proposition IX. 3.1], $J\Gamma$ is generated as a right $Z^e\Gamma$-module by $\{(\gamma^* \otimes 1 - 1^* \otimes \gamma) | \gamma \in \Gamma\}$.

If A is a $\mathfrak{B}^e\Pi$-module, and $\Gamma \to \Pi \in |(\mathfrak{B}, \Pi)|$, then A becomes a $\mathfrak{B}^e\Gamma$-module, by pullback, and $\mathrm{Der}(\Gamma, A)$ denotes the set of derivations of Γ into A, i.e. maps $\delta: \Gamma \to A$ satisfying $(\gamma_1\gamma_2)\delta = \gamma_1(\gamma_2\delta) + (\gamma_1\delta)\gamma_2$; these form an abelian group under addition. If A is a right $\mathfrak{B}\Pi$-module, the usual definition of a derivation of a group into a module is regained by assuming that Π acts trivially on A on the left.

Lemma 1.2. *If $\Gamma \to \Pi \in |(\mathfrak{B}, \Pi)|$ and A is a right $\mathfrak{B}^e\Pi$-module, then there is an isomorphism* $\mathrm{Der}(\Gamma, A) \cong \mathrm{Hom}_{\mathfrak{B}^e\Pi}(D^e\Gamma, A)$ *which is natural in Γ and in A.*

Proof. It is easy to verify, using Lemma 1.1, that every derivation of Γ into A may be written uniquely as a composite $\Gamma \xrightarrow{\delta} D^e\Gamma \xrightarrow{\beta} A$, where δ is defined by $\gamma\delta = (\gamma^* \otimes 1 - 1 \otimes \gamma^*) \otimes_{Z^e\Gamma} 1$ and β is a $\mathfrak{B}^e\Pi$-module homomorphism. The lemma then follows at once.

Regard $\mathfrak{B}\Pi$ either as a ring or as a left $\mathfrak{B}^e\Pi$-right $\mathfrak{B}\Pi$-module, the left $\mathfrak{B}^e\Pi$-module structure arising from letting Π act by multiplication on the left and trivially on the right, the right $\mathfrak{B}\Pi$-module structure being the natural one; the context will make it clear which meaning is intended. Let A be a right $\mathfrak{B}\Pi$-module. Then ${}_\epsilon A = \mathrm{Hom}_{\mathfrak{B}\Pi}(\mathfrak{B}\Pi, A)$, where the (right) $\mathfrak{B}^e\Pi$-module structure on A arises from the left $\mathfrak{B}^e\Pi$-module structure on $\mathfrak{B}\Pi$. So

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{B}\Pi}(D\Gamma, A) &= \mathrm{Der}(\Gamma, A) = \mathrm{Der}(\Gamma, {}_\epsilon A)\\ &= \mathrm{Hom}_{\mathfrak{B}^e\Pi}(D^e\Gamma, \mathrm{Hom}_{\mathfrak{B}\Pi}(\mathfrak{B}\Pi, A)) = \mathrm{Hom}_{\mathfrak{B}\Pi}(D^e\Gamma \otimes_{\mathfrak{B}^e\Pi} \mathfrak{B}\Pi, A).\end{aligned}$$

From this we deduce

Lemma 1.3. *With the above module structure on $\mathfrak{B}\Pi$, $D\Gamma$ and $D^e\Gamma \otimes_{\mathfrak{B}^e\Pi} \mathfrak{B}\Pi$ are naturally isomorphic right $\mathfrak{B}\Pi$-modules.*

Proof. If a right Π-module A is regarded as a split extension of A by Π, and hence, using the canonical surjection onto Π, as an object in $(\mathfrak{B}, \Pi)$, then by the above remarks $\mathfrak{M}_{\mathfrak{B}\Pi}$ becomes a coreflective subcategory of $(\mathfrak{B}, \Pi)$ in the sense of [26], and both $\Gamma \to D\Gamma$ and $\Gamma \to D^e\Gamma \otimes_{\mathfrak{B}^e\Pi} \mathfrak{B}\Pi$ define coreflexions.

It was proved in [H I, Lemma 1.2] that if F is $\mathfrak{B}$-freely generated by a set $\mathfrak{X}$ then DF is freely generated as a right $\mathfrak{B}\Pi$-module by $\{(1 - x) \otimes 1 | x \in \mathfrak{X}\}$. The proof depended on the way that $D\Gamma$ represents the functor $\mathrm{Der}(\Gamma, -)$ from right $\mathfrak{B}\Pi$-modules to **Ab**. A similar argument proves

Proposition 1.4. *If F is $\mathfrak{B}$-freely generated by a set $\mathfrak{X}$, then D^eF is freely generated as a right $\mathfrak{B}^e\Pi$-module by $\{x^* \otimes 1 - 1^* \otimes x | x \in \mathfrak{X}\}$.*

