

A DECOMPOSITION FOR COMBINATORIAL GEOMETRIES

BY

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ABSTRACT. A construction based on work by Tutte and Grothendieck is applied to a decomposition on combinatorial pregeometries in order to study an important class of invariants. The properties of this *Tutte decomposition* of a pregeometry into a subgeometry $G \setminus e$ and contraction G/e is explored in a categorically integrated view using factored strong maps. After showing that direct sum decomposition distributes over the Tutte decomposition we construct a universal pair (R, t) where R is a free commutative ring with two generators corresponding to a loop and an isthmus; and t , the *Tutte polynomial* assigns a ring element to each pregeometry. Evaluations of $t(G)$ give the Möbius function, characteristic polynomial, Crapo invariant, and numbers of subsets, bases, spanning and independent sets of G and its Whitney dual. For geometries a similar decomposition gives the same information as the chromatic polynomial throwing new light on the critical problem.

A basis is found for all linear identities involving Tutte polynomial coefficients. In certain cases including Hartmannis partitions one can recover all the Whitney numbers of the associated geometric lattice $L(G)$ from $t(G)$ and conversely.

Examples and counterexamples show that duals, minors, connected pregeometries, series-parallel networks, free geometries (on which many invariants achieve their upper bounds), and lower distributive pregeometries are all characterized by their polynomials. However, inequivalence, Whitney numbers, and representability are not always invariant.

Applying the decomposition to chain groups we generalize the classical two-color theorem for graphs to show when a geometry can be imbedded in binary affine space. The decomposition proves useful also for graphical pregeometries and for unimodular (orientable) pregeometries in the counting of cycles and co-boundaries.

1. Introduction. The present work is born from the application to combinatorial theory of some techniques that have been successfully used in the field of commutative algebra and algebraic topology [4]. In this introduction we shall attempt to outline the motivation, as well as the main results that we have obtained and possible future areas of exploration suggested by these results.

Our basic starting point is the notion of a combinatorial geometry or more generally a combinatorial pregeometry (matroid) for whose motivation we refer to [10]

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or [20]. Some of the basic definitions and elementary propositions from combinatorial geometries which we will need are summarized in the next section. Briefly, the concept of the combinatorial geometry abstracts the notion of linear dependence of sets of points in projective or affine space. As first pointed out by Whitney, this abstraction acts as a powerful unifying concept in a variety of combinatorial structures including graphs, totally unimodular matrices, and extremal combinatorial problems.

The study of such abstract projective configurations is only at its beginning. After the basic work of Whitney and Birkhoff, and some isolated contributions by Dilworth and Rado, the main results of the theory are due to Tutte, and pertain to the representation of abstractly given combinatorial geometries as concrete sets of points in projective space [19].

It would appear at first that the notions of linear algebra apply to the study of combinatorial geometries. It turns out, however, outside of certain very obvious and rather simple results, such as the notions of linear independence and bases and the pertaining result that all bases have the same number of elements etc., very little of linear algebra can be carried to arbitrary pregeometries without some a priori knowledge of a projective space into which the geometry can be embedded.

In a more general context, the study of commutative algebras and modules has led to a characterization of the trace of a matrix by the well-known work of Grothendieck. At present, one associates with every Abelian category a Grothendieck group (and sometimes a Grothendieck ring), which possesses all the information which is contained in the classical matrix case in the notion of trace and determinant.

This is precisely our starting point: by taking an abstract point of view, we carry over to combinatorial pregeometries constructions which resemble the Grothendieck group and ring [4]. We add at the outset however that this construction was first intuited in a profound paper by W. Tutte written in 1947 [18]; therefore, credit for the idea of a Grothendieck group and ring should be shared with Tutte.

Motivated by the way the classical Grothendieck group arises from the invariance of the trace of a linear operator on certain vector space decompositions, we are led to explore two decompositions which have as an invariant not only the classical characteristic polynomial but also another generating function which may be evaluated for any pregeometry to give its number of subsets, bases, independent sets, and spanning sets.

These two decompositions—direct sum and another reminiscent of subobject-quotient object decomposition—are compatible in a certain sense resembling the distributive compatibility of ring operations. It is exactly this compatibility in addition to the uniqueness of these two decompositions which allows us to construct a free commutative ring called the Tutte-Grothendieck ring and canonical map called the

Tutte polynomial which composes with ring homomorphisms to give a 1-1 correspondence between such homomorphisms and all functions which, like the characteristic polynomial, are invariant for the above decompositions. One makes this construction since it allows us to compute the Tutte polynomial which then serves as a universal invariant for the decomposition as well as to characterize all such invariant functions and to let knowledge of one invariant give information about another.

In §3 we show that the category of combinatorial pregeometries has the requisite properties for the construction of a Tutte-Grothendieck ring. Our main result here is that this ring is isomorphic to a polynomial ring over the integers in two variables. When geometries are considered instead of pregeometries, one obtains a ring in one variable, which embodies all the information contained in the classical characteristic polynomial. Finally, one may consider pregeometries with basepoints and in this case a polynomial in four variables results. Naturally, constructions associated with more general pairs (G, H) suggest themselves and we will study them in a future paper.

In §4 we study those invariants which can be obtained by evaluating the Tutte polynomial of a pregeometry at various values of the two variables. It is remarkable that several classical combinatorial invariants can be obtained very easily by such evaluations. A table is given at the end of the section. These include the Möbius function and an invariant recently discovered by Crapo as well as the ones alluded to previously. Our basic result allows us to obtain these and all such invariants as evaluations of the Tutte polynomial. By viewing Whitney duality and upper lattice truncation as functors in the decomposition category, it is shown that the Tutte polynomial also allows one to compute the chromatic polynomial of a truncated pregeometry as well as all the above invariants for the Whitney dual. In §5 we use the Higgs lift construction to prove that Tutte decompositions correspond to factored bijective unit rank decreasing strong maps thereby correlating our constructions with the categorical description of geometries in [10].

In §6 we make a detailed study of the polynomial associated with every pregeometry in the Tutte-Grothendieck ring, which turns out to coincide with the Tutte polynomial studied by Tutte in the case of graphs [18] and by Crapo generally [8]. We obtain a complete set of linear identities which are satisfied by all but a finite number of Tutte polynomials. We also relate the coefficients of the Tutte polynomials with the various classes of closed sets or flats of combinatorial geometries. Perhaps the most interesting result in this connection is the inequality we obtain relating the Tutte polynomial of a connected geometry with the Tutte polynomial of any of its minors, showing that the coefficients are always decreasing.

As examples of our theories in §7 we consider several classes of geometries which can be completely characterized and fruitfully studied in terms of their Tutte polynomials. They include series-parallel networks and partitions of type n . These

latter partitions play an important role as not only are they conjectured to predominate in an enumeration of n -point pregeometries, but also they are exactly those pregeometries whose Tutte coefficients give and are derivable from a set of doubly indexed Whitney numbers. We conjecture that the Tutte polynomial of a pregeometry G can be factored in the extended Tutte-Grothendieck ring if and only if G is separable. In light of Stanley's results about factorization of the characteristic polynomial relative to a modular flat ([16] or [3]) it becomes an interesting open question as to whether the Tutte polynomial of a connected (nonseparable) pregeometry is always prime.

In §8 we also show by simple counterexamples that the Tutte polynomial not only does not characterize a geometry up to isomorphism, but also cannot always give information about representability and Whitney numbers.

In §§9–12 we introduce the notion of hereditary classes of pregeometries as subdecompositions, and show that for certain of these hereditary classes special invariants can be obtained from the Tutte-Grothendieck ring. Perhaps the most remarkable examples are the relationships with the critical problem, as described by Crapo and Rota, and with the problem of coloring a graph. In this connection our main new result is a generalization of a classical two-color theorem to arbitrary binary pregeometries. An application of these methods of decomposition give Cayley's formula for the number of trees in a complete graph [5].

We conclude with a brief survey of possible application to unimodular pregeometries, which barely touches the subject and which we hope to take up in later publications.

In closing, we shall not hide the fact that our main motivation for study of the Tutte-Grothendieck ring has been and is the critical problem, which is the natural generalization of the problem of coloring a graph to a wider class of geometries. We hope that this work will begin a new trend in the study of the coloring problem, lifting it away from the classical purely graphical methods in which it has been enmeshed in the past, and bring it into the mainstream of contemporary algebra and combinatorial theory.

We would especially like to thank Professor Gian-Carlo Rota who first suggested that Tutte's techniques could be fruitfully generalized to pregeometries [15] as well as for his many suggestions and discussions on these methods. Also we are indebted to Professors Henry Crapo, Ladnor Geissinger, Curtis Greene, Robert Norman and the referee for their valuable advice.

2. Basic definitions. This section surveys the relevant notions of the underlying category for our work, \mathcal{G} , the category of finite *combinatorial pregeometries* and *strong maps* discussed in Crapo and Rota [10]. We then define the basic concepts in the theory of decompositions as explored in [4]. The reader is advised to read these basic definitions rapidly and refer to them as they come up in the paper.

A finite *pregeometry* or *matroid*, G , is a finite set of *points* with a closure operator $J_G(\cdot)$ or satisfying the *exchange property*: For any points $p, q \in G$ and any subset $P \subseteq G$, if $p \in P \cup \{q\}$ but $p \notin \overline{P}$, then $q \in \overline{P} \cup \{p\}$. When confusion might arise we denote by $\{G\}$ the points of G (without the closure structure). A *geometry* is a pregeometry in which the empty set and each point is closed. The lattice, L , of closed sets or *flats* of a pregeometry is called a *geometric lattice* and is characterized as a finite, semimodular, point lattice. Flats covering 0 in L are called *atoms*, and flats covered by 1 are called *coatoms* or *copoints*. A *bond* B of G is the set complement of a copoint. In such lattices, each lattice element x is the supremum of atoms representing closures of points and each has a well-defined *rank*, $r(x)$, equal to the length of any maximal chain from the 0 element (representing the closure of the empty set) to x . The semimodular law for L states that for all $x, y \in L$, $r(x) + r(y) \geq r(x \wedge y) + r(x \vee y)$. Flats x and y form a *modular pair* if the latter inequality is an equality; x is a *modular flat* if it forms a modular pair with every other flat. $r(A)$, the rank of a set of points $A \subseteq G$, is defined as $r(\overline{A})$ in the associated geometric lattice. Hence, $r(G)$, the *rank of the pregeometry* is $r(1)$ in the lattice.

For any subset A , the cardinality of A , $|A|$ denotes the number of points it contains, the *corank* of A is the nonnegative integer $r(G) - r(A)$, and the *nullity* of A is the nonnegative integer $|A| - r(A)$. A set of points $A \subseteq G$ represents a *spanning set* for G if $\overline{A} = G$. A set of points, A , is *independent* if $r(A) = |A|$. Otherwise, $r(A) < |A|$ and A is *dependent*. An independent spanning set is called a *basis*.

A strong map from a pregeometry G into H is a function $f: \{G\} \cup \{0\} \rightarrow \{H\} \cup \{0\}$ (where "0" stands for the empty set in G and H respectively) such that $f(0) = 0$ and the inverse image of any closed set in H is closed in G . Pregeometries G and H are *isomorphic* denoted $G \cong H$ if there is a 1-1 correspondence, f , between the points of G and H and the closed sets of G and H such that for any point p and closed set K , $p \in K$ iff $f(p) \in f(K)$. An *isomorphism class* or *equivalence class* of pregeometries denoted $[G]$ is the class of all pregeometries isomorphic to G .

A pregeometry on the point set $\{G\}$ can be uniquely determined by $\underline{C}(G)$, the family of minimal dependent sets or *circuits* of G . A family \underline{F} of subsets is the circuit set for some pregeometry if no subset in \underline{F} properly contains another and the subsets satisfy the *circuit elimination property* \underline{C}^* : If C_1 and C_2 are two distinct elements of \underline{F} and $e \in C_1 \cap C_2$ then the set difference $(C_1 \cup C_2) \setminus \{e\}$ is dependent and contains an element $C_3 \in \underline{F}$.

G may also be uniquely determined from its set of bases, $\underline{B}(G)$. A family \underline{F} of incomparable subsets is the set of bases for some pregeometry if \underline{F} satisfies the *basis exchange axiom* \underline{B}^* : For all B_1 and B_2 in \underline{F} and $p \in B_1$ there exists $q \in B_2$ such that $(B_1 \setminus \{p\}) \cup \{q\}$ is also in \underline{F} .

The (Whitney) *dual* of G , \tilde{G} , is the unique pregeometry on the same point set with a set of bases consisting of base complements of G . Hence $B \in \underline{B}(\tilde{G})$ iff $\{G\} \setminus B \in \underline{B}(G)$. A pregeometry is *self-dual* if $G \simeq \tilde{G}$.

G is the *direct sum* of two pregeometries: $G_1 \oplus G_2$ if the points of G , $\{G\}$; and circuits of G , $\underline{C}(G)$, are the disjoint union $\{G_1\} \cup \{G_2\}$ and $\underline{C}(G_1) \cup \underline{C}(G_2)$ respectively (equivalently $\underline{B}(G)$ is the set cartesian product $\underline{B}(G_1) \times \underline{B}(G_2)$). G_1 is then said to be a *direct sum factor* of G , and G is termed *separable* with the flats G_1 and G_2 in L as *separators*. If no such nontrivial *direct sum decomposition* exists, any two distinct points of G are contained in a circuit and G is termed *connected*. A one point direct sum factor, p , is an *isthmus* if it is in no circuits of G and a *loop* if it is itself a circuit. A point is a *nonfactor* if it is neither a loop nor an isthmus. A *pre-Boolean algebra* B_{nm} is a pregeometry which is the direct sum of n isthmuses and m loops. A *Boolean algebra* B_n is a geometry which is the direct sum of n isthmuses.

If $p \in G$ we define two derived pregeometries on the point set $\{G\} \setminus \{p\}$: the *deletion*, $G \setminus p$; and the *contraction*, G/p . If $A \subseteq \{G\} \setminus \{p\}$, and \bar{A} denotes its closure in G : then the closure on A in $G \setminus p$ is defined as $\bar{A} \setminus \{p\}$ while its closure in G/p is defined as $\overline{A \cup \{p\}} \setminus \{p\}$. If $D \subseteq G$, the *subgeometry* $G \setminus D$ is defined as a sequence of deletions by points in D . Similarly we define the contraction G/D as a sequence of contractions. An arbitrary sequence of contractions and deletions is called a *minor*.

An *invariant* is a function f defined on the class of all pregeometries such that $f(G) = f(H)$ if $G \simeq H$. Examples of invariants used in this paper include $\underline{c}(G)$, the *complexity* or number of bases of G ; $\underline{I}(G)$, the number of independent sets; the *subgeometry generating function* $S(G) = \sum_{i,j} a_{ij} u^i v^j$ where a_{ij} is the number of subgeometries (subsets) of G with corank i and nullity j ; and $\mu(G)$, the *Möbius function* which is defined as $\mu(0, 1)$ evaluated on the geometric lattice L associated with G , where for $x \leq y$, $\mu(x, y)$ is given by the recursion: $\mu(x, x) = 1$, $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$. Two other invariants evaluated on L are the *characteristic polynomial*, $\chi(G, \lambda) = \chi(G) = \sum_{x \in L} \mu(0, x) \lambda^{r(1)-r(x)}$; and $\beta(G)$, the *Crapo invariant* which is explored in [5] with distinguishing properties: $\beta(G) = \beta(G \setminus e) + \beta(G/e)$ if $e \in G$ is neither an isthmus nor a loop, $\beta(G) \geq 0$, and $\beta(G) = 0$ iff G is separable.