Now consider the sequence of functors

$$(*) \qquad (\dot{\mathfrak{B}}, \Pi) \xrightarrow{D^e} {}_{\mathfrak{B}\Pi}\mathfrak{M}_{\mathfrak{B}\Pi} \xrightarrow{P} \mathfrak{M}_{\mathfrak{B}\Pi} \xrightarrow{Q} \mathrm{Ab}$$

where $P = - \otimes_{\mathfrak{B}^e\Pi} \mathfrak{B}\Pi$, $Q = - \otimes_{\mathfrak{B}\Pi} A$, and A is a fixed left $\mathfrak{B}\Pi$-module. By Lemma 1.3, $D^e P = D$. Also, $PQ = - \otimes_{\mathfrak{B}^e\Pi} A_\epsilon$, and $D^e PQ = \mathrm{Diff}(-, A)$. Moreover, by Proposition 1.4, D^e takes $\mathfrak{B}$-free groups over Π to free $\mathfrak{B}^e\Pi$-modules. Now if $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, $\mathcal{A}$ having enough projectives, and if S: $(\mathfrak{B}, \Pi) \to \mathcal{A}$ and T: $\mathcal{A} \to \mathcal{B}$ are functors such that the nth derived functor of T vanishes, for $n > 0$, on the image in $\mathcal{A}$ of a $\mathfrak{B}$-free group over Π, then there is a spectral sequence

$$L_p T(\mathfrak{B}_q(\Pi, S)) \underset{p}{\Rightarrow} \mathfrak{B}_n(\Pi, ST),$$

as in [13, Theorem 2.26], where $L_p T$ denotes the pth left derived functor of T. Recall that if A is a left $\mathfrak{B}\Pi$-module, then $\mathfrak{B}_q(\Pi, \mathrm{Diff}(-, A))$ is abbreviated to $\mathfrak{B}_q(\Pi, A)$; in particular $\mathfrak{B}_q(\Pi, D) = \mathfrak{B}_q(\Pi, \mathfrak{B}\Pi)$. Applying this to the sequence $(*)$, with one functor omitted or two functors composed as above, spectral sequences

(i) $\mathrm{Tor}_p^{\mathfrak{B}^e\Pi}(\mathfrak{B}_q(\Pi, D^e), \mathfrak{B}\Pi) \Rightarrow_p \mathfrak{B}_n(\Pi, \mathfrak{B}\Pi)$,

(ii) $\mathrm{Tor}_p^{\mathfrak{B}^e\Pi}(\mathfrak{B}_q(\Pi, D^e), A_\epsilon) \Rightarrow_p \mathfrak{B}_n(\Pi, A)$, and

(iii) $\mathrm{Tor}_p^{\mathfrak{B}\Pi}(\mathfrak{B}_q(\Pi, D), A) \Rightarrow_p \mathfrak{B}_n(\Pi, A)$

are obtained.

Similarly, since P is additive and takes projective $\mathfrak{B}^e\Pi$-modules to projective $\mathfrak{B}\Pi$-modules and Q is additive and right exact, there is, by [28, Theorem 2.4.1], a spectral sequence

(iv) $\mathrm{Tor}_p^{\mathfrak{B}\Pi}(\mathrm{Tor}_q^{\mathfrak{B}^e\Pi}(D^e\Pi, \mathfrak{B}\Pi), A) \Rightarrow_p \mathrm{Tor}_n^{\mathfrak{B}^e\Pi}(D^e\Pi, A_\epsilon)$.

(iii) has already been studied in [H II] where a homomorphism $\theta_n^{\mathfrak{B}}(\Pi, A)$: $\mathfrak{B}_n(\Pi, A) \to \mathrm{Tor}_n^{\mathfrak{B}\Pi}(D\Pi, A)$ was constructed which appears as an edge homomorphism of (iii). Similarly one may construct homomorphisms from $\mathfrak{B}_n(\Pi, A)$ to $\mathrm{Tor}_n^{\mathfrak{B}^e\Pi}(D^e\Pi, A_\epsilon)$ and from $\mathrm{Tor}_n^{\mathfrak{B}^e\Pi}(D^e\Pi, A_\epsilon)$ to $\mathrm{Tor}_n^{\mathfrak{B}\Pi}(D\Pi, A)$ whose composite is $\theta_n^{\mathfrak{B}}(\Pi, A)$ and which appear as edge homomorphisms in the spectral sequences (ii) and (iv), at least if these are constructed analogously to the construction of (iii) in [H II]. This factorisation of θ is the sense in which $\mathrm{Tor}_n^{\mathfrak{B}^e\Pi}(D^e\Pi, A_\epsilon)$ approximates more closely to $\mathfrak{B}_n(\Pi, A)$ than does $\mathrm{Tor}_n^{\mathfrak{B}\Pi}(D\Pi, A)$. In dimension 1, θ factorises as a product of surjections $\mathfrak{B}_1(\Pi, A) \to \mathrm{Tor}_1^{\mathfrak{B}^e\Pi}(D^e\Pi, A) \to \mathrm{Tor}_1^{\mathfrak{B}\Pi}(D\Pi, A)$, so that the approximation is better in this dimension in a more concrete sense. Analogues of the above remarks may, of course, be made for cohomology.