A *decomposition* $\underline{D}(S)$ of a set S is a rule whereby one may decompose an element s in S into a (finite) multisubset (subset with repetitions) M of S denoted $s \leq M$. A decomposition is *unique* if every $s \leq S$ can be decomposed into a unique multiset of *irreducible* elements (elements which cannot be further decomposed). A *decomposition invariant* function f with domain S takes values in abelian group and is such that if s decomposes into a multiset M then $f(s) = \sum_{m \in M} f(m)$. If two decompositions \underline{D}_1 and \underline{D}_2 (compatibly) decompose the same set they form a *bidecomposition*. This bidecomposition is *distributive* if (1) both \underline{D}_1 and \underline{D}_2 are

unique; (2) any element which decomposes in \underline{D}_1 into a multiset of irreducible elements is itself irreducible (for \underline{D}_2); and (3) \underline{D}_2 decomposition commutes with \underline{D}_1 decomposition in a manner resembling general distributivity for addition and multiplication respectively. An element which cannot be decomposed in \underline{D}_1 is termed *indecomposable* and hence an element which cannot be decomposed for either decomposition is an *irreducible indecomposable*. A *bidecomposition invariant* function f is a function into a commutative ring which is invariant for \underline{D}_2 and also is such that for every \underline{D}_1 decomposition of s into a multiset M' , $f(s) = \prod_{m \in M'} f(m)$.

Associated with a decomposition is its *Tutte-Grothendieck group* A which along with a function mapping S into A is universal for decomposition invariant functions. A is isomorphic to the free abelian group on S factored by the subgroup generated by elements $s - \sum_{m \in M} m$ for any decomposition of s into M . Associated with a bidecomposition is its *Tutte-Grothendieck ring* R which is the free commutative ring on S factored by the ideal generated as above by all (multiplicative) \underline{D}_1 decompositions and (additive) \underline{D}_2 decompositions. When R turns out to be a free commutative ring the image of an element s under the natural map t from S into R is called the *Tutte polynomial* of s .

3. The Tutte bidecomposition.

Definition 3.1 On S , the set of isomorphism classes of finite pregeometries, let $\underline{D}_1(S)$ be direct sum decomposition and let \underline{D}_2 , the *Tutte decomposition*, be generated by decompositions of the form $G \leq G \setminus e + G/e$ for all nonfactors $e \in G$. Then since both \underline{D}_1 and \underline{D}_2 decompose pregeometries into those of smaller cardinality, the two decompositions form a bidecomposition \underline{T} called the *Tutte bidecomposition*.

Definition 3.2. Let Q be the class of polynomials whose variables are pregeometries and for a point p , let $D_p: Q \rightarrow Q$ be the partial function whose domain is all polynomials q in which each term contains exactly one factor F_j such that $p \in F_j$; and for such polynomials $q = \sum_j \prod_i G_{ij} F_j$, let $D_p(q) = \sum_j \prod_i G_{ij} F'_j$ where

$$F'_j = \begin{cases} (p)(F_j \setminus p) & \text{if } p \text{ is an isthmus or loop,} \\ (F_j \setminus p) + (F_j / p) & \text{otherwise.} \end{cases}$$

For any G and ordering $O = (p_1, \dots, p_n)$ on the points of G let $D_O(G) = D_{p_n} \circ \dots \circ D_{p_1}(G)$ which is well defined since D_{p_i} is defined for all i on each term of the polynomial $D_{p_{i-1}} \circ \dots \circ D_{p_1}(G)$. If $q = \sum \prod_i G_{ij}$ and $q' = \sum \prod_i G'_{ij}$, define an equivalence relation \sim on Q such that $q \sim q'$ if for some ordering of the terms, the pregeometries G_{ij} and G'_{ij} are isomorphic. Then Q/\sim is the set of all polynomials of equivalence classes of pregeometries. Denote by $[D_O(G)]$ the equivalence class of the polynomial $D_O(G)$ and denote by x and z the equivalence classes of

a loop and isthmus respectively.

Lemma 3.3. *For all pregeometries G and orderings O and O' on its points*

- (a) $[D_O(G)]$ is a polynomial with positive integer coefficients in z and x ; and
- (b) $[D_O(G)] = [D_{O'}(G)]$.

Proof. (a) Using induction on the number of points of G , we note that a one point pregeometry is necessarily an isthmus or loop and (a) holds in this case.

Assume (a) for all orderings of cardinality $n - 1$ and consider $D_O(G)$ for $O = (p_1, \dots, p_n)$. Then if $O' = (p_2, \dots, p_n)$, $D_O(G) = D_{O''}(D_{p_1}(G)) = D_{O''}((G \setminus p_1)(p_1))$ or $D_{O''}(G \setminus p_1 + G/p_1)$. But by the hypothesis $[D_{O''}(G \setminus p_1)]$ and $[D_{O''}(G/p_1)]$ are both positive polynomials in z and x and the result follows for $[D_O(G)]$ by multiplication by z or x or polynomial addition respectively.

(b). We can effect any reordering by interchanging consecutive points so we need only show that $D_O(G) = D_{O'}(G)$ for $O = (p_1, \dots, p_k, p_{k+1}, \dots, p_n)$ and $O' = (p_1, \dots, p_{k+1}, p_k, \dots, p_n)$. By the definition of the operator D_O , we need only check that $D_{p_k} \circ D_{p_{k+1}}(G_{ij}) = D_{p_{k+1}} \circ D_{p_k}(G_{ij})$ on the relevant variables G_{ij} of $D_{p_{k-1}} \circ \dots \circ D_{p_1}(G)$.

Let G_{ij} be any pregeometry containing p_k and p_{k+1} . There are three cases to consider:

Case I. If p_k is a nonfactor of G_{ij} , and p_{k+1} is a nonfactor of both pregeometries $G_{ij} \setminus p_k$ and G_{ij}/p_k ; then neither p_k nor p_{k+1} is an isthmus or loop of G_{ij} , and $\{p_k, p_{k+1}\}$ is not a two point circuit in either G_{ij} or \tilde{G}_{ij} . Hence, p_{k+1} is a nonfactor of G_{ij} and p_k is a nonfactor of both $G_{ij} \setminus p_{k+1}$ and G_{ij}/p_{k+1} . Considering the set of bases we have $(G_{ij} \setminus p_k) \setminus p_{k+1} = (G_{ij} \setminus p_{k+1}) \setminus p_k$; $(G_{ij} \setminus p_k)/p_{k+1} = (G_{ij}/p_{k+1}) \setminus p_k$; $(G_{ij}/p_k)/p_{k+1} = (G_{ij}/p_{k+1})/p_k$; and $(G_{ij}/p_k) \setminus p_{k+1} = (G_{ij} \setminus p_{k+1})/p_k$. For example, both sides of the last identity define the pregeometry P on the point set $\{G_{ij} \setminus (p_k \cup p_{k+1})\}$ with bases those subsets of P which when adjoined with p_k form a basis for G_{ij} .

Case II. If p_k is a loop (isthmus) of G_{ij} , then it is a loop (isthmus) of $G_{ij} \setminus p_{k+1}$ and G_{ij}/p_{k+1} . In any event,

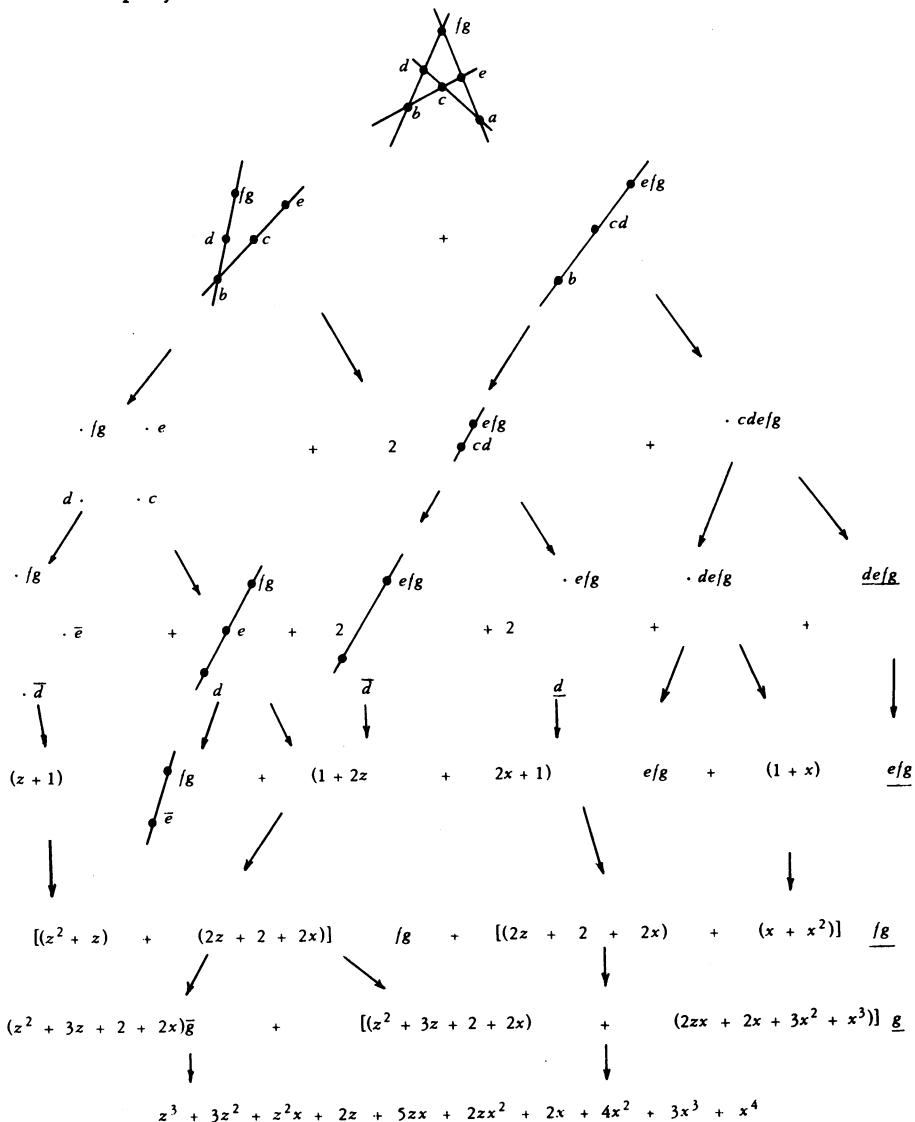
$$((D_{p_{k+1}}(G_{ij})) \setminus p_k)(p_k) = D_{p_{k+1}}((G_{ij} \setminus p_k)(p_k)).$$

Case III. If p_k and p_{k+1} are both nonfactors of G_{ij} but p_{k+1} is a loop of G_{ij}/p_k (p_{k+1} is an isthmus of $G_{ij} \setminus p_k$), then $\{p_{k+1}, p_k\}$ forms a circuit in G_{ij} (in \tilde{G}_{ij}). But this means that in $G_{ij}(G_{ij})$, $\bar{p}_k = \bar{p}_{k+1}$ and there is a strong map automorphism of G_{ij} which interchanges p_k and p_{k+1} . Clearly then $[D_O(G)] = [D_{O'}(G)]$.

Since these are all the possible cases, we may denote $[D_O(G)]$ unambiguously by $D(G)$, the *Tutte polynomial* of G .

Example 3.4. We illustrate the computation of a Tutte polynomial: $D_O(P'_4)$

where P'_4 is a planar pregeometry of seven points, two in the same atom, whose geometric lattice is the lattice of partitions of a four element set. It is decomposed relative to the ordering $O = (a, b, c, d, e, f, g)$. Points and lines represent closed sets while \overline{p} indicates an isthmus, \underline{p} a loop. Arrows descending to the left represent deletions; those to the right represent contractions; and vertical arrows show factorization. Note that in the third row two isomorphic pregeometries are identified to simplify the calculation.



Theorem 3.5. *The Tutte bidecomposition is distributive (these terms are defined in §2) and the equivalence classes of an isthmus and a loop are the irreducible indecomposables.*

Proof. The decomposition \underline{D}_1 (direct sum decomposition) is unique with connected pregeometries as the indecomposables [10]. The irreducible elements of \underline{D}_2 are clearly pre-Boolean algebras (those pregeometries made up of entirely isthmuses and loops: $B_{nm} = \bigoplus_{i=1}^n z \oplus \bigoplus_{j=1}^m x$). The irreducible indecomposables are then either x or z and if an element has a \underline{D}_1 decomposition into these elements it is of the form B_{nm} and is (\underline{D}_2) indecomposable. Further, $\underline{D}_2(S)$ is a unique decomposition since we may sharpen the proof of (3.3) to show that $[D_O(G)]$ is invariant under rearrangements in the orderings of the partial terms of the decomposition. (For example, $D_O(G) = D_{p_n} \circ \dots \circ D_{p_2}((G \setminus p_1) + (G/p_1)) = D_{p_n} \circ \dots \circ D_{p_2}(G \setminus p_1) + D_{p_n} \circ \dots \circ D_{p_4} \circ D_{p_2} \circ D_{p_3}(G/p_1)$.) But \underline{D}_2 decomposition corresponds to partial rearrangement in which at all stages of the decomposition points which are neither isthmuses nor loops precede points which are. So as in the proof of (3.3) full \underline{D}_2 -decomposition is unique and equals the Tutte polynomial $D(G)$ with $z^n x^m$ replaced by B_{nm} .

We will be done when we show distributivity for a basic decomposition $G = G_1 \oplus G_2$ of \underline{D}_1 and a basic decomposition $G_1 \leq (G_1 \setminus e) + (G_1/e)$ of \underline{D}_2 . But if e is a nonfactor of G_1 then it is a nonfactor of $G = G_1 \oplus G_2$ since direct sum decomposition is unique, so \underline{D}_2 contains the decomposition $G \leq (G \setminus e) + (G/e)$. Also, \underline{D}_1 contains the decompositions $G \setminus e = (G_1 \setminus e) \oplus G_2$ and $G/e = (G_1/e) \oplus G_2$ since the closure operator of a direct sum works componentwise, and hence for all $A_1 \subseteq \{G_1\} \setminus \{e\}$ and $A_2 \subseteq \{G_2\}$,

$$J_{[G \setminus e]}(A_1 \cup A_2) = \overline{A_1 \cup A_2} \setminus e = (\overline{A_1} \setminus \{e\}) \cup \overline{A_2} = J_{[(G_1 \setminus e) \oplus G_2]}(A_1 \cup A_2);$$

and

$$\begin{aligned} J_{[G/e]}(A_1 \cup A_2) &= \overline{A_1 \cup A_2 \cup \{e\}} \setminus \{e\} \\ &= (\overline{A_1 \cup \{e\}} \setminus \{e\}) \cup \overline{A_2} = J_{[(G_1/e) \oplus G_2]}(A_1 \cup A_2). \end{aligned}$$

Theorem 3.6. *There is a Tutte-Grothendieck ring (R, t) for pregeometries associated with the Tutte bidecomposition, \underline{T} . It is isomorphic to $P[z, x]$, the ring of polynomials in two variables without constant term over the integers (where the variable z represents the coset of an isthmus and x represents a loop).*

Further, if S is the set of isomorphism classes of pregeometries and $t: S \rightarrow R$ sends a pregeometry to its coset representation in R , then $t(G) = D(G)$, the Tutte polynomial of G defined in (3.3).

Proof. The first part of the theorem follows from (3.5) and the results in [4] where it is shown that the universal Tutte-Grothendieck ring $T = F[S]/I$ for a distributive bidecomposition is isomorphic to a free commutative ring with basis the irreducible indecomposable elements of the bidecomposition. It is further shown that the image of the universal invariant t which maps a pregeometry G into the

Tutte-Grothendieck ring is equal to the polynomial corresponding to G 's (unique) decomposition first into (the additively written multiset of) D_2 -indecomposable elements and then for each indecomposable its respective decomposition in D_1 into (its multiplicatively written multiset of) indecomposable irreducible elements.

The second paragraph of the theorem is a consequence of the fact that the point decomposition operator D_{p_i} corresponds to a decomposition of \underline{T} and in the free ring $F[S]$, $G \equiv D(G) \pmod{I}$ since the ideal I is generated by all identities which decompose G into $D(G)$.

We will henceforth refer to the above ring for pregeometries as the *Tutte-Grothendieck ring*. We also have (for pregeometries) the *Tutte-Grothendieck group* associated with the Tutte decomposition \underline{D}_2 of \underline{T} .

Corollary 3.7. *The Tutte-Grothendieck group (A, t_A) is isomorphic to the free abelian group generated by pre-Boolean algebras. In addition $t_A(G) = D(G)|_{z^n x^m \rightarrow B_{nm}}$.*

Proof. A theorem in [4] states that A is group isomorphic to R if \underline{D}_1 decomposition is bijective between irreducible elements of \underline{D}_2 and multisets of irreducible indecomposable elements. The corollary then follows from (3.6).

Definition 3.8. Let U be the functor which takes an equivalence class of pregeometries G (of rank greater than 0) to the equivalence class of its underlying geometry \overline{G} (i.e. the geometry with the same lattice of closed sets). Then the Tutte bidecomposition \underline{T} induces a decomposition \overline{T} on $\overline{\mathcal{S}}$, the set of equivalence classes of geometries, called the *characteristic bidecomposition*. The basic decompositions of \overline{T} are $G_1 \oplus G_2 \leq G_1 \times G_2$ and $G \leq G \setminus e + G/e$ for all geometries G_1, G_2 and G , and each nonisthmus $e \in G$.

Theorem 3.9. *On the set $\overline{\mathcal{S}}$ of nonempty geometries, the characteristic bidecomposition is distributive and its associated ring $(\overline{R}, \overline{t})$ called the characteristic ring is isomorphic to $P[z]$, the ring of polynomials in one variable over the integers. If $t(G)$ is the Tutte polynomial of G , then $\overline{t}(G) = t(G)|_{x=0}$.*

Proof. The pair $(\overline{R}, \overline{t} \circ U)$ is \underline{T} -invariant, where \overline{R} is the characteristic ring and $\overline{t} \circ U$ sends a pregeometry G to the image in \overline{R} under \overline{t} of \overline{G} , its associated geometry (or 0 if \overline{G} is empty). Hence by the universality of R and (3.6) there is a ring homomorphism $b: R \rightarrow \overline{R}$ where $b(z) = z$ and $b(x) = 0$, since an isthmus is a geometry and the empty geometry underlies a loop. The rest of the theorem now follows.

Corollary 3.10. *The additively written decompositions of \overline{T} form a unique decomposition called the characteristic decomposition. The associated group \overline{A} is the free abelian group generated by Boolean algebras B_n , while $\overline{t}_A = \overline{t}_R|_{z^n \rightarrow B_n}$.*

Proof. Apply the arguments of (3.7) to (3.9).

Proposition 3.11. Let f be a \underline{T} -invariant with associated ring homomorphism b . Then a necessary and sufficient condition for f when restricted to geometries to be a $\overline{\underline{T}}$ -invariant is that $f(x) = 0$ (or equivalently $b(x) = 0$). Similarly a function which is group invariant for the Tutte decomposition is invariant for the characteristic decomposition iff the associated group homomorphism b has its support on Boolean algebras.

Conversely a function \overline{g} which is a $\overline{\underline{T}}$ -invariant can be extended uniquely to a \underline{T} -invariant g by defining

$$g(G) = \begin{cases} 0 & \text{if } G \text{ has any loops,} \\ \overline{g}(G) & \text{otherwise,} \end{cases}$$

with similar extensions for group invariants.

Proof. We will prove the proposition for ring invariants. Proofs for group invariants are entirely analogous.

If $b(x) = 0$ then, since $t(G) = t(G)|_{x=0}$, $f(G) = b' \circ \overline{t}(G)$ for all geometries G where b' is the restriction of b to polynomials in the variable z . Hence by (3.9) f is a $\overline{\underline{T}}$ -invariant.

Conversely, if f is both a $\overline{\underline{T}}$ -invariant and a T -invariant on geometries, then if z represents an isthmus, x a loop, and L_3 the three point line: $f(L_3) = (f(z))^2 + f(z) = (f(z))^2 + f(z) + f(x)$. Hence $f(x) = 0$.

If \overline{g} is a $\overline{\underline{T}}$ -invariant which is to be extended to a $\overline{\underline{T}}$ -invariant g then by the above, $g(G) = 0$ if G has any loops. Otherwise, proceeding by induction on the number n of points of G minus the number of atoms ($|G| - |\overline{G}|$) if $n = 0$ then $G = \overline{G}$ and certainly $g(G)$ must equal $g(\overline{G})$. But if $n > 0$ and a point p is in the closure of another point q then G/p has a loop and must be equal to 0 while $\overline{G} \setminus p = \overline{G}$ and hence g is an invariant if we define $g(G) = g(G \setminus p) + g(G/p) = g(\overline{G}) + 0$.

An application of the characteristic decomposition to the study of modular flats can be found in [3].

The category \underline{G}' of pointed pregeometries (F, p) and strong maps $f: (F, p) \rightarrow (F', p')$ such that $f(p) = p'$ and $f^{-1}(p') = p$ was defined in [2]. Three functors were also defined from \underline{G}' to \underline{G} , the category of pregeometries and strong maps $T((F, p)) = F$, $T'((F, p)) = F \setminus p$, and $T''((F, p)) = F/p$.

This category provides a natural setting for the study of series-parallel networks and the Tutte bidecomposition of such point pregeometries facilitates the calculation of some important network invariants as shown in [2].

Definition 3.12. Let S' be the set of equivalence classes of pregeometries S and pointed pregeometries (where no pregeometry is equivalent to a pointed pregeometry). The *pointed Tutte bidecomposition* $T'(S')$ includes the decompositions of the Tutte subbidecomposition $T(S)$ as well as basic decompositions of the form

$(F, p) \leq ((F, p) \setminus e) + ((F, p)/e) \in \underline{D}_2'$ and $(F, p) \oplus G \leq (F, p) \times G \in \underline{D}_1'$ for all pointed pregeometries (F, p) and pregeometries G , and for all nonfactors $e \in \{F\} \setminus \{p\}$ where $(F, p) \oplus G$ is the pointed pregeometry $(F \oplus G, p)$ and $(F, p) \setminus e$ $((F, p)/e)$ is the pointed pregeometry $(F \setminus e, p) ((F/e, p))$. A function invariant for this decomposition is termed a \underline{T}' -invariant. Examples of \underline{T}' -invariants include $f \circ T$ where T is the forgetful functor which sends a pointed pregeometry (F, p) to its underlying pregeometry F and f is a \underline{T} -invariant.

Theorem 3.13. *The pointed Tutte bidecomposition \underline{T}' is distributive and with it we may associate the pointed Tutte-Grothendieck ring (R', t') where $R'[z, x, z', x']$ is isomorphic to the polynomial ring over the integers in four variables corresponding to the equivalence classes of an isthmus, loop, pointed isthmus, and pointed loop respectively. Further, the pointed Tutte polynomial $t'(F)$ equals $z'f_1(z, x) + x'f_2(z, x)$ for all pointed pregeometries F , and $t'(G) = f(z, x)$ for all nonpointed pregeometries G .*

Proof. These statements are all proved in [2].

The following theorem summarizes some applications of the pointed Tutte-Grothendieck ring to the theory of series-parallel networks. All proofs can be found in [2].

Theorem 3.14. *If (F, p) is a pointed pregeometry and T is the forgetful functor: $T((F, p)) = F$, then by universality of R' , $(t \circ T)((F, p)) = zf_1 + xf_2$ where the pointed Tutte polynomial of (F, p) , $t'((F, p))$, equals $z'f_1 + x'f_2$. The Tutte polynomial also commutes with the deletion and contraction functors: $t(F \setminus p) = t(F/p) = f_1 + f_2$ if p is a factor of (F, p) while if p is a nonfactor, $t(F \setminus p) = (z - 1)f_1 + f_2$ and $t(F/p) = f_1 + (x - 1)f_2$. In the category of pointed pregeometries there is a direct sum called the parallel connection, $P(G, H)$. This corresponds to the graph theoretic concept when G and H are graphical. In addition one may use Whitney duality to define the series connection, $S(G, H) = P(\hat{G}, \hat{H})$. If p is a nonfactor of G and $t'(G) = z'g_1 + x'g_2$ while $t(H) = z'b_1 + x'b_2$, then*

$$t'(S(G, H)) = z'[(z - 1)g_1b_1 + g_1b_2 + g_2b_1] + x'g_2b_2$$

and

$$t'(P(G, H)) = z'g_1b_1 + x'[(x - 1)g_2b_2 + g_1b_2 + g_2b_1].$$

4. Decomposition invariants. In this section we explore the Tutte bidecomposition \underline{T} in more detail and give examples of some useful functions which are invariant on T .

Proposition 4.1. (a) *For all pregeometries G and points $e \in G$, if A is a subset of $\{G\} \setminus \{e\}$, then the rank $r(A)$ of A in the associated geometric lattice $L(G)$*

is the same as the rank $r'(A)$ computed in the lattice $L(G \setminus e)$ associated with the deletion of e .

(b) There is a lattice isomorphism from the lattice of the contraction $L(G/e)$ into $L(G)$, $f: [0, G/e] \cong [e, G]$ where for all closed sets x in G/e , $f(x) = x \cup e$, a closed set of G . Hence if $e \in A \subseteq G$, then $r(A)$ computed in $L(G)$ is equal to $r''(A \setminus \{e\}) + 1$ computed in $L(G/e)$ if e is not a loop.

(c) A point e of G is therefore an isthmus iff $r(G) = r(\{G\} \setminus \{e\}) + 1 = r'(G \setminus e) + 1$. Otherwise $r(G) = r'(G \setminus e)$. Also $e \in G$ is a loop iff $r(G) = r''(G/e)$ and $r(G) = r''(G/e) + 1$ otherwise.

Proof. (a) Since for subsets A of $G \setminus e$, the closure of A is $\overline{A \setminus \{e\}}$, the lattice of closed sets of $G \setminus e$, $L(G \setminus e)$ can be formed by taking suprema (in G) of points in $\{G\} \setminus \{e\}$. Hence on these subsets, rank is invariant in G and $G \setminus e$. But $e \in G$ is an isthmus iff it is a bond iff $\{G\} \setminus \{e\}$ is a copoint of G .

(b) The closure operator J in G/e for the set \underline{A} of subsets A of $\{G\} \setminus \{e\}$ is $\underline{A} \cup \{e\} \setminus \{e\}$ and so is isomorphic to the closure operator J' in G for the set \underline{A}' , subsets A' of G containing e by the isomorphism $f(A) = A \cup \{e\}$, $f^{-1}(A') = A' \setminus \{e\}$ in the following commutative diagram.

$$\begin{array}{ccc} \underline{A} & \xrightarrow{J} & \underline{A} \\ f \downarrow \wr & & \downarrow \wr f \\ \underline{A}' & \xrightarrow{J'} & \underline{A}' \end{array}$$

(c) A point $e \in G$ is an isthmus iff e is a bond iff $\{G\} \setminus \{e\}$ is a copoint of G iff $r(G) = r(\{G\} \setminus \{e\}) + 1 = r'(G \setminus e) + 1$ (since rank is invariant in G and $G \setminus e$ on subsets of $\{G\} \setminus \{e\}$). Otherwise $G \setminus e = G$ and $r'(G \setminus e) = r(\{G\} \setminus \{e\}) = r(G)$.

The rank of G/e equals by (b), the length of a maximal chain in $L(G)$ between e and G . This chain has length $r(G) - 1$ or $r(G)$ depending on whether e is (in) an atom or in the closure of the empty set (i.e. a loop) respectively.

Theorem 4.2. Let $\underline{K}(G)$ denote the closed sets of G and let

$$\chi(G, \lambda) = \chi(G) = \sum_{x \in \underline{K}(G)} \mu(0, x) \lambda^{r(G) - r(x)}$$

be the characteristic polynomial of G . Then the invariant defined by

$$\begin{cases} 0 & \text{if } G \text{ has a loop,} \\ (-1)^{r(G)} \chi(G) & \text{otherwise,} \end{cases}$$

is an invariant of the Tutte bidecomposition \underline{T} . (From now on when we apply the geometric invariant χ to pregeometries we will define $\chi(G)$ to be 0 if G has a loop.)

Proof.

$$\begin{aligned}
 & (-1)^{r(G_1 \oplus G_2)} \chi(G_1 \oplus G_2) \\
 &= (-1)^{r(G_1 \oplus G_2)} \sum_{x \in \underline{K}(G_1 \oplus G_2)} \mu(0, x) \lambda^{r(G_1 \oplus G_2) - r(x)} \\
 &= (-1)^{r(G_1) + r(G_2)} \sum_{x' \in \underline{K}(G_1)} \sum_{x'' \in \underline{K}(G_2)} \mu(0, x' \vee x'') \lambda^{r(G_1) + r(G_2) - r(x') - r(x'')} \\
 &= \left[(-1)^{r(G_1)} \sum_{x' \in \underline{K}(G_1)} \mu(0, x') \lambda^{r(G_1) - r(x')} \right] \\
 &\quad \cdot \left[(-1)^{r(G_2)} \sum_{x'' \in \underline{K}(G_2)} \mu(0, x'') \lambda^{r(G_2) - r(x'')} \right] \\
 &= [(-1)^{r(G_1)} \chi(G_1)] [(-1)^{r(G_2)} \chi(G_2)]
 \end{aligned}$$

since in lattice cartesian products, $\mu(0, x' \vee x) = \mu(0, x')\mu(0, x'')$ [14].

By (3.11) and noting that $\chi(G)$ depends only on the geometric lattice we need only show that the theorem holds for geometries. In that case using spanning sets y of atoms for the subgeometry x defined by all closed sets $x \in \underline{K}(G)$, we obtain by order dualizing Proposition 5.1 from [14] the formula $\mu(0, x) = \sum_{y \subseteq x} (-1)^{|y|}$. Hence if $e \in G$ is not an isthmus:

$$(-1)^{r(G)} \chi(G) = (-1)^{r(G)} \sum_{y \subseteq G} (-1)^{|y|} \lambda^{r(G) - r(y)}$$

(since the closed sets $x \in \underline{K}(G)$ partition subsets of G into spanning subsets for x). But summing separately over those subsets which do not contain e and those which do, the formula for the characteristic polynomial becomes

$$\begin{aligned}
 & (-1)^{r(G)} \sum_{e \notin y} (-1)^{|y|} \lambda^{r(G) - r(y)} + (-1)^{r(G)} \sum_{e \in y} (-1)^{|y|} \lambda^{r(G) - r(y)} \\
 &= (-1)^{r'(G \setminus e)} \sum_{y \subseteq G \setminus e} (-1)^{|y|} \lambda^{r'(G \setminus e) - r'(y)} \\
 &\quad - (-1)^{r''(G/e)} \sum_{y' \subseteq G/e} -(-1)^{|y'|} \lambda^{r''(G/e) - r''(y')} \\
 &= (-1)^{r'(G \setminus e)} \chi(G \setminus e) + (-1)^{r''(G/e)} \chi(G/e)
 \end{aligned}$$

using the results and isomorphism f of (4.1).

Corollary 4.3. *For all pregeometries G , if the Tutte polynomial $t(G)$ equals*

$f(z, x)$, then $\chi(G) = (-1)^{r(G)} f(1 - \lambda, 0)$. In particular, $\mu(G) = (-1)^{r(G)} f(1, 0)$.