2. **Metabelian of class c.** The object of this paragraph is to calculate $\mathfrak{B}_2(\Pi, Z)$ (approximately) where $\mathfrak{B} = \mathfrak{N}_c \cap \mathfrak{A}^2$, the variety of all metabelian groups

that are nilpotent of class at most c, where $c \geq 3$, and Π is a finitely generated abelian group. The method of proof is the same as in the corresponding calculation in the case $\mathfrak{B} = \mathfrak{N}_c$ carried out in [H III]. The essence of the problem is an understanding of the Schur multiplier of the $\mathfrak{B}$-free groups. We can get away with less than the full structure.

If F is freely generated by the finite or infinite set $x_1, x_2, \cdots$, a simple basic commutator of weight m is a commutator of the form $[x_{i_1}, x_{i_2}, \cdots, x_{i_m}]$, where $i_1 > i_2 \leq i_3 \leq \cdots \leq i_m$. A dopple basic commutator of weight $m+2$ is a commutator of the form $[[x_{i_1}, x_{i_2}, \cdots, x_{i_m}], [x_i, x_j]]$, where $[x_{i_1}, x_{i_2}, \cdots, x_{i_m}]$ is a simple basic commutator; $i > j$, $j \leq i_3$ and either $i_2 < j$, or $i_2 = j$ and $i_1 < i$, or $i_2 = j$ and $i_1 = i$ and $m > 2$.

Lemma 2.1. *Let F be freely generated by the finite or infinite set $x_1, x_2, \cdots$, and let $c \geq 3$. Then the Schur multiplier $F''F_{c+1}/[F''F_{c+1}, F]$ of the $(\mathfrak{N}_c \cap \mathfrak{A}^2)$-free group $F/F''F_{c+1}$ is the direct product of three subgroups L, M and N, where L is freely generated, as an abelian group, by the images in $F''F_{c+1}/[F''F_{c+1}, F]$ of the dopple basic commutators of weight 4, M is generated by the images of the dopple basic commutators of weight greater than 4, and N is freely generated, as an abelian group, by the images of the simple basic commutators of weight $c+1$.*

Proof. Note that $[F''F_{c+1}, F] = [F'', F]F_{c+2}$, since $[F'', F]$ is a normal subgroup of F. That $F''F_{c+2}/[F'', F]F_{c+2}$ is generated by L and M is clear from §5 of [30]. Hence $F''F_{c+1}/[F''F_{c+1}, F]$ is generated by L, M and N. To show that $F''F_{c+1}/[F''F_{c+1}, F]$ is the direct product of LM and N, and that the given generators of N are linearly independent, take the natural map $F''F_{c+1}/[F'', F]F_{c+2} \to F''F_{c+1}/F''F_{c+2}$; by a theorem of Magnus (Hanna Neumann [12, Theorem 36.32]) the second group is freely generated, as an abelian group, by the images of the simple basic commutators of weight $c+1$, and L and M are clearly contained in the kernel. It remains to show that $F''F_{c+1}/[F_{c+1}, F]$ is the direct product of L and MN, and that the given generators of L are linearly independent. Consider the natural map $F''F_{c+1}/[F''F_{c+1}, F] \to F_4/F_5$, which is freely generated, as an abelian group, by the images of the simple and dopple basic commutators of weight 4. This completes the proof.

We shall also need the following commutator identity, valid in any group G.

Lemma 2.2. *If a, b, c, $d \in G$, and if n is any positive integer, then*

$$\begin{aligned}[][[a^n, b], [c, d]] &\equiv [[a, b^n], [c, d]] \equiv [[a, b], [c^n, d]] \\ &\equiv [[a, b], [c, d^n]] \equiv [[a, b], [c, d]]^n \mod [G_2, G_3].\end{aligned}$$

Proof. Immediate by induction.