Proof. By (3.6) and (4.2), we need only compute $(-1)^{r(G)} \chi(G)$ for an isthmus z and a loop x . But $(-1)^{r(z)} \chi(z) = (-1)(\lambda - 1)$ while $(-1)^{r(x)} \chi(x) = 0$. Hence $f(1 - \lambda, 0) = (-1)^{r(G)} \chi(G)$. In particular $\mu(G) = \mu(0, G) = \chi(G)|_{\lambda=0} = (-1)^{r(G)} f(1 - \lambda, 0)|_{\lambda=0}$.

It is the above universal description of χ which allows one to show in [3] that $\chi(x)$ divides $\chi(G)$ for any geometry G and modular flat x of G .

Corollary 4.4. A function f is an invariant (of both \underline{T} and \overline{T}) if and only if it is an evaluation of $\chi(G)$.

Proof. By (3.11), f is invariant under \overline{T} and \underline{T} iff $f(x) = 0$ iff f is an evaluation of $t(G)|_{x=0}$ iff f is an evaluation of $t(G)|_{x=0, z=1-\lambda}$.

Theorem 4.5. The subgeometry generating function, $S(G) = \sum a_{ij} u^i v^j$, is a \underline{T} -invariant where a_{ij} is the number of subgeometries of G with corank i and nullity j (hence if $r(G) = n$, $a_{n-k, m-k}$ counts the number of subgeometries of G with rank k and cardinality m).

Proof. If $G = G_1 \oplus G_2$, the rank of any subset of G is the sum of the ranks of the subset when intersected with the two sets $\{G_1\}$ and $\{G_2\}$. This holds in particular for G , hence it holds for corank as well as cardinality and hence nullity. Thus

$$\begin{aligned} S(G_1 \oplus G_2) &= \sum a_{ij} u^i v^j = \sum_{A \subseteq G_1 \oplus G_2} u^{r(G_1 \oplus G_2) - r(A)} v^{|A| - r(A)} \\ &= \sum_{A \subseteq (G_1 \cup G_2)} u^{r(G_1) + r(G_2) - r(A \cap G_1) - r(A \cap G_2)} v^{|A \cap G_1| + |A \cap G_2| - r(A \cap G_1) - r(A \cap G_2)} \\ &= \sum_{A' \subseteq G_1} u^{r(G_1) - r(A')} v^{|A'| - r(A')} \sum_{A'' \subseteq G_2} u^{r(G_2) - r(A'')} v^{|A''| - r(A'')} \\ &= S(G_1)S(G_2). \end{aligned}$$

If $e \in G$ is neither an isthmus nor a loop

$$\begin{aligned} S(G) &= \sum_{A \subseteq G} u^{r(G) - r(A)} v^{|A| - r(A)} \\ &= \sum_{A \subseteq (G \setminus e)} u^{r(G \setminus e) - r(A)} v^{|A| - r(A)} + \sum_{A; e \in A} u^{r''(G/e) - r(A) - 1} v^{|A \setminus e| - r(A) - 1}. \end{aligned}$$

But the sinister is equal to $S(G \setminus e)$ while the isomorphisms f^{-1} from $[e, G]$ to $[0, G/e]$ shows that the dexter is equal to $S(G/e)$.

Corollary 4.6. *For all pregeometries G , if the Tutte polynomial $t(G)$ equals $f(z, x)$, then $S(G) = f(u + 1, v + 1)$. In particular the number of spanning sets of G equals $f(1, 2)$; the number of independent sets is $f(2, 1)$; the number of bases equals $f(1, 1)$; and the number of subsets equals $f(2, 2)$. More generally, if n is the rank of G , the number of independent sets of cardinality k is given by replacing $z^i x^j$ by $\binom{i}{n-k}$ in $t(G)$, and the number of spanning sets of cardinality k by replacing $z^i x^j$ by $\binom{j}{k-n}$.*

Proof. By (3.6) and (4.5) we need only compute $S(G)$ for an isthmus z and a loop x . But $S(z) = u + 1$ (both subgeometries have nullity 0) while $S(x) = v + 1$ (both subgeometries have rank and corank 0). Hence $s(G) = f(u + 1, v + 1)$. In particular $f(1, 2) = S(G)|_{u=0, v=1} = \sum_j a_{0,j}$ = the number of subgeometries with corank 0, i.e. the spanning sets; $f(2, 1) = S(G)|_{u=1, v=0} = \sum_i a_{i,0}$ = the number of subgeometries with nullity 0, i.e. the independent sets; $f(1, 1) = a_{0,0}$, the bases; while $f(2, 2) = \sum a_{ij}$, the number of subsets ($2^{|G|}$).

Some further applications of the previous theorems can be found in [2] where, in conjunction with (3.14), the invariants of (4.2) and (4.5) are computed across a series and a parallel connection. As a corollary to these results, it is shown that a series-parallel network (a pregeometry which contains neither a four point line nor P_4 , the geometry of partitions of a four element set, as minors) is three colorable.

Certain operators on pregeometries are functors from the Tutte decomposition into itself (i.e. preserve decompositions) and lead to important invariants.

Theorem 4.7. *Letting \tilde{G} denote the (Whitney) dual of the pregeometry G , then if $\tilde{t}(G) = t(\tilde{G})$, the Tutte polynomial of the dual pregeometry, then \tilde{t} is a \underline{T} -invariant. In addition, if $f(z, x)$ is the Tutte polynomial of G , then $f(x, z)$ is the Tutte polynomial of \tilde{G} .*

Proof. If $G = G_1 \oplus G_2$ then the set of bases of G , $\underline{B}(G)$ is the cartesian product $\underline{B}(G_1) \times \underline{B}(G_2)$. But for all subsets $B_1 \subseteq G_1$ and $B_2 \subseteq G_2$: $B_1 \cup B_2 \in \underline{B}(G)$ iff $G \setminus (B_1 \cup B_2) \in \underline{B}(G)$ iff $(G_1 \setminus B_1) \cup (G_2 \setminus B_2) \in \underline{B}(G)$ iff $\tilde{G}_1 \setminus \tilde{B}_1 \in \underline{B}(\tilde{G}_1)$ and $\tilde{G}_2 \setminus \tilde{B}_2 \in \underline{B}(\tilde{G}_2)$ iff $B_1 \in \underline{B}(\tilde{G}_1)$ and $B_2 \in \underline{B}(\tilde{G}_2)$. Hence $\underline{B}(\tilde{G}) = \underline{B}(\tilde{G}_1) \times \underline{B}(\tilde{G}_2)$ and $\tilde{G}_1 \oplus \tilde{G}_2 = \tilde{G}_1 \oplus \tilde{G}_2$.

By the previous paragraph, $G = (G \setminus e) \oplus e$ iff $\tilde{G} = (\tilde{G} \setminus e) \oplus \tilde{e}$ and so e is a nonfactor in G iff it is one in \tilde{G} ; and if this is the case: $B \in \underline{B}(G \setminus e)$ iff $e \notin B \in \underline{B}(\tilde{G})$ iff $e \in (G \setminus B) \in \underline{B}(G)$ iff $(G \setminus B) \setminus e \in \underline{B}(G/e)$ iff $B \in \underline{B}(\tilde{G}/e)$. Hence $\tilde{G} \setminus e = \tilde{G}/e$ and dualizing, $\tilde{G}/e = \tilde{G} \setminus e$. Thus if G decomposes into $G \setminus e$ and G/e ; \tilde{G} decomposes into \tilde{G}/e and $\tilde{G} \setminus e$. This shows that \tilde{t} is a \underline{T} -invariant (taking values in R). The dual of an isthmus (with the isthmus as the only basis) is a loop (with the empty set as the only basis) and conversely.

Lemma 4.8. *If $t(G) = \sum b_{ij} z^i x^j$ is the Tutte polynomial of G , then the mapping $t_A(G)$ of G into the Tutte-Grothendieck group, A , is equal to $\sum b_{ij} B_{ij}$, and if $|G| \geq 2$, then $b_{10} = b_{01}$.*

Proof. The fact that the coefficients of $z^i x^j$ and B_{ij} are the same was shown in (3.7). If $|G| \geq 2$, the only way to get a one point pregeometry which would contribute to either b_{01} or b_{10} is to decompose a two point pregeometry by a morphism in \underline{D}_2 . But the only pregeometry of two points which is connected (hence has a point which is not a one point factor) is the two point circuit C_2 and the only decomposition available is $C_2 \leq x + z$ (or $C_2 \leq B_{10} + B_{01}$).

This linear identity involving the coefficients of the Tutte polynomial is the first in a basis for all such identities developed later (6.6).

Theorem 4.9. *The Crapo "Betsy" invariant $\beta(G) = (-1)^r(G) \sum_{A \subseteq G} (-1)^{|A|} r(A)$ defined in [6] is an invariant of the Tutte decomposition where if $t_A(G) = \sum b_{ij} B_{ij}$, $\beta(G) = b_{10}$. Further, for all $|G| \geq 2$, and for any invariant f which, like β , vanishes on separable pregeometries, f is a multiple of β and by properties of t_A , $f(G) = f(\tilde{G})$.*

Proof. That $\beta(G) = \beta(G \setminus e) + \beta(G/e)$ for all nonfactors e and that $\beta(G) = 0$ for all separable G was proved in [6]. Thus β is an evaluation of $t_A(G)$. But $\beta(B_{ij}) = 0$ for all $i + j > 1$ and $\beta(B_{10}) = 1$ while $\beta(B_{01}) = 0$. Hence $\beta(G) = b_{10}$. If f is any other Tutte decomposition invariant with support on connected pregeometries, then $f(B_{ij}) = 0$ for all $i + j > 1$ and hence $f(G) = k_1 b_{10} + k_2 b_{01}$. But by (4.8), $f(G) = (k_1 + k_2)b_{10} = (k_1 + k_2)\beta(G)$.

Corollary 4.10. $\beta(G)$ is an invariant of the characteristic decomposition and $\beta(G) = d(\chi(G))/d\lambda|_{\lambda=1}$.

Proof. From (4.9) and (3.11), $\beta(G)$ is an invariant of the characteristic decomposition. In addition $\beta(G) = b_{10} = \partial(t(G))/\partial z|_{x=0, z=0} = d(\chi(G))/d\lambda|_{\lambda=1}$.

Although upper truncation is a useful operation on geometries there seems to be no nontrivial category in which it is a functor. The following proposition shows that perhaps the characteristic decomposition is the proper category in which to consider truncation.

Proposition 4.11. *If G is a geometry and $T_1(G)$ denotes the upper rank one truncation of G (i.e. the geometry associated with the geometric lattice formed from $L(G)$ by removing the copoints and letting the colines become the new copoints) then $T_1(G)$ preserves characteristic decomposition and if $\bar{t}_A(G) = \sum b_i B_i$ then $\bar{t}_A(T_1(G)) = \sum_{i>0} (\sum_{j>i} b_j) B_i$.*

Proof. If G has rank one, $T_1(G)$ is the empty geometry. If G has rank two, $T_1(G)$ is an isthmus. Otherwise G and $T_1(G)$ have the same number of atoms. If A and $G \setminus A$ are two nontrivial flats of G , $r(A) + r(G \setminus A) > r(G) > r(T(G))$. Hence $T(G)$ can have no separators and is connected. So $e \in T(G)$ is never an isthmus or loop.

By the remarks on the closed sets of G/e and $G \setminus e$ in (4.1), $L(G/e) \simeq [e, G]$ in $L(G)$. Hence copoints in $G \setminus e$ are in 1-1 correspondence with copoints con-

taining e in G , and $L((T_1(G))/e) = L(T_1(G/e))$.

Rank and cardinality of subsets of $\{G\} \setminus e$ are invariant in G and $G \setminus e$. Hence $L(T_1(G \setminus e)) = ((T_1(G)) \setminus e)$ since both sides represent the lattice of closed sets of the geometry P whose family of independent sets $\underline{L}(P)$ is given by $\underline{L}(G \setminus e) \setminus \underline{B}(G \setminus e)$ (the independent nonbases of $G \setminus e$).

A theorem in [4] states that an operator T preserves decompositions if and only if there is an endomorphism b of A which reflects the action of T (i.e. $t_A \circ T = b \circ T$). Hence one may compute $t_A \circ T_1(G)$ if $t_A(G)$ is known. For the Boolean algebra B_n , $T_1(B_n) = C_n$ the n point circuit. For any $e \in C_m$, $C_m \setminus e = B_{m-1}$ and $C_m/e = C_{m-1}$. Therefore $t_A \circ T_1(B_n) = \sum_{i=1}^{n-1} B_i$ and the result follows.

Table 4.12. Interpretations of some of the evaluations of the Tutte polynomial $t(G)$ are summarized below:

$x =$	$-i$	0	1	2
$z = -i$	$(-1)^{r(G)} \chi(G, i+1)$ (nowhere zero coboundaries)			
0	$(-1)^{r(\tilde{G})} \chi(\tilde{G}, i+1)$ (nowhere zero cycles)	0	$ \mu(\tilde{G}) $	
1	$ \mu(G) $	complexity (number of bases)	number of spanning sets	
2		number of independent sets	number of subgeometries (subsets) = $2^{ G }$	

In addition, $\partial t(G)/\partial z|_{z=x=0} = \beta(G)$, and the number of independent (spanning) sets of cardinality $k = t(G)|_{x^i x^j} = \binom{i}{r(G)-k} (= \binom{j}{k-r(G)})$.

5. Decomposition and strong map factorization. In this section we examine the Tutte decomposition in the context of the category of pregeometries and strong maps. We make use of a lift construction of D. Higgs [12] in which a strong map is factored as an injection followed by a contraction.

Lemma 5.1. *If $f: G' \rightarrow G''$ is a strong map epimorphism from the geometry G' onto the geometry G'' then there exists a geometry G and strong maps $f': G' \rightarrow G$*

and $f'': G \rightarrow G''$ such that f' is the injection of a subgeometry G' into G , f'' is a contraction, and $f = f'' \circ f'$.

Regarding G , G' , and G'' as geometric lattices then G is isomorphic to a quotient of $G' \oplus Q$ (a geometric sublattice of $G' \oplus Q$ associated with a closure operator $\overline{}^G$) where Q is an arbitrary geometric lattice of rank $r(G') - r(G'')$. Denoting the elements of $G' \oplus Q$ by (x, y) where $x \in G'$ and $y \in Q$, the interval $[(0, 1)^G, (1, 1)^G]$ in G is isomorphic to G'' and f'' is contraction of G by the element $(0, 1)^G$. The injection $x \rightarrow (x, 0)$ of G' into $G' \oplus Q$ followed by G -closure as a strong map onto G is an injection of G' into G .

In the special case in which $r(G') - r(G'') = 1$, then G is the sublattice consisting of those elements $\{(x_i, 0)\} \cup \{(y_j, 1)\}$ of $G' \oplus Q$ in which x_i is the supremum of all elements of G' with given image and y_j is a minimal element with given image.

Proof. A proof of this theorem can be found in [10] as well as [12].

Theorem 5.2. Two pregeometries G' and G'' are the Tutte decomposition of a pregeometry G if and only if they differ in rank by one and there exists a 1-1 onto strong map f between them. In this case the map f may be viewed as the composition of injection of $G \setminus e$ into G followed by contraction of G onto G/e for some point $e \in G$ where $G \setminus e \cong G'$ and $G/e \cong G''$. Further this G is unique.

Proof. If $G < G \setminus e + G/e \in \mathcal{D}_2$ then by (4.1), $r'(G \setminus e) = r''(G/e) + 1$ while the composite of the injection $G \setminus e$ into G followed by the contraction of G by e is strong map which is certainly 1-1 and onto on $\{G\} \setminus \{e\}$.

Conversely, assume $f: G' \rightarrow G''$ is a 1-1 strong map from G' onto G'' and assume $r(G') = r(G'') + 1$. Then associated with f there is an onto strong map $\bar{f}: \bar{G}' \rightarrow \bar{G}''$ between the underlying geometries \bar{G}' and \bar{G}'' . This is so since all loops must be mapped into loops so that the 0 element of \bar{G}' is sent to the 0 element of \bar{G}'' and also all points in the same atom in \bar{G}' must be mapped to the same lattice element in \bar{G}'' since inverse images of closed sets are closed. By the Higgs theorem (4.1) we may factor f through \bar{G} . The geometry \bar{G} may be viewed as a quotient of $\bar{G}' \oplus B_1$ where B_1 is the two element lattice.

At most one atom of \bar{G}' is mapped to the 0 element of \bar{G}'' since rank is decreased by one. There are then two cases to consider:

If no atoms are sent to 0, then using (5.1) the atoms of \bar{G} are the elements $\{(a_i, 0)\} \cup (0, 1)$ where the a_i 's are the atoms of G' , since the 0 element of \bar{G}' is the only preimage of the 0 element of \bar{G}'' . We construct the pregeometry G from \bar{G} on the set of points $\{G'\} \cup \{e\}$ in which the atom $(0, 1)$ represents the point e and the atom $(a_i, 0)$ (and 0 element) of G contains the points of the atom a_i (and 0 element) of G' . Then by (5.1), $G' \cong G \setminus e$ and $G'' \cong G/e$. Further, $r'(G \setminus e) - r''(G/e) = 1$ so e is not an isthmus or loop of G .

If on the other hand an atom a' of \bar{G}' is sent to 0, the atoms of \bar{G} are the ele-

ments $(a', 1) \cup \{(a_i, 0)\}$ for all atoms $a_i \neq a'$ of G' since a' is the supremum of the two preimages of 0: a' and 0, while a' is the only atom of \bar{G}' which is not minimal for some given image. Construct the pregeometry G from \bar{G}' on the point set $\{G'\} \cup \{e\}$ in which for all $a_i \neq a'$, the atom $(a_i, 0)$ (and 0 element) of G contains the points of the atom a_i (and 0 element) of G' ; and in which the atom $(a', 1)$ contains the points of $a' \in G'$ as well as the point e . Then since $(0, 1)^G = (a', 1)$ we are done as in the first case.

If $G \setminus e$ and G/e are pregeometries on the same points then using (4.1) the copoints of G/e are exactly those copoints of the constructed G which contain the point e with e then deleted. Also the closed sets of $G \setminus e$ are exactly those sets $A \subseteq \{G\} \setminus \{e\}$ such that A or $A \cup \{e\}$ is closed. Hence the set of copoints, \underline{D} , of G is equal to the union $(\underline{D}' \setminus \underline{D}'') \cup (\underline{D}'' \times \{e\})$ where \underline{D}'' is the family of copoints of G/e and \underline{D}' is the family of copoints of $G \setminus e$. Hence G is unique up to isomorphism in the above factorization for any 1-1 onto unit rank decreasing strong map $f: G' \rightarrow G''$.

We mention in passing another consequence of the above theorem.

Corollary 5.3. *The partial order P_n of isomorphism classes of pregeometries on n points ordered by strong maps ($G \leq H$ if there exists a strong map $f: G \rightarrow H$) is self-dual where $i: \tilde{P}_n \cong P_n$ takes a pregeometry to its (Whitney) dual.*

Proof. The covering pairs in P_n are the unit rank decreasing strong maps. But G'' covers G' iff there exists a pregeometry G on $n+1$ points such that $G' \simeq G \setminus e$ and $G'' \simeq G/e$ by (5.2) iff $\tilde{G}' \simeq G/e$ and $\tilde{G}'' \simeq G \setminus e$ by (4.7) iff \tilde{G}' covers \tilde{G}'' by (5.2).

6. The Tutte polynomial. In §§3 and 4 we noted the importance of the function $t(G)$ taking a pregeometry into the Tutte-Grothendieck ring. In this chapter we will cite some of the properties of this polynomial, called the Tutte polynomial after previous work by Henry Crapo [8].

We note in passing that $t(G) = \sum b_{ij} z^i x^j$ regarded as a generating function has a combinatorial interpretation wherein b_{ij} counts the number of bases with internal activity i and external activity j . Relative to a linear ordering O on the points of a pregeometry G , a point $p_1 \in B$ is *internally active* relative to the basis B if p_1 is greater than any other point in the unique bond contained in $(G \setminus B) \cup \{p_1\}$. Dually, $p_2 \in G \setminus B$ is *externally active* relative to the basis B (and order O) if p_2 is greater than any other point in the unique circuit contained in $B \cup \{p_2\}$. The internal (external) activity of B is then the number of points internally (externally) active relative to B (and O). This interpretation, dependent as it is on the particular ordering, is explored no further in the present paper.

Theorem 6.1. *Assume $t(G) = \sum_{i=0}^n \sum_{j=0}^m b_{ij} z^i x^j$ is the Tutte polynomial of a pregeometry G where $\sum_j b_{nj} > 0$ and $\sum_i b_{im} > 0$. Then the following facts hold:*

(a) The rank of G , $r(G)$ is equal to n while $n(G)$, the nullity of G is equal to m . Hence $|G| = n + m = \log_2 (t(G)|_{z=x=2})$.

(b) $b_{nj} = \delta(j, k)$ iff G has k loops and $b_{im} = \delta(i, b)$ iff G has b isthmuses.

If G' has j loops then $G' = \bigoplus_{k=1}^j x \oplus G$ where G has no loops and $t(G') = x^j t(G)$. Hence no generality is lost if we now assume G to be loopless so $\sum_j b_{nj} = b_{n0} = 1$.

(c) The number of atoms of G (flats of rank one) is equal to $n + b_{n-1, 0}$.

(d) $\sum_j b_{n-1, j} = m$ while if G has no isthmuses, $\sum_i b_{i, m-1} = n$.

Proof. In (a), (b), and (d) the second statement follows from the first and (4.7) which states that $t(G) = \sum_j b_{ji} z^i x^j$.

To prove (a) and (b) we use induction on the number of nonfactors. If $G = \bigoplus_{i=1}^n z \oplus \bigoplus_{j=1}^m x$ then $t(G) = z^n x^m$ while G has rank n . In addition $b_{nj} = \delta(j, m)$ while G has m loops.

Assuming the result for all pregeometries with m nonfactors, let G have $m+1$ nonfactors one of which is e . Then $t(G) = t(G \setminus e) + t(G/e)$. G has rank n with k loops iff $G \setminus e$ has rank n with k loops and G/e has rank $n-1$ iff (by induction) in $G \setminus e$, the coefficient of $z^n x^j$ is $\delta(j, k)$ while the coefficient for $z^i x^j$ is 0 for all $i > n$ in $G \setminus e$ and for all $i \geq n$ in G/e . The result follows from adding the polynomials $t(G \setminus e)$ and $t(G/e)$. Since $|G| = r(G) + n(G)$ we are done by (4.6) wherein it was proved that $t(G)|_{z=x=2} = 2^{|G|}$.

We now assume that G is a loopless pregeometry. If G has no nonfactors then $t(G) = z^n$ and (c) and (d) hold since G has n atoms while $b_{n-1, 0} = \sum_j b_{n-1, j} = 0 = m$.

Inductively, let $t(G) = t(G \setminus e) + t(G/e)$. If e is closed in G , then $G \setminus e$ has one less atom than G while G/e is loopless so the coefficient of z^{n-1} in G/e is one. If $e' \in \bar{e}$ in G , then $G \setminus e$ has the same number of atoms as G while e' is a loop in G/e so that coefficient of z^{n-1} in G/e is 0. In any event, by induction $n(G \setminus e) = r(G/e) = m-1 = \sum_j b'_{n-1, j}$ in $t(G \setminus e)$, while in $t(G/e)$, $\sum_j b''_{n-1, j} = 1$. Both (c) and (d) follow by adding the polynomials $t(G \setminus e)$ and $t(G/e)$.

Lemma 6.2. Let G be a pregeometry of rank n . If f_n^{kj} counts the number of flats of corank k and nullity j in a rank n pregeometry, and if e is a nonfactor of G then:

$$f_n^{kj}(G) = f_n^{kj}(G \setminus e) + f_{n-1}^{kj}(G/e) - \bar{f}_{n-1}^{k, j+1}(G/e)$$

where $\bar{f}_{n-1}^{k, j+1}(G/e)$ counts only those closed sets x of G/e which are not closed in G .

In particular, if e is a nonfactor of all flats of corank k and nullity $j+1$ in G then $\bar{f}_{n-1}^{k, j+1}(G/e) = f_{n-1}^{k, j+1}(G/e)$.

Proof. We will use properties of the Tutte decomposition (4.1) and will account for all contributions to the above equation which we denote by $(a) = (b) + (c) - (d)$.

The closed sets of corank k and nullity j in G counted in (a) either contain e and are equicardinal with (c) (since e is not a loop) or do not contain e and are counted in (b). The other flats counted in (b) are those subsets y such that $y \cup e$ is closed in G but y is not. Hence e is not an isthmus of $y \cup e$ and hence $y \cup e$ has corank k and nullity $j+1$ in G/e . Such sets y are exactly those sets counted in (d).

Theorem 6.3. Let G be a pregeometry of rank n with Tutte polynomial $t(G) = \sum b_{ij} z^i x^j$. If all p element subsets of G are independent, then the number of flats of G of corank k and nullity j is counted for all j by

$$f_n^{kj}(G) = \sum_{s=k}^n \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} b_{s,j+t}$$

if $k \geq n-p$ or if $k=0$.

(Note that for $k > n-p$ this gives the obvious: $f_n^{kj}(G) = \delta(0, j) \binom{n}{k}$.)

Proof. For all pre-Boolean algebras, $f_n^{kj}(B_{nm}) = \binom{n}{k} \delta(j, m)$. But $t(B_{nm}) = z^n x^m$, hence $b_{s,j+t}$ is zero unless $s=n$ and $t=m-j$ in which case the formula gives

$$(-1)^0 \binom{0}{m-j} \binom{n}{k} = \binom{n}{k} \delta(j, m).$$

We will use induction on the number of nonfactors. Since we have proved the theorem for 0 nonfactors above we may assume that the equality holds for all pre-geometries with fewer than q nonfactors and consider G with q nonfactors including e . We will show that the recursion of (6.2) is valid for all k in the theorem with f replaced by f . Surely it holds if $k=0$.

By (4.1) a subset S containing a nonfactor e is independent in G iff $S \setminus \{e\}$ is independent in G/e and a subset S disjoint from e is independent in G iff it is independent in $G \setminus e$. Hence all subsets of corank $n-p$ are independent in G iff all subsets of corank $n-p$ are independent in $G \setminus e$ and G/e . But all subsets of corank $n-p$ are independent iff all subsets of corank $n-p+1$ are (independent and) closed. If x is a closed set of corank greater than or equal to $n-p+1$, the subgeometry x is a Boolean algebra while if x has corank $n-p$, the subgeometry is the truncation of a Boolean algebra (i.e. free). In any event the recursion of (6.2) for f_n^{kj} holds for all $k \geq n-p$ (since in all such subgeometries isthmuses are only of flats of nullity 0) and the induction hypothesis may be extended to $G \setminus e$ and G/e . Simplifying the dexter we get successively if $t(G \setminus e) = \sum b'_{ij} z^i x^j$ and $t(G/e) = \sum b''_{ij} z^i x^j$:

$$\begin{aligned}
& f_n^{kj}(G \setminus e) + f_{n-1}^{kj}(G/e) - f_{n-1}^{k,j+1}(G/e) \\
&= f_n^{kj}(G \setminus e) \\
&+ \sum_{s=k}^{n-1} \sum_{t=0}^{n-s} \left[(-1)^t \binom{n-s-1}{t} \binom{s}{k} b''_{s,j+t} - (-1)^{t-1} \binom{n-s-1}{t-1} \binom{s}{k} b''_{s,j+t} \right] \\
&= \sum_{s=k}^n \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} b'_{s,j+t} + \sum_{s=k}^{n-1} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} b''_{s,j+t} \\
&= \sum_{s=k}^n \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} (b'_{s,j+t} + b''_{s,j+t}) - \binom{n}{k} b''_{nj} \\
&= f_n^{kj}(G) - 0.
\end{aligned}$$

Corollary 6.4. Under the hypotheses of the above theorem the number of flats of corank k (the Whitney number of the second kind w_{n-k}) is given by

$$w_{n-k} = \binom{n}{k} + \sum_{s=k}^{n-1} \sum_{i=0}^{n-s-1} (-1)^i \binom{n-s-1}{i} \binom{s}{k} b_{si}$$

(Again $w_i = \binom{n}{i}$ for all $i < p$.) In particular, if G is a geometry (i.e. $p \geq 2$), $w_2 = \binom{n}{2} + b_{n-2,0} - b_{n-2,1} + (n-1)b_{n-1,0}$.

Proof.

$$\begin{aligned}
w_{n-k} &= \sum_{s=k}^n \binom{s}{k} \sum_{j=0}^{\infty} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} b_{s,j+t} \\
&= \sum_{s=k}^n \binom{s}{k} \sum_{i=0}^{\infty} \sum_{t=0}^i (-1)^t \binom{n-s}{t} b_{si} \\
&= \sum_{s=k}^{n-1} \binom{s}{k} \sum_{i=0}^{\infty} (-1)^i \binom{n-s-1}{i} b_{si} + \binom{n}{k} \sum_{i=0}^{\infty} b_{ni}.
\end{aligned}$$

Corollary 6.5. If all p element subsets of G are independent then for all $s \geq n-p$,

$$b_{sj} = \sum_{v=0}^{\infty} \binom{n-1-s+v}{n-1-s} \sum_{q=s}^{\infty} (-1)^{q-s} \binom{q}{s} f^{q,j+v}$$

where $r(G) = n$, f^{kj} is the number of flats of G of corank k and nullity j , and b_{sj}

is the coefficient of $z^s x^j$ in the Tutte polynomial $t(G)$.