For a positive integer r, define

$$r_{[4]} = (r+1)r(r-1)(r-2)/8.$$

For positive integers r, c, define

$$\psi(r, c) = c\binom{c+r-1}{c+1} \quad (r > 1), \qquad \psi(1, c) = 0.$$

Lemma 2.3. *$\psi(r, c)$ is the number of simple basic commutators of weight $c+1$ on r symbols.*

Proof. We may assume $r > 1$. Fix the value of i_2. Suppose $i_2 = r - s + 1$. Then the number of sequences $i_2 \le i_3 \le \cdots \le i_{c+1} \le r$ is the number of ways in which $c - 1$ objects may be selected from s large heaps, $s > 0$, objects being indistinguishable if and only if they come from the same heap. This number is $\binom{s+c-2}{c-1}$, see [27, §II.5]. Hence, for the given value of i_2, the number of sequences $i_1, i_2, \cdots, i_{c+1}$ satisfying the given inequalities is $(s-1)\binom{s+c-2}{c-1}$, so that the total number of sequences is $\Sigma_{s=2}^{r}(s-1)\binom{s+c-2}{c-1} = c\binom{c+r-1}{c+1}$, as is immediate by induction on r.

We can now prove

Proposition 2.4. *If Π is a finitely generated abelian group of rank $r > 0$ and type $(s; n_1, \cdots, n_t)$ and $\mathfrak{B} = \mathfrak{N}_c \cap \mathfrak{A}^2$, $c \ge 3$, then $\mathfrak{B}_0(\Pi, H_2(-, Z))$ is of rank ρ and type $(\sigma; \nu_1, \cdots, \nu_\tau)$ where $\rho = \psi(r, c) + r_{[4]}$, $\sigma = \psi(s, c) + s_{[4]}$, and $\nu_\tau = n_t$ if $s > 0$ and $t > 0$, $\nu_\tau = n_{t-1}$ if $s = 0$ and $t > 1$. In particular, $\rho = 0$ if $r = 1$.*

Proof. The proof is based on that of [H III, Lemma 3.1], q.v. However, in this case the problem is made harder by the more complicated structure of the Schur multiplier of the $\mathfrak{B}$-free groups. Let $\Pi = C(a_1) \times \cdots \times C(a_s) \times C_{n_1}(a_{s+1}) \times \cdots \times C_{n_t}(a_r)$, let P_0 be $\mathfrak{B}$-freely generated by $x_1, \cdots, x_r$, and define $f\colon P_0 \to \Pi$ by $x_i f = a_i$, $i = 1, \cdots, r$. If R is the kernel of f, R is generated, *qua normal subgroup*, by $\{x_i^{n_{i-s}} \mid i = s+1, \cdots, r\} \cup \{[x_j, x_k] \mid 1 \le j < k \le r\}$. Now the split extension of R by P_0 is generated by all pairs (a, b), where a runs over a set of generators of R qua normal subgroup and b runs over a set of generators of P_0. Then the fibre product $P_0 \times_\Pi P_0$ is generated by all pairs (ba, b). Hence we take P_1 to be $\mathfrak{B}$-freely generated by $\{y_i \mid 1 \le i \le r\} \cup \{z_i \mid s+1 \le i \le r\} \cup \{v_{jk} \mid 1 \le j < k \le r\}$, and define $(g_1, g_2)\colon P_1 \to P_0 \times_\Pi P_0$ by

$$y_i g_1 = x_i, \quad z_i g_1 = x_i^{n_{i-s}}, \quad v_{jk} g_1 = [x_j, x_k], \quad y_i g_2 = x_i, \quad z_i g_2 = 1, \quad v_{jk} g_2 = 1.$$

Then (g_1, g_2) is a surjection, and $\mathfrak{B}_0(\Pi, H_2(-, Z))$ is the cokernel of $H_2(g_1, Z) - H_2(g_2, Z)\colon H_2(P_1, Z) \to H_2(P_0, Z)$. (To accord with the dissonant conventions of group theory and homology, $H_2(P_1, Z)$ and $H_2(P_0, Z)$ will be written sometimes additively and sometimes multiplicatively.) Lemma 2.1 describes $H_2(P_0, Z)$ as the direct product of three subgroups L_0, M_0 and N_0. Similarly, $H_2(P_1, Z)$ is the direct product of three subgroups L_1, M_1 and N_1 which are defined once the generators of P_1 have been ordered; this we do by taking the y_i in the natural order followed by the z_i and v_{jk} in any order. The lemma then specifies generators for the groups L_0, L_1 etc., the generators of which will be described as canonical. It is clear that any canonical generator of $H_2(P_1, Z)$ is mapped to the identity by $H_2(g_2, Z)$ unless all the generators of P_1 appearing in the expression for the given generator are elements y_i, i.e. no z_i or v_{jk} appears. In the case when only generators y_i appear, the image of the given generator of $H_2(P_1, Z)$ is the same under $H_2(g_1, Z)$ and $H_2(g_2, Z)$. Thus we are reduced to looking at the quotient group of $H_2(P_0, Z)$ by the subgroup generated by the images under $H_2(g_1, Z)$ of the canonical generators of $H_2(P_1, Z)$ in which at least one z_i or v_{jk} appears. We first prove that M_0 is in the subgroup generated by the above images; this is why the precise structure of M_0 is not needed. Let $[[x_{i_1}, x_{i_2}, \cdots, x_{i_m}], [x_i, x_j]]$ define a canonical generator of M_0 so $m > 2$. Now $[[v_{i_1 i_2}, y_{i_3}, \cdots, y_{i_m}], [y_i, y_j]]$ defines an element of $H_2(P_1, Z)$ (though not a canonical generator) whose image under $H_2(g_1, Z) - H_2(g_2, Z)$ is clearly the given canonical generator of M_0. Thus M_0 makes no contribution to $\mathfrak{B}_0(\Pi, H_2(-, Z))$. Since L_1 and M_1 are mapped by $H_2(g_1, Z) - H_2(g_2, Z)$ into $L_0 \times M_0$, and N_1 is mapped into N_0, the contributions of L_0 and N_0 may be calculated separately. Every canonical generator of L_1 and M_1 will be mapped into M_0 except for those represented by elements $[[a, b], [c, d]]$, where each of a, b, c and d is a y_i or a z_i with at least one z_i. By Lemma 2.2 it is sufficient to consider commutators involving exactly one z_i in which case, in view of our ordering of the generators of P_1, the commutator will be of the form