In particular, if m is the nullity of G , for all $t < p$, $j > 0$,

$$b_{n-t,0} = \binom{m+t-1}{t}; \quad b_{n-t,j} = 0;$$

$$b_{n-p,0} = \sum_{\nu=0}^{\infty} \binom{p-1+\nu}{p-1} f^{n-p,\nu} + \binom{m+p-1}{p} - \binom{n+m}{p};$$

and

$$b_{n-p,j} = \sum_{\nu=0}^{\infty} \binom{p-1+\nu}{p-1} f^{n-p,\nu+j}.$$

Proof. Since the formula involves only terms $f^{q,j+\nu}$ for $q \geq s \geq n-p$, we need only check that the formula inverts the equation for f_n^{kj} in (6.3). We will use the symbolic methods of the umbral calculus.

$$\begin{aligned} f^{kj} &= u^k v^j = \sum_{s=k}^{\infty} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} x^s y^{j+t} \\ &= (1-y)^n y^j \sum_{s=k}^{\infty} \binom{s}{k} \left(\frac{x}{1-y} \right)^s \\ &= (1-y)^n y^j \left(\frac{x}{1-y} \right)^k \left(1 - \frac{x}{1-y} \right)^{-k-1} \\ &= \frac{(1-y)^{n+1} x^k y^j}{(1-y-x)^{k+1}} \quad (x^s y^t = b_{st}). \end{aligned}$$

Similarly,

$$\begin{aligned} b_{sj} &= x^s y^j = \sum_{r=0}^{\infty} \binom{n-1-s+r}{n-1-s} \sum_{q=s}^{\infty} (-1)^{q-s} \binom{q}{s} u^q v^{j+r} \\ &= \frac{u^s v^j}{(1+u)^{s+1} (1-v)^{n-s}} \quad (u^k v^j = f^{kj}). \end{aligned}$$

Composing we get:

$$\begin{aligned} &\sum_{s=k}^{\infty} \sum_{t=0}^{n-s} (-1)^t \binom{n-s}{t} \binom{s}{k} \frac{u^s v^{j+t}}{(1+u)^{s+1} (1-v)^{n-s}} \\ &= \sum_{s=k}^{\infty} \binom{s}{k} \frac{u^s v^j (1-v)^{n-s}}{(1+u)^{s+1} (1-v)^{n-s}} \\ &= \frac{u^k v^j}{(1+u)^{k+1}} \left(1 - \frac{u}{u+1} \right)^{-k-1} = u^k v^j = f^{kj}. \end{aligned}$$

Conversely,

$$\begin{aligned} & \sum_{r=0}^{\infty} \binom{n-1-s+r}{n-1-s} \sum_{q=s}^{\infty} (-1)^{q-s} \binom{q}{s} \frac{(1-y)^{n+1} x^q y^{j+r}}{(1-y-x)^{q+1}} \\ & = y^j (1-y)^{n+1} \left(\frac{x}{1-y-x} \right)^s \left(1 + \frac{x}{1-y-x} \right)^{-s-1} (1-y)^{-n+s} (1-y-x)^{-1} \\ & = x^s y^j = b_{sj}. \end{aligned}$$

Changing variables in the formula for b_{sj} if the nullity of G is m ,

$$b_{n-q,j} = \sum_{\nu=0}^{\infty} \binom{q-1+\nu}{q-1} \sum_{r=0}^q (-1)^{q-r} \binom{n-r}{q-r} f^{n-r, j+\nu}.$$

But every subset of $p-1$ or less elements is closed, hence $f^{n-r, j+\nu} = \binom{n+m}{r} \delta(j+\nu, 0)$ for all $r \leq p-1$ and

$$b_{n-q,j} = \sum_{r=0}^q (-1)^{q-r} \binom{n-r}{q-r} \binom{n+m}{r} \delta(j, 0) = \binom{m+q-1}{q} \delta(j, 0)$$

for all $q < p$ as can be seen by comparing the coefficient of x^q on either side of the identity $(1+x)^{-(n-q+1)} (1+x)^{n+m} = (1+x)^{m+q-1}$. Further

$$\begin{aligned} b_{n-p,j} &= \sum_{\nu=0}^{\infty} \binom{p-1+\nu}{p-1} f^{n-p, j+\nu} + \left(\sum_{r=0}^{p-1} (-1)^{p-r} \binom{n-r}{p-r} \binom{n+m}{r} \right) \delta(j, 0) \\ &= \sum_{\nu=0}^{\infty} \binom{p-1+\nu}{p-1} f^{n-p, j+\nu} + \left(\binom{m+p-1}{p} - \binom{n+m}{p} \right) \delta(j, 0). \end{aligned}$$

Theorem 6.6. If $|G| \geq n$ and the Tutte polynomial $t(G)$ equals $\sum_{s,t} b_{st} z^s x^t$ then the following identity holds among the coefficients b_{st} :

$$I_n(G) = \sum_{s=0}^{n-1} \sum_{t=0}^{n-s-1} (-1)^t \binom{n-s-1}{t} b_{st} = 0.$$

Furthermore, in the vector space V of linear combinations of Tutte coefficients $\{b_{st}\}$, the vectors $\{I_1, \dots, I_n, \dots\}$ form a basis for the subspace of all linear identities which hold for all but a finite number of Tutte polynomials.

Proof. It is sufficient for an inductive proof to show that $I_n(G) = 0$ for all $|G| = n$ since if $I_n(G) = 0$ for all $|G| = k \geq n$ then it also holds for all pre-Boolean algebras B_{ij} , $i+j = k+1 > n$, since the only coefficients involved in I_n are

those $b_{s,t}$ such that $s+t < n$. On the other hand if $|G| = k+1$ has a nonfactor e then $t(G) = t(G \setminus e) + t(G/e)$, and the latter two polynomials satisfy the induction hypothesis so that $I_n(G) = I_n(G \setminus e) + I_n(G/e) = 0$.

Now assume $r(G) = n'$ and $n(G) = m'$ where $n' + m' = n$. $I_n(B_n) = 0$ vacuously so assume $m' > 0$. The only flat of corank 0 is G , hence using the notation of (6.3), $f_n^0, i = 0, 1, \dots, m'-1$, so we have

$$\begin{aligned} 0 &= \sum_{i=0}^{m'-1} (-1)^i \binom{m'-1}{i} f_n^0, i = \sum_{s=0}^{n'} \sum_{i=0}^{m'-1} \sum_{t=0}^{n'-s} (-1)^{i+t} \binom{m'-1}{i} \binom{n'-s}{t} b_{s,i+t} \\ &= \sum_{s=0}^{n'} \sum_{j=0}^{n-s-1} (-1)^j \sum_{t=0}^j \binom{m'-1}{j-t} \binom{n'-s}{t} b_{sj} \quad (j = i+t) \\ &= \sum_{s=0}^{n'} \sum_{j=0}^{n-s-1} (-1)^j \binom{n-s-1}{j} b_{sj} \\ &= I_n(G) \text{ since } b_{sj} = 0 \text{ for all } s > n'. \end{aligned}$$

Certainly the I_n are independent in V since $I_{n+1} = \sum_{s=0}^n (-1)^{n-s} b_{s,n-s} + \sum_{i+j < n} a_{ij} b_{ij}$ is independent of $\{I_j\}_{j \leq n}$. On the other hand if $I'_{n+1} = \sum_{i+j \leq n} a'_{ij} b_{ij}$ is a relation which is 0 almost everywhere then $-a'_{s,n-s} = a'_{s-1,n-s+1}$ for all $s = 1, 2, \dots, n$ since for the infinite class of pregeometries $B_{s-1,n-s} \oplus C$ (C any connected pregeometry), $b_{s,n-s} = b_{s-1,n-s+1}$ and $b_{ij} = 0$ for all other $i+j \leq n$.

We now list some useful properties of the series and parallel connection (3.14). The proofs may all be found in [2].

Lemma 6.7. (a) A pointed pregeometry (F, p) is a series connection $F = S(G, H)$ if and only if $F \setminus p = (G \setminus p) + (H \setminus p)$. Dually, F is a parallel connection $P(G, H)$ iff $F/p = G/p + H/p$.

(b) If H is a loop then $S(G, H) = G$. If H is an isthmus then $P(G, H) = G$.

(c) If G and H each have at least two points then $S(G, H)$ ($P(G, H)$) is connected if and only if G and H both are.

(d) If $e \in \{G\} \setminus \{p\}$, then $S(G, H)/e = S(G/e, H)$; $S(G, H) \setminus e = S(G \setminus e, H)$; $P(G, H)/e = P(G/e, H)$; and $P(G, H) \setminus e = P(G \setminus e, H)$.

The following proposition order generalizes Corollary 14.2 of [10].

Proposition 6.8. If M_1 is a connected minor of a connected pregeometry G , then there exists a sequence of connected minors $M_1, M_2, \dots, M_m = G$ such that $M_{i-1} = M_i \setminus e_i$ or $M_{i-1} = M_i/e_i$ for some $e_i \in M_i$.

Proof. We proceed by induction on $|G|$. If $|G| = 1$, then $M_1 = G$ and we are done. Assume that the theorem holds for all $|G'| < n$ and let M_1 be a connected minor of a pregeometry G , $|G| = n$. Then by definition there exists a sequence of minors $M_1 = M'_1, \dots, M'_m = G$ such that $M'_{i-1} = M'_i \setminus e_i$ or $M'_{i-1} = M'_i / e_i$. If M'_{m-1} is connected we are done by induction since then there would also exist a connected sequence $[M_1, M'_{m-1}]$. So assume $M'_{m-1} = G \setminus e = M^{(1)} \oplus \dots \oplus M^{(k)}$ where each $M^{(k)}$ is connected. Then M_1 is contained in exactly one of the $M^{(i)}$, assume M_1 is a minor of $M^{(1)}$. By (6.7(a)), (G, e) is a (connected) series connection, $G = S(N^{(1)}, \dots, N^{(k)})$ where each $N^{(i)}$ is connected and $N^{(i)} \setminus e = M^{(i)}$. By the induction hypothesis we may reduce $(N^{(2)}, e)$ to a loop e by a connected sequence which induces a connected sequence for G by (6.7(c)) and (6.7(d)). But by (6.7(b)) the sequence is $[G, S(N^{(1)}, N^{(3)}, \dots, N^{(k)})]$ and again using the induction hypothesis we can extend the sequence to M_1 .

Corollary 6.9. *If M_1 is a minor of a connected pregeometry G , then $t(M_1) \leq t(G)$ (i.e. each coefficient b_{ij} of $t(M_1)$ is less than or equal to the corresponding coefficient of $t(G)$).*

Proof. Proceeding by induction, if $|G| = 1$, $G = M_1$ and $t(G) = t(M_1)$. If the theorem holds for all $|G'| < n$ and $|G| = n$, then, as in the proof of (6.8), let $M_1, M_2, \dots, M_m = G$ be a sequence of minors such that $M_{i-1} = M_i / e_i$ or $M_i \setminus e_i$. If M_{m-1} is connected the result follows by induction: if for example $M_{m-1} = M_m / e_m$ then $t(M_1) \leq t(M_{m-1}) = t(G) - t(M_m \setminus e_m)$ each coefficient of the latter term being positive.

On the other hand, if, for example, $M_m = P(M^{(1)}, \dots, M^{(k)})$ and $M_1 = M_1^{(1)} \oplus \dots \oplus M_1^{(j)}$ we may assume each $M^{(i)}$ contains a nonempty direct sum $M_1^{(i_1)} \oplus \dots \oplus M_1^{(i_s)}$ since otherwise we could reduce $M^{(i)}$ to the isthmus e_m through a sequence of connected geometries (which would hence have no loops or isthmuses) and the result would follow from the induction hypothesis and the observation that if $e \in M_m$ were neither an isthmus nor a loop, then $t(M_m \setminus e) \leq t(M_m)$ and $t(M_m / e) \leq t(M_m)$.

Thus each direct sum factor of M_1 is partitioned by the $M^{(i)} / e_m$ in which it is a minor, hence

$$\begin{aligned} t(M_1) &= \prod_i t(M_1^{(i)}) = \prod_j M_1^{(i)} \prod_{M^{(j)} / e_m} t(M_1^{(i)}) \\ &\leq \prod_j t(M^{(j)} / e_m) = t(G / e_m) \leq t(G) \end{aligned}$$

where the first inequality follows from induction on each factor.

A similar argument holds if G is a series connection.

Proposition 6.10. *If in $t(G)$, $b_{kj} = 0$ for all $(k, j) \geq (s, t)$ then f^{kj} , the number of closed sets of corank k and nullity j , equals 0 for all $(k, j) \geq (s, t)$.*

Proof. If $t(G) = z^n x^m$, then $b_{kj} = 0$ only for all $(k, j) \geq (s, m+1)$ or all $(k, j) \geq (n+1, t)$ and there are no closed sets of nullity greater than m or corank greater than n .

Proceeding by induction on the number of nonfactors assume $t(G) = t(G \setminus e) + t(G/e)$. If in $t(G)$, $b_{kj} = 0$ for all $(k, j) \geq (s, t)$ then in $t(G \setminus e)$ and $t(G/e)$ respectively $b'_{kj} = b''_{kj} = 0$ for all $(k, j) \geq (s, t)$. If there were a closed set x of corank u and nullity v in G , there would either be a closed set of corank u and nullity v in G/e (i.e. $x \setminus e$) if $e \in x$ or there would be a closed set of corank u and nullity v in $G \setminus e$ (i.e. x) if $e \notin x$. In either case by induction we are assured that $(u, v) \not\leq (s, t)$.

7. Characterizations and examples. Certain classes of pregeometries may be characterized by properties of their Tutte polynomials. Conversely, for certain classes of pregeometries many of the coefficients of the Tutte polynomials can be computed directly from the cardinality of various families of flats.

Proposition 7.1. All the points of a pregeometry G are loops or isthmuses iff it is completely separable iff it has no nonfactors iff its Tutte polynomial, $t(G)$, has one term.

The following proposition first appeared in [5]. The proof is new.

Proposition 7.2. A pregeometry G is connected if and only if G is a loop or in $t(G)$, $b_{10} > 0$.

Proof. If G is separable ($G = G_1 \oplus G_2$) then $t(G) = t(G_1)t(G_2)$. The latter two polynomials have no constant term, hence $t(G)$ can have no first degree terms.

Conversely, if G is connected and not a loop, then it has a minor isomorphic to an isthmus, z , and $t(G) \geq t(z)$ by (6.9).

Proposition 7.3. A connected pregeometry G of two or more points is a series-parallel network [2] (i.e. G has no minor isomorphic to L_4 , the four point line, or P_4 , the lattice of partitions of a four element set) if and only if $b_{10} = 1$ in $t(G)$.

Proof. This characterization is proved in [2].

Corollary 7.4. A pregeometry G without loops or isthmuses is a direct sum of n series-parallel networks iff in $t(G)$, $b_{ij} = \binom{i+j}{i}$ for all (equivalently for some) $i + j = n$ but $b_{st} = 0$ for all $s + t < n$.

Proof. This is a result of (7.2), (7.3) and the fact that $b_{01} = b_{10}$ (4.8).

Proposition 7.5. In a pregeometry G , all p element subsets are independent (i.e. $L(G)$ is distributive up to level $p-1$) if $b_{n-q,j} = 0$ for all $q < p$, $j > 0$.

Proof. This follows in one direction from (6.5) and in the other by (6.10).

The preceding proposition together with the explicit calculations in (6.5) give us characterizations of wide classes of pregeometries by their Tutte polynomials.

Proposition 7.6. A pregeometry G of rank n is loopless iff in $t(G)$, $b_{ij} = 0$ for all $(i, j) \geq (n, 1)$. In addition for such pregeometries $b_{n, 0} = 1$, $b_{n-1, 0} = f^{n-1} - n$; and $b_{n-1, j} = \sum_{v=0}^{\infty} f^{n-1, v+j}$ for all $j > 0$ where $f^{n-1, v}$ is the number of atoms with $v + 1$ points, and f^{n-1} is the number of atoms of G .

Proof. A pregeometry is loopless iff all one element subsets are independent. We apply (6.5) and (7.5).

Similar results hold for pregeometries with loops if we first factor them out as in (6.1).

Proposition 7.7. A pregeometry G of rank n is a geometry iff in $t(G)$, $b_{ij} = 0$ for all $(i, j) \geq (n-1, 1)$. In addition if the nullity of the geometry G equals m then $b_{n, 0} = 1$, $b_{n-1, 0} = m$, $b_{n-2, 0} = \sum_{v=0}^{\infty} (v+1)f^{n-2, v} + \binom{m+1}{2} - \binom{m+n}{2}$, and $b_{n-2, j} = \sum_{v=0}^{\infty} (v+1)f^{n-2, v+j}$; where $f^{n-2, v}$ is the number of $(v+2)$ -point lines.

Proof. A pregeometry is a geometry iff all two element subsets are independent. We apply (6.5) and (7.5).

Definition 7.8. A covering π of a set S is a family of subsets with union S . A covering is a partition of type n (or n -partition) of S if every element of π or block has at least n elements and every n -element subset is in a unique block.

The set S , the blocks, and all subsets of S with fewer than n elements form the flats of rank $n+1$ pregeometry. In this pregeometry all subsets of cardinality n are independent. These partitions of type n are discussed in [11] and [10] where they are conjectured to predominate in an enumeration of n -point geometries.

Partitions of type n are special in our setting, as their Tutte polynomials are completely determined from the cardinalities of the blocks.

Proposition 7.9. G is a partition of type $n-1$ iff $r(G) = n$ and $b_{ij} = 0$ for all $(i, j) \geq (2, 1)$. In addition if $|G| = n+m$ and a_t represents the number of blocks of cardinality t ($t = n-1, n, \dots, n+m$), then

$$b_{i0} = \binom{m+n-i-1}{n-i} \quad \text{for all } i \geq 2;$$

$$b_{10} = \sum_{v=0}^{\infty} \binom{n-2+v}{n-2} a_{v+n-1} + \binom{m+n-2}{n-1} - \binom{m+n}{n-1};$$

and for all $j > 0$,

$$b_{1j} = \sum_{v=0}^{\infty} \binom{n-2+v}{n-2} a_{v+j+n-1},$$

and

$$b_{0j} = \binom{n+m-j-1}{n-1} - \sum_{v=0}^{\infty} \binom{n-1+v}{n-1} a_{v+j+n-1}.$$

Proof. The first statement follows from (7.5). Assume G is a partition of type $n - 1$. Then the formula in (6.3) for f_n^{kj} is valid not only for $k \geq 1$ (since every $(n - 1)$ -element subset is independent) but also the formula is valid for $k = 0$. Hence (6.3) holds for all (k, j) and may be inverted so that the formula for b_{sj} in (6.5) is valid for all (s, j) . Substituting i for $n - q$ in (6.5), $b_{ij} = \binom{m+n-i-1}{n-i} \delta(0, j)$ for all $i \geq 2$. Substituting $n - 1$ for p and $a_{v+j+n-1}$ for $f^{1, v+j}$ the formulas for b_{1j} follow from those in (6.5). Finally, for all $j > 0$,

$$\begin{aligned} b_{0j} &= \sum_{q=0}^{\infty} (-1)^q \sum_{\nu=0}^{\infty} \binom{n-1+\nu}{n-1} f^{q, j+\nu} \\ &= \sum_{\nu=0}^{\infty} \binom{n-1+\nu}{n-1} f^{0, j+\nu} - \sum_{\nu=0}^{\infty} \binom{n-1+\nu}{n-1} f^{1, j+\nu} + \sum_{j>0, i \geq 2} c_{ij} f^{ij}. \end{aligned}$$

But $f^{0, j+\nu} = \delta(j + \nu, m)$ hence the first sum equals $\binom{n+m-j-1}{n-1}$; $f^{1, j+\nu} = a_{v+j+n-1}$ in the second sum; and each f^{kj} in the third sum equals zero.

We may apply (7.9) to some important special cases: partitions, planar geometries, and free geometries.

The isomorphism classes of loopless rank two pregeometries are in 1-1 correspondence with integer partitions $p = p_1 + \dots + p_k$, $p_1 \geq p_2 \geq \dots \geq p_k > 0$ where p_i corresponds to the cardinality of an atom and where $k \geq 2$. To every such partition there corresponds a unique Ferrers dual partition, $p = p'_1 + \dots + p'_{p_1}$ where p'_i is the number of integers p_j greater than or equal to i .

Proposition 7.10. If $t(G) = b_{ij} z^i x^j$ is the Tutte polynomial of the pregeometry corresponding to the integer partition $p = p_1 + p_2 + \dots + p_k$ with Ferrers dual $p = p'_1 + \dots + p'_{p_1}$ then $b_{2j} = \delta(0, j)$; $b_{10} = p'_1 - 2$; $b_{1j} = p'_j$ and $b_{0j} = p - j - 1 - \sum_{u \geq 1} u a_{u+j} = p - j - 1 - \sum_{i>j} p'_i$.

Proposition 7.11. A planar (rank 3) geometry G of nullity m with a_i lines of cardinality i is characterized by its Tutte polynomial: $b_{3j} = \delta(0, j)$; $b_{2j} = m\delta(0, j)$; $b_{10} = \sum_{\nu=0}^{\infty} (\nu + 1) a_{\nu+2} + \binom{m+1}{2} - \binom{m+n}{2}$; and for all $j > 0$,

$$b_{1j} = \sum_{\nu=0}^{\infty} (\nu + 1) a_{\nu+j+2}; \quad b_{0j} = \binom{m-j+2}{2} - \sum_{\nu=0}^{\infty} \binom{\nu+2}{2} a_{\nu+j+2}.$$

Using (6.4) or summing the a_i 's one sees that there are $n + b_{20}$ points and $2b_{20} + b_{10} - b_{11} + n$ lines. Hence, subtracting we get $b_{20} + b_{10} - b_{11} = 2b_{01} - b_{02} \geq 0$ and is equal to 0 if and only if G is a projective plane (modular).

Proposition 7.12. For $n > 1$ let F^{nm} represent the partition of type $n - 1$ of cardinality $n + m$ in which every $(n - 1)$ -element subset is a block. We call F^{nm} the free geometry of rank n and nullity m . Note that F^{nm} represents the m th truncation of the Boolean algebra B_{n+m} . Free geometries are characterized as those for which $b_{ij} = 0$ for all $ij \neq 0$ in $t(G)$, and in particular

$$t(F^{nm}) = \sum_{p=0}^{n-1} \binom{m-1+p}{m-1} z^{n-p} + \sum_{q=0}^{m-1} \binom{n-1+q}{n-1} x^{m-q}.$$

Proposition 7.13. *Let f be any Tutte (group) invariant such that f is greater when evaluated on a free (pre)geometry of rank n and nullity m than on the pre-Boolean algebra of the same rank and nullity for all $nm > 0$ (i.e. $f(F^{nm}) > f(B_{nm})$). Then $f(F^{nm}) > f(G)$ for all other pregeometries G of rank n and nullity m . (Similar results hold for any other relation compatible with addition.)*

Proof. If G is not a pre-Boolean algebra it has a nonfactor e and $f(G) = f(G \setminus e) + f(G/e)$. But F^{nm} decomposes into $F^{n,m-1}$ and $F^{n-1,m}$ (and by (7.12) it is the only pregeometry which does so). Hence by induction on cardinality, $f(F^{n,m-1}) \geq f(G \setminus e)$ and $f(F^{n-1,m}) \geq f(G/e)$ with at least one of the inequalities strict.

For given rank and cardinality the number of bases, spanning sets and independent sets are clearly maximized by free geometries; $|\mu|$ and β are also.

Corollary 7.14. *For all pregeometries of rank $n > 1$ and cardinality $n+m$,*

$$|\mu(G)| \leq \binom{m+n-1}{n-1} \quad \text{and} \quad \beta(G) \leq \binom{m+n-2}{n-1}.$$

Further, equality holds if and only if $G = F^{nm}$.

Proof. Evaluating the Tutte polynomial of (7.12), if $m > 0$, $\beta(F^{nm}) = b_{10} = \binom{n+m-2}{n-1} > 0$; and $|\mu(F^{nm})| = t(F^{nm})|_{z=1, x=0} = \sum_{p=0}^{n-1} \binom{m-1+p}{p} = \binom{m+n-1}{n-1} > 1$. We may now apply (7.13).

We conjecture that the Tutte polynomial $t(G)$ of a connected pregeometry is prime in the unique factorization domain $Z[z, x]$ although the characteristic polynomial is not. This would imply that F or another pregeometry with the same Tutte polynomial is a direct sum factor of G iff $t(F)$ divides $t(G)$.

A small partial result in this direction is that if G is a series-parallel network (or any pregeometry G such that $\beta(G)$ is odd) it could not be factored into positive polynomials since $t(G)|_{z=2, x=2} = 2^{n+m}$ by (4.6) so that any nonconstant factor when evaluated at $z = x = 2$ would be a power of 2 and hence would have to have an even constant term.

8. Two counterexamples. We may now ask just how strong an invariant of a pregeometry G the Tutte polynomial $t(G)$ is. Its strength is attested to by the fact that if G is a rank two pregeometry and $t(G) = t(H)$ then $G \cong H$. (This follows from factoring out loops as in (6.1) and deriving the dual partition numbers p'_j from the coefficients b_{1j} (7.10).) Tutte equivalence, however, is weaker than isomorphism. (Hence converses to propositions like (6.9) are generally false.)

Proposition 8.1. *Let G be the rank three geometry with points a_1, a_2, \dots, a_6*

and two nontrivial copoints $\{a_1, a_2, a_3\}$ and $\{a_4, a_5, a_6\}$. Let H be the rank three geometry on the same set of points with the nontrivial copoints $\{b_1, b_2, b_3\}, \{b_1, b_4, b_5\}$. Then $G \neq H$ but $t(G) = t(H)$.

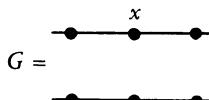
No smaller pregeometries G' , H' have this same property. Neither do any other geometries of the same cardinality.

Proof. Surely $G \neq H$ since every point of G is on exactly one three point line while in H , b_1 is on two three point lines (and b_6 is on none). We will prove that $t(G) = t(H)$ using three methods by way of illustrating some of the previous material.

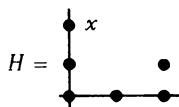
(i) $G \setminus a_4$ and $H \setminus a_4$ are isomorphic, each being the planar five point geometry with one three point line. Also G/a_4 and H/a_4 are isomorphic, each corresponding to the integer partition $2 + 1 + 1 + 1 = 5$. Hence $t(G) = t(G \setminus a_4) + t(G/a_4) = t(H \setminus a_4) + t(H/a_4) = t(H)$.

(ii) G and H are both geometries on six points with exactly two three point lines and nine two point lines. But by (7.11) all the coefficients b_{ij} in the Tutte polynomial of a planar geometry are derivable from the ranks and cardinalities of the various flats.

(iii) Embedded in the real plane,



and

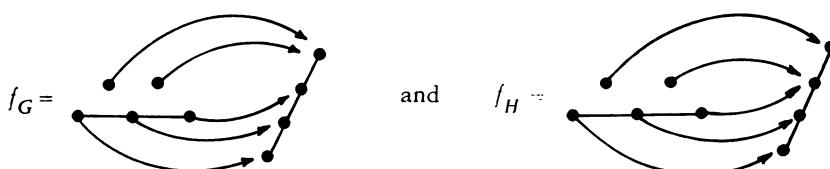


Deleting and contracting the point x gives the desired result.

Since $t(G) = t(H)$ iff $\tilde{t}(G) = \tilde{t}(H)$ by (4.7) one sees that no pair of smaller non-isomorphic pregeometries have the same Tutte polynomial since if $|G| \leq 5$, either \tilde{G} or \tilde{H} has rank less than or equal to two and by the remarks preceding this proposition, G is characterized up to isomorphism by its Tutte polynomial.

It is also easy to check that among the planar six point geometries only G and H have the same number of nontrivial lines of the same cardinality.

It is instructive to show in light of (5.2) how the composite strong map f_G : $G \setminus a_4 \rightarrow G/a_4$ resulting from the decomposition $G < G \setminus a_4 + G/a_4$ differs from the corresponding strong map f_H . Pictorially,



so the two maps differ since in f_G the preimage of the two point atom is disjoint from the three point line and in f_H it is not.

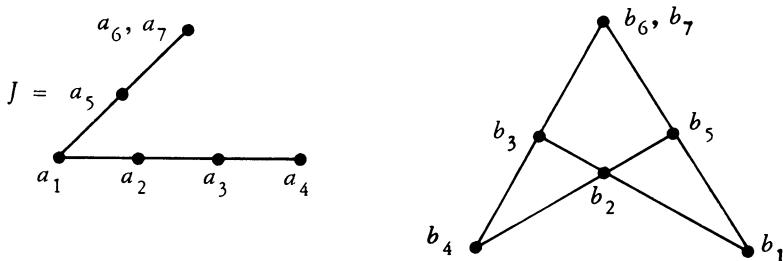
Since series-parallel networks may be characterized by the Tutte polynomial, it becomes a relevant question to ask if other important classes of geometries (binary, graphic, unimodular, planar graphic, transversal, etc.) may be so characterized. One may also ask if from the Tutte polynomial one can enumerate the number of flats, f^{kj} , indexed by corank and nullity as one can for n -partitions (7.9). All these hopes are damped by the following example.

Proposition 8.2. *Let J be the rank three pregeometry with points a_1, a_2, \dots, a_7 and nontrivial lines $\{a_1, a_2, a_3, a_4\}, \{a_1, a_5, a_6, a_7\}, \{a_2, a_6, a_7\}, \{a_3, a_6, a_7\}$, and $\{a_4, a_6, a_7\}$.*

Let P'_4 be the pregeometry with one atom of two points whose underlying geometry is the geometry of partitions of a four element set.

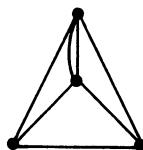
Then $t(J) = t(P'_4)$, the Tutte polynomial calculated in (3.4). But J is not binary (and hence not unimodular, graphic, or planar) but P'_4 is all of these. Also J is transversal while P'_4 is not even the contraction of a transversal pregeometry. In addition P'_4 and J have different numbers of circuits, two point lines, lines, and flats. Such differences also appear in \tilde{J} and \tilde{P}'_4 both of which are geometries.

Proof. In an embedding in the real plane:



The deletions $J \setminus a_4$ and $P'_4 \setminus b_4$ are isomorphic by the strong map $f(a_i) = b_i$, $i \neq 4$. In addition J/a_4 and P'_4/b_4 both represent the integer partition $3 + 2 + 1 = 6$ and hence are isomorphic. Therefore, $t(J) = t(J \setminus a_4) + t(J/a_4) = t(P'_4 \setminus b_4) + t(P'_4/b_4) = t(P'_4)$.

The subset $\{a_1, a_2, a_3, a_4\}$ of J is a four atom line and hence by the representation theorem for binary lattices [10], J is not binary (and hence not graphic). On the other hand, we shall see later (11.1) that P'_4 is the geometry of contractions of the planar graph K'_4 :



and so P'_4 is planar graphic and hence unimodular and binary. It is well-known that P_4 (and hence P'_4) is not transversal, while $\{a_1, a_2, a_3, a_5\}, \{a_2, a_3, a_4\}$ and $\{a_5, a_6, a_7\}$ present J as a transversal pregeometry.

Although J and P'_4 have the same number of atoms (6), four point lines (2), and three point lines (3); J has the three two point lines $\{a_2, a_5\}, \{a_3, a_5\}$, and $\{a_4, a_5\}$ while $\{b_4, b_1\}$ and $\{b_3, b_5\}$ are the only two point lines of P'_4 . Also, P'_4 has 5 four point circuits while J has six.

Since neither P'_4 nor J have any copoint of five or more points, \tilde{P}'_4 and \tilde{J} are geometries which are Tutte equivalent but have differences in circuits, copoints, and representability.

We remark that an interesting problem would be to find other conditions on pregeometries which would guarantee that Tutte equivalence would be as strong as isomorphism. For instance it is not known whether two inequivalent graphic (11) (or even binary) geometries can have the same Tutte polynomial (they can have the same characteristic polynomial). Such investigation should lend insight into such areas as the Ulam graph reconstruction conjecture.

In the next four sections we explore the induced Tutte-Grothendieck ring for some important hereditary classes of pregeometries—chain groups over finite fields (especially $GF(2)$, the two element field), graphic pregeometries, and orientable (unimodular) pregeometries. (An hereditary class of pregeometries is one closed under arbitrary minors, i.e. if G is in the class and $p \in G$ then $G \setminus p$ and G/p are also in the class.)

For these classes the subbidecomposition inherited from \underline{T} can be looked at in a new perspective and classic theorems concerning certain invariants can be furnished with new proofs and sometimes generalized by use of the inherited decomposition structure of $\underline{T}(S')$ (i.e. the Tutte decomposition restricted to one of above classes).