$$[[z_{i_1}, y_{i_2}], [y_i, y_j]], \quad i > j,\ i_2 < j, \quad \text{or} \quad [[y_{i_1}, y_{i_2}], [z_i, y_j]], \quad i_1 > i_2,\ i_2 \leq j.$$

It follows, using Lemma 2.2 again, that the contribution of L_0 to $\mathfrak{B}_0(\Pi, H_2(-,Z))$ consists of an abelian group with generators represented by

$$\{[[x_{i_1}, x_{i_2}], [x_i, x_j]] \mid i_1 > i_2,\ i > j, \text{ and either } i_2 < j \text{ or } i_2 = j \text{ and } i_1 < i\}$$

and relators represented by $\{[[x_{i_1}, x_{i_2}], [x_i, x_j]]^k\}$, where $k = k(i_1, i_2, i, j) = 0$ if all of $a_{i_1}, a_{i_2}, a_i, a_j$ are of infinite order, and otherwise is the least of these orders. This is a group of rank $r_{[4]}$ and type $(s_{[4]}; \lambda_1, \cdots, \lambda_\alpha)$, where $\lambda_\alpha = n_t$ if $s > 1$, $\lambda_\alpha = n_{t-1}$ if $s = 1$, and $\lambda_\alpha = n_{t-2}$ if $s = 0$, since at least three distinct generators of P_0 are needed to define a generator of L_0. (If $r = s + t < 3$, $r_{[4]} = 0$.) The contribution of N_0 is generated by elements represented by all simple basic commutators $[x_{i_1}, x_{i_2}, \cdots, x_{i_{c+1}}]$ with $\{[x_{i_1}, x_{i_2}, \cdots, x_{i_{c+1}}]^k\}$ representing the relators where $k = k(i_1, i_2, \cdots, i_{c+1}) = 0$ if all of $a_{i_1}, \cdots, a_{i_{c+1}}$ have infinite order, and is otherwise the least of these orders. This is a group of rank $\psi(r, c)$, by Lemma 2.3, and is clearly of type $(\psi(s, c); \mu_1, \cdots, \mu_\beta)$, where

$\mu_\beta = n_t$ if $s > 0$, and $\mu_\beta = n_{t-1}$ if $s = 0$. Thus the proposition is proved.

We now deduce, exactly as in the proof of [H III, Theorem 3.2], the following

Theorem 2.5. *If* Π *is a finitely generated abelian group of rank* $r > 0$ *and type* $(s; n_1, \cdots, n_t)$, *and* $\mathfrak{V} = \mathfrak{N}_c \cap \mathfrak{A}^2$, $c \geq 3$, *then* $\mathfrak{V}_2(\Pi, Z)$ *is of rank* ρ *and type* $(\sigma; \nu_1, \cdots, \nu_\tau)$ *where*

$0 \leq \rho - \psi(r, c) - r_{[4]} \leq t + r(r-1)(r-2)/6$ (*cf. pp.* 297–298),

$0 \leq \sigma - \psi(s, c) - s_{[4]} \leq s(s-1)(s-2)/6$,

ν_τ *divides* n_{t_2}, *and if* $s = 0$, ν_τ *divides* $n_{t-1}n_t$ $(n_0 = 1)$.

Note that this theorem coincides with [H III, Theorem 3.2] in the case $c = 3$ since in this case $\mathfrak{N}_c \cap \mathfrak{A}^2 = \mathfrak{N}_c$.

Corollary 2.6. *If* Π *is a free abelian group of rank* 3 *then* $\mathfrak{V}_2(\Pi, Z) \ncong \operatorname{Tor}_2^{\mathfrak{V}\Pi}(D\Pi, Z)$ *in at least one of the cases* $\mathfrak{V} = \mathfrak{N}_4$, $\mathfrak{V} = \mathfrak{N}_4 \cap \mathfrak{A}^2$. *Also,* $\mathfrak{V}_2(\Pi, Z) \ncong \operatorname{Tor}_2^{\mathfrak{V}^e\Pi}(D^e\Pi, Z)$ *in at least one of these cases.*

Proof. Since Π is abelian, if A is any Π-module the split extension of A by Π lies in one of these varieties if and only if it lies in the other. Thus $\mathfrak{V}\Pi$ is the same for either variety, and so are $D\Pi$, $\mathfrak{V}^e\Pi$ and $D^e\Pi$ (cf. [H I, Proposition 1.5]). Now let ρ_1 and ρ_2 be the ranks of $\mathfrak{V}_2(\Pi, Z)$ in the cases $\mathfrak{V} = \mathfrak{N}_4$ and $\mathfrak{V} = \mathfrak{N}_4 \cap \mathfrak{A}^2$ respectively. Then by [H III, Theorem 3.2], $\rho_1 \geq \gamma(3, 4) = 48$, and by Theorem 2.5 of this paper, $\rho_2 \leq \psi(3, 4) + 3_{[4]} + 1 = 28$. Thus $\rho_1 \neq \rho_2$, and the result follows.

3. **Metabelian and centre-by-metabelian.** In this paragraph we calculate $\mathfrak{V}_2(\Pi, Z)$ exactly for Π a finitely generated abelian group, and $\mathfrak{V}$ the varieties $\mathfrak{A}^2$ and $[\mathfrak{A}^2, \mathfrak{E}]$ of all metabelian and centre-by-metabelian groups. The free groups of the former variety of infinite rank, in fact of rank greater than one, have trivial centre, and so if Π is any metabelian group and A is any $\mathfrak{V}\Pi$-module, where $\mathfrak{V} = \mathfrak{A}^2$, $\mathfrak{V}^2(\Pi, A)$ can be interpreted in terms of obstruction theory, see Orzech [32]. If Π acts trivially on A, $\mathfrak{V}^2(\Pi, A)$ may be calculated from $\mathfrak{V}_2(\Pi, Z)$ by universal coefficients [H I, Lemma 4.1]. The free groups of $[\mathfrak{A}^2, \mathfrak{E}]$, on the other hand, have enormous centres; in fact those of rank greater than one do not satisfy the maximal condition on normal subgroups. Now the 'construction pas à pas' of André [1] or the Tierney-Vogel construction [18] can be used to produce a simplical resolution of any finitely generated group in $\mathfrak{V}$ by finitely generated $\mathfrak{V}$-free groups, provided the finitely generated groups of $\mathfrak{V}$ satisfy the maximal condition on normal subgroups. Hence in such varieties, for example in nilpotent or metabelian varieties, $\mathfrak{V}_n(\Pi, A)$ is finitely generated provided that Π and A are. However, if $\mathfrak{V} = [\mathfrak{A}^2, \mathfrak{E}]$ and Π is free-metabelian of rank greater than one, $\mathfrak{V}_1(\Pi, Z) = H_2(\Pi, Z)$ is not finitely generated; but if Π is finitely generated abelian, it turns out that $\mathfrak{V}_2(\Pi, Z)$ (and of course $\mathfrak{V}_1(\Pi, Z)$) is finitely generated; see also the remarks at the end of this paper.

Let Π be any group, and let $\mathfrak{B}$ contain the product variety $\mathfrak{A}$ var Π. Then, by [H I, Proposition 1.4], $\mathfrak{B}\Pi = Z\Pi$; so if A is any $Z\Pi$-module and $\theta_n = \theta_n^{\mathfrak{B}}(\Pi, A)$ as in [H II, §1],

$$\theta_n : \mathfrak{B}_n(\Pi, A) \longrightarrow \mathrm{Tor}_n^{\mathfrak{B}\Pi}(D\Pi, A) = \mathrm{Tor}_n^{Z\Pi}(I\Pi, A) = \mathfrak{U}_n(\Pi, A),$$

where $\mathfrak{U}$ is the universal variety. Also, if $\phi_n = \phi_n(\mathfrak{U}, \mathfrak{B}, \Pi, A)$ as in [H III, §1], then $\phi_n : \mathfrak{U}_n(\Pi, A) \longrightarrow \mathfrak{B}_n(\Pi, A)$. The composite $\phi_n\theta_n : \mathfrak{U}_n(\Pi, A) \longrightarrow \mathfrak{U}_n(\Pi, A)$ may be regarded as a δ-morphism of δ-functors from $\mathfrak{M}_{Z\Pi}$ to **Ab**, and since $\phi_0\theta_0$ is the identity and $\mathfrak{U}_n(\Pi, A)$ is trivial for $n > 0$ and A projective, it follows that $\phi_n\theta_n$ is the identity for all $n > 0$. But ϕ_1 is a surjection, by [H III, Corollary 2.2], and so is an isomorphism; thus, by the same corollary, coker $\phi_2 =$ $\mathfrak{B}_0(\Pi, H_2(-, A))$. Hence

Proposition 3.1. *Let $\mathfrak{B}$ contain $\mathfrak{A}$ var Π, so that $\mathfrak{B}\Pi = Z\Pi$, and let A be any Π-module. Then for all $n > 0$,*

$$\mathfrak{B}_n(\Pi, A) \cong H_{n+1}(\Pi, A) \oplus K_n(\Pi, A)$$

where $K_n(\Pi, A) = \ker \theta_n = \operatorname{coker} \phi_n$. Moreover

$$K_1(\Pi, A) = 0 \quad \textit{and} \quad K_2(\Pi, A) \cong \mathfrak{B}_0(\Pi, H_2(-, A)).$$

It would be interesting to know if there are any nonabelian varieties of finite homological dimension; i.e. varieties $\mathfrak{B}$ such that $\mathfrak{B}_n(\Pi, A) = 0$ for all n greater than some n_0, and for all Π in $\mathfrak{B}$ and all $\mathfrak{B}\Pi$-modules A. The following result rules out 'large' varieties.

Corollary 3.2. *Let $\mathfrak{B}$ contain the product variety $\mathfrak{A}\mathfrak{X}$ for some nontrivial variety $\mathfrak{X}$. Then there is a group Π in $\mathfrak{B}$ and a $\mathfrak{B}\Pi$-module A such that $\mathfrak{B}_n(\Pi, A)$ $\neq 0$ for all $n > 0$.*

Proof. Take Π to be a nontrivial finite cyclic group in $\mathfrak{X}$, choose A so that $H_n(\Pi, A) \neq 0$, and apply Proposition 3.1.

Using Proposition 3.1 we calculate $\mathfrak{B}_2(\Pi, Z)$ for Π a finitely generated abelian group and $\mathfrak{B} = \mathfrak{A}^2$. Information about the structure of the Schur multipliers of the $\mathfrak{B}$-free groups is needed; for their precise structure see [29].

Lemma 3.3. *Let F be freely generated by the finite or infinite set $x_1, x_2, \cdots$. Write $K = [F'', F]$. Then the Schur multiplier F''/K of the free-metabelian group F/F'' is the direct product of two subgroups L and M, where L is freely generated, as an abelian group, by*

$$S = \{[[x_{i_1}, x_{i_2}], [x_i, x_j]]K \mid [[x_{i_1}, x_{i_2}], [x_i, x_j]] \text{ is a dopple basic commutator of weight } 4\},$$

and M *is generated by*

$$T = \{[[x_{i_1}, x_{i_2}, x_{i_3}^{\epsilon_3}, \cdots, x_{i_m}^{\epsilon_m}], [x_i, x_j]]K|$$

$$\epsilon_k = \pm 1, \text{ all } k;\ m > 2;\ i > j,\ i_1 > i_2 \leq i_3 \leq \cdots \leq i_m,\ j \leq i_3;$$

$$\text{if } i_p = i_{p+1} \text{ and } p \geq 3, \text{ then } \epsilon_p = \epsilon_{p+1};$$

$$\text{and either } i_2 < j \text{ or } i_2 = j \text{ and } i_1 \leq i\}.$$

Proof. This follows from §7 of [29], where it is shown that F''/K is freely generated as an abelian group by the set $S \cup T_1$ where T_1 is a subset of T. The proof is long and complicated; however, the fact that $S \cup T$ generates F''/K, while not quite sufficient for our purposes, may be readily deduced from the first three paragraphs of the proof of [29, Proposition 7.8].

Theorem 3.4. *If* Π *is a finitely generated abelian group of rank* $r > 0$ *and type* $(s; n_1, \cdots, n_t)$, *and* $\mathfrak{B}$ *is the variety of all metabelian groups, then* $\mathfrak{B}_2(\Pi, Z)$ *is isomorphic to* $H_3(\Pi, Z) \oplus K_2(\Pi, Z)$, *where* $K_2(\Pi, Z)$ *is of rank* $r_{[4]}$ *(see p. 297) and type* $(s_{[4]}; n_1, n_1, \cdots, n_t, n_t)$, *where* n_i *is repeated* $\frac{1}{2}(r-i+1)(r-1)(r-i-1)$ *times,* $i = 1, \cdots, t$.

Note that $H_3(\Pi, Z)$ is of rank $t + r(r-1)(r-2)/6$ and type $(s(s-1)(s-2)/6;$ $n_1, n_1, \cdots, n_t, n_t)$ where n_i is repeated $1 + \frac{1}{2}(r-i)(r-i-1)$ times, $i = 1, \cdots, t$ (cf. [H III, the proof of Theorem 3.2]).

Proof. By Proposition 3.1, $K_2(\Pi, Z) \cong \mathfrak{B}_0(\Pi, H_2(-, Z))$, which is calculated as in the case of $\mathfrak{B} = \mathfrak{N}_c \cap \mathfrak{A}^2$ (Proposition 2.4), except that analogues of the groups N_0 and N_1 do not arise.

We now turn to the variety $[\mathfrak{A}^2, \mathfrak{E}]$ beginning with information about the Schur multipliers of the relatively free groups; for their precise structure see [29].

Lemma 3.5. *Let* F *be freely generated by the finite or infinite set* $x_1, x_2, \cdots$. *Write* $K = [F'', F, F]$. *Then the Schur multiplier* $[F'', F]/K$ *of the free-centre-by-metabelian group* $F/[F'', F]$ *is the direct product of two subgroups* L *and* M, *where* L *is freely generated, as an abelian group, by* $\{[b, x_k]K|\ b$ *a dopple basic commutator of weight* $4\}$, *and* M *is generated by* $\{[c, x_k]K|\ c$ *a dopple commutator of weight greater than* 4 *as used to define a generator of the group* M *in Lemma* 3.3$\}$.

Proof. The result follows from §8 of [29].

Theorem 3.6. *If* Π *is a finitely generated abelian group of rank* $r > 0$ *and type* $(s; n_1, \cdots, n_t)$, *and* $\mathfrak{B}$ *is the variety of all centre-by-metabelian groups, then* $\mathfrak{B}_2(\Pi, Z)$ *is isomorphic to* $H_3(\Pi, Z) \oplus K_2(\Pi, Z)$, *where* $K_2(\Pi, Z)$ *is of rank* $r \times r_{[4]}$ *(see p. 297) and type* $(s \times s_{[4]}; n_1, n_1, \cdots, n_t, n_t)$, *where* n_i *is repeated*

$(r-i+1)(r-i)(r-i-1)(5r-5i+2)/8$ *times*, $i=1,\cdots,t$.

Proof. The proof of Theorem 3.4 needs only slight alteration, an extra term being tagged onto the end of every commutator.

If Π is an abelian group, $\mathfrak{V}\Pi$, $D\Pi$, $\mathfrak{V}^e\Pi$ and $D^e\Pi$ are the same in each of the varieties $\mathfrak{A}^2$, $[\mathfrak{A}^2, \mathfrak{E}]$ and the universal variety; this produces a similar situation to that of Corollary 2.9, except that the three homology theories compared in §1 are known to coincide in the latter variety, so we can point a finger at the villains.

Corollary 3.7. *If* Π *is a finitely generated group of rank* $r \geq 3$, *then* $\mathfrak{V}_2(\Pi, Z) \ncong \mathrm{Tor}_2^{\mathfrak{V}\Pi}(D\Pi, Z)$ *and* $\mathfrak{V}_2(\Pi, Z) \ncong \mathrm{Tor}_2^{\mathfrak{V}^e\Pi}(D^e\Pi, Z)$ *if* $\mathfrak{V}$ *is the variety of all metabelian groups and if* $\mathfrak{V}$ *is the variety of all centre-by-metabelian groups.*

We end with a note on some other varieties. Let

$$\mathfrak{V} = \left[\mathfrak{A}^2, \underset{n}{\mathfrak{E}}\right]$$

be the variety defined by the law $[[[x_1, x_2], [x_3, x_4]], x_5, \cdots, x_{n+4}]$. Then the Schur multiplier of the $\mathfrak{V}$-free group obtained by dividing out the free group on $x_1, x_2, \cdots$ by this law is generated by elements represented by $\{[[[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]], x_{i_5}, \cdots, x_{i_{n+4}}] \mid [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]]$ is a dopple basic commutator$\}$ together with commutators of higher weight. We do not know what relations hold between generators of the displayed form, but we have seen there are none in the cases $n=0$ or 1, and conjecture that there are none if $n=2$. In any case, for this variety, and Π a finitely generated abelian group of rank r, $K_2(\Pi, Z)$ is of rank at most $r^n r_{[4]}$, with equality for $n=0$ or 1 and, if our conjecture is true, for $n=2$. This may be compared with [H III, Theorem 3.2], according to which, if ρ is the rank of $\mathfrak{V}_2(\Pi, Z)$ where $\mathfrak{V} = \mathfrak{N}_c$ and Π is as above, then $\rho \sim r^{c+1}/(c+1)$ as $c \to \infty$.

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