9. Chain groups and the critical problem.

Definition 9.1. Let V_n be a vector space of dimension n over a fixed finite field $GF(q)$. For a spanning set of vectors $S \subseteq V_n$ not including the zero vector (i.e. S is contained in no proper subspace and $0 \notin S$) we define $G(S)$, the *geometry spanned by S*. The closed sets of $G(S)$ are the subspaces of V_n spanned by subsets of S (those subspaces which have a basis in S). Such geometries are called *chain groups*.

If $s, s' \in S$ are both in the same one-dimensional subspace or atom (i.e. $s' = \alpha s, \alpha \in q \setminus \{0\}$), then $G(S) = G(S \setminus \{s\})$ since any basis B for a subspace containing s may be replaced by $(B \setminus \{s\}) \cup \{s'\}$. Hence $G(S) = G(S')$ where $S' \subseteq S$ has no two dependent vectors. If $S = V'_n = V_n \setminus \{0\}$ (or $\lambda \cap S \neq \emptyset$ for all atoms $\lambda \in V_n$) then $L(G(S))$ is the modular lattice of the *projective space* associated with V_n . Thus

any chain group $G(S)$ can be thought of as a subgeometry (sequence of deletions of \sim dimensional subspaces) of $G(V'_n)$ and we will henceforth consider it as such.

Proposition 9.2. *Let S be a spanning subset of lines of V_n , and let $G(S)$ be the associated geometry. Then if $\lambda \in S$ is not an isthmus of $G(S)$, then $G(S \setminus \lambda) = G(S) \setminus \lambda$ and $G(S/\lambda) = G(S)/\lambda$ where S/λ is the projection of $\{S\} \setminus \{\lambda\}$ from the atom (one dimensional subspace) λ . S_1 and S_2 lie in complementary subspaces of V_n (i.e. $S_1 \subseteq V'$, $S_2 \subseteq V''$, and $V = V' \oplus V''$) if and only if $G(S_1 \cup S_2) = G(S_1) \oplus G(S_2)$.*

Proof. By definition $G(S)$ is the subgeometry $G(V'_n) \setminus (\{V'_n\} \setminus S)$, hence

$$G(S) \setminus \lambda = (G(V'_n) \setminus (\{V'_n\} \setminus S)) \setminus \lambda = G(V'_n) \setminus (\{V'_n\} \setminus (S \setminus \lambda)) = G(S \setminus \lambda).$$

The flats of rank k in the contraction $G(S)/\lambda$ are the flats of rank $k+1$ in the vector space V_n which have a basis in S and contain the atom λ . But this is the geometry $G(S/\lambda)$ whose closed sets are subspaces of $V_n/\lambda \simeq V_{n-1}$ spanned by atoms of V_n/λ which are projections of two dimensional subspaces in V_n containing λ and another atom of S .

If $S_1 \subseteq V'$, $S_2 \subseteq V''$, $V = V' \oplus V''$, and $S_1 \cup S_2$ spans V then for every pair of bases $B_1 \subseteq G(S_1)$ and $B_2 \subseteq G(S_2)$, $B_1 \cup B_2$ is independent and spans V , hence is a basis for $G(S_1 \cup S_2)$. Hence $G(S_1 \cup S_2) = G(S_1) \oplus G(S_2)$. Conversely if $G(S_1 \cup S_2) = G(S_1) \oplus G(S_2)$ then let $V' = \overline{S}_1$ and $V'' = \overline{S}_2$ in V_n . Then $V = V' + V''$ since $S_1 \cup S_2$ spans V_n . But $r(V) = r(S_1 \cup S_2) = r(S_1) + r(S_2) = r(V') + r(V'')$ so $V = V' \oplus V''$.

Definition 9.3. The *critical problem* as formulated in [10] for a set of vectors S in a vector space V_n ($0 \notin S$) is the extraction of the *critical exponent*, c —the minimal number of projective hyperplanes H_1, \dots, H_c such that $H_1 \cap \dots \cap H_c \cap S = \emptyset$. Since every nonzero linear functional has a hyperplane for its kernel and every hyperplane is the kernel for some linear functional, the critical problem is equivalent to finding the minimal number c of linear functionals, f_1, \dots, f_c which distinguish the set S , i.e. which for all $s \in S$ there is a functional f_i ($0 \leq i \leq c$) such that $f_i(s) \neq 0$.

Lemma 9.4. *Let S be a spanning set of vectors for a vector space V and let $N(S, k)$ be the set of ordered k -tuples of linear functionals of V which distinguish the set S . Then $(-1)^{r(V)} |N(S, k)|$ is a Tutte bidecomposition invariant on the geometry $G(S)$.*

Proof. For any two dependent vectors $s, s' \in S$ and any linear functional f ; $f(s) = 0$ iff $f(s') = 0$. Hence (f_1, \dots, f_k) distinguishes S iff it distinguishes $S \setminus s$. But by the remarks following (9.1), $G(S) = G(S \setminus s)$, so we may delete any point of S which is in an atom with another point of S and change neither $G(S)$ nor $N(S, k)$. We may thus assume that at most one $s \in S$ lies in any atom of V_n .

If $\lambda \in S$ is not an isthmus of $G(S)$, we may partition $N(S \setminus \lambda, k)$ into two sets—those which vanish on λ and those k -tuples that contain an f_i such that $f_i(\lambda) \neq 0$. The latter set is exactly those k -tuples which distinguish λ as well as $S \setminus \lambda$ and hence form $N(S, k)$. The former set is equicardinal with $N(S/\lambda, k)$ as is shown below.

Extend λ to a basis B for V_n . We then have the dual basis B^* which includes the functional f_λ . It is readily seen that $B^* \setminus \{f_\lambda\}$ can be viewed as a basis for $V_n/q\lambda$ (where $q\lambda$ is the atom of V_n spanned by λ) as well as a basis for those functionals on V_n which vanish on λ . A k -tuple of such functionals distinguishes the points $S \setminus \{\lambda\}$ in V_n iff in $G(V_n')$ their respective hyperplanes intersect S in λ iff the corresponding hyperplanes in $V_n/q\lambda$ distinguish S/λ .

Since S and $S \setminus \lambda$ span the n -dimensional space V_n and S/λ spans an $(n-1)$ -dimensional space we see that the identity $(-1)^n |N(S, k)| = (-1)^n |N(S \setminus \lambda, k)| + (-1)^{n-1} |N(S/\lambda, k)|$ as shown above proves the lemma.

If $V = V' \oplus V''$, $S_1 \subseteq V'$, and $S_2 \subseteq V''$, then picking a basis for V (and hence for the dual space V^*) from $V' \cup V''$, and functional f on V is equal to $f' + f''$ where $f^{(i)}$ is a functional on $V^{(i)}$ and $(f' + f'')(s) = f^{(i)}(s)$ if $s \in S^{(i)}$, $i = 1, 2$.

Such a functional f distinguishes $S_1 \cup S_2$ iff f' distinguishes S_1 and f'' distinguishes S_2 . Hence

$$(-1)^{r(V)} |N(S, k)| = (-1)^{r(V')} |N(S_1, k)| (-1)^{r(V'')} |N(S_2, k)|.$$

Theorem 9.5. *If S spans V , then the number of k -tuples of functionals on V which distinguish S , $|N(S, k)|$, equals $\chi(S, q^k)$ where $\chi(S, \lambda)$ is the characteristic polynomial of $G(S)$ (4.2). Hence if $k = c$, then the number of subsets of hyperplanes equals $\chi(S, q^c)/c! (q-1)^c$.*

Proof. By the previous lemma $(-1)^{r(V)} |N(S, k)|$ is a Tutte bidecomposition invariant. If $G(S)$ is a loop x , it is the origin of a 0-dimensional vector space and no functional can distinguish it. Hence $(-1)^0 |N(x, k)| = 0$. If $G(S)$ is an isthmus z , all q^k k -tuples of linear functionals distinguish it except the k -tuple of all 0 functionals, hence $(-1)^1 |N(z, k)| = 1 - q^k$. By the fundamental Tutte-Grothendieck theorem (3.6),

$$(-1)^{r(V)} |N(S, k)| = t(G)|_{z=1-q^k, x=0} = (-1)^{r(V)} \chi(S, \lambda)|_{\lambda=q^k}.$$

When k is the critical exponent c , the c functionals correspond to distinct hyperplanes.

10. Binary pregeometries. A new characterization of binary pregeometries (those which can be coordinatized over the field $GF(2)$) results in a theorem generalizing the two-color theorem for planar graphs in its full strength to binary pregeometries.

Proposition 10.1 Let G be a binary pregeometry and let $C = \{c_1, \dots, c_n\}$ be a circuit of G . Then for all points $e \in G \setminus C$, C is either a circuit or a direct sum of two circuits C_1 and C_2 in the contraction G/e . In the latter case both $C_1 \cup \{e\}$ and $C_2 \cup \{e\}$ were circuits of G .

Proof. Let $G = C \cup \{e\}$ where C is a circuit. If e is an isthmus or loop of G then $G/e = C$ and the proposition holds. Otherwise G has nullity two and its dual \tilde{G} is a binary pregeometry of rank two. In \tilde{G} , C is a bond which makes e a copoint (atom). A binary pregeometry has been characterized [10] as one in which every coline contains at most three copoints. Hence \tilde{G} has two other atoms C_1 and C_2 which partition $\tilde{G} \setminus e = C$ (since e is not a loop of G hence not an isthmus of G). Therefore in the pregeometry $\tilde{G} \setminus e$, both are bonds while in G , $C_1 \cup \{e\}$ and $C_2 \cup \{e\}$ were both bonds. But this means that in $\tilde{G} \setminus e = G/e$, C is partitioned as the direct sum of the two circuits C_1 and C_2 and both $C_1 \cup \{e\}$ and $C_2 \cup \{e\}$ were circuits of G .

For a more general G , if C is a circuit of G and $e \in \{G\} \setminus C$, then $(G/e) \setminus (\{G\} \setminus (C \cup \{e\})) = (G \setminus (\{G\} \setminus (C \cup \{e\}))) / e$ since both are defined by the derived closure operator $J(A) = \overline{A \cup e} \setminus (\{G\} \setminus C)$ for all $A \subseteq C$ where $\overline{A \cup e}$ is the closure in G . C is a circuit of G iff it is a circuit of any subgeometry of G containing it (a circuit is defined wholly internally in terms of rank and cardinality of its subsets) hence we may delete everything except C and e before we contract e and the above special case applies.

Note that if L_4 is the four point line and $e \in L_4$, then $L_4 \setminus e$ is a circuit but in L_4/e all three two point subsets are circuits. Hence G is binary iff (10.1) holds for all minors of G .

Lemma 10.2. Let G be a binary pregeometry. Then

$$f(G) = \begin{cases} (-1)^{r(G)} & \text{if } G \text{ has all even circuits,} \\ 0 & \text{if } G \text{ has an odd circuit,} \end{cases}$$

is a Tutte invariant.

Proof. If $G = G_1 \oplus G_2$ then the circuit set of G , $\underline{C}(G)$ is the disjoint union of $\underline{C}(G_1)$ and $\underline{C}(G_2)$. Hence $\underline{C}(G)$ contains all even circuits iff $\underline{C}(G_1)$ and $\underline{C}(G_2)$ both do iff $f(G_1)$ and $f(G_2)$ are both nonzero in which case $(-1)^{r(G_1)}(-1)^{r(G_2)} = (-1)^{r(G_1 \oplus G_2)} = (-1)^{r(G)}$. Otherwise, $f(G_1) = 0$ or $f(G_2) = 0$ and $f(G) = f(G_1)f(G_2)$.

We must prove that $f(G) = f(G \setminus e) + f(G/e)$ for any nonfactor $e \in G$. Since $r(G) = r(G \setminus e) = r(G/e) + 1$, dividing the equation through by $(-1)^{r(G)}$ we get

$$(**) \quad |f(G)| = |f(G \setminus e)| - |f(G/e)|.$$

Assume $|f(G)| = 1$. Then G has all even circuits and so does $G \setminus e$ (since for all $C \subseteq G \setminus e$, C is a circuit of G if and only if it is a circuit of $G \setminus e$). Since e is

not an isthmus, e is in some (even) circuit of G and so C/e is a circuit (of odd length) in G/e and (***) becomes $1 = 1 - 0$.

If $f(G) = 0$ and $f(G \setminus e) = 0$ then G must contain an odd circuit C disjoint from e . By the above proposition (10.2) the set C is either a circuit in G/e or the direct sum of two circuits C_1 and C_2 in which case $|C_1| + |C_2| = |C|$. Since C is odd one of the summands and hence one of the circuits in G/e is odd and (***)) is evaluated as $0 = 0 - 0$.

Finally let $f(G) = 0$ and $|f(G \setminus e)| = 1$. Then all circuits not containing e are even but there is an odd circuit C containing e in G . Let C' be any other circuit in G containing e . Then the symmetric difference $\Delta(C, C')$ is the disjoint union of circuits (this follows from the dual of Theorem 15.2 in [10]) all of which must be even. But cardinality of symmetric difference is mod 2 addition, hence C' must also be odd and so in G every circuit containing e is odd while every circuit not containing e is even.

But if D is a circuit in G/e , the rank of D in G/e , $r''(D)$, equals $|D| - 1$ so that in G , $r(D \cup \{e\}) \leq |D \cup \{e\}| - 1$ hence $D \cup \{e\}$ is dependent and contains a circuit D' . If $D' \setminus \{e\}$ were properly contained in D , then D could not be a minimal dependent set in G/e , so that D or $D \cup \{e\}$ is a circuit of G . In either case D is an even circuit and (***)) becomes $0 = 1 - 1$.

Theorem 10.3. *If G is a binary pregeometry and $\chi(G, \lambda)$ is the characteristic polynomial of G , then $\chi(G, 2) = 1$ if G has all even circuits and is 0 otherwise.*

Proof. Let $\bar{f}(G) = 1$ if G has all even circuits and 0 otherwise. Then $(-1)^{r(G)} \bar{f}(G)$ is the Tutte invariant $f(G)$ of (10.2). If z is an isthmus, then z has no circuits and $f(z) = -1$ while if x is a loop, x is an odd circuit and $f(x) = 0$. By the fundamental theorem (3.6),

$$(-1)^{r(G)} \bar{f}(G) = t(G)|_{x=-1, z=0} = (-1)^{r(G)} \chi(G, \lambda)|_{1-\lambda=-1}.$$

Corollary 10.4. *A spanning subset S of vectors in a vector space V over $GF(2)$ will be distinguished by a unique functional (equivalently will be contained in a unique bond of $L(V)$ or will be disjoint from a unique hyperplane) if and only if S has all even circuits.*

Hence a geometry of rank n can be imbedded in the affine space of dimension n over $GF(2)$ if and only if it has no four point lines as minors and no odd circuits.

Proof. This follows from (10.3) and (9.5).

11. Graphic pregeometries. Under certain conditions a pregeometry G has the same point-circuit incidence relationship as the edge-circuit incidence relationship of a multigraph M (graph in which multiple edges and loops are allowed). In that

case we say the pregeometry is *graphic* and we may give a special interpretation to the decomposition and many of the invariants of §4.

We now list some of the basic definitions and properties of multigraphs and their associated pregeometries.

Definition 11.1. A *multigraph* M is a function from a set E into the set of unordered pairs $V \& V$ (i.e. $V \times V / (v_i, v_j) \sim (v_j, v_i)$). The set E is called the *edges* of M and V is called the *vertices*. The restriction $M' = M(E')$ of M to a subset $E' \subseteq E$ is called a *submultigraph* of M .

Denoting the preimage of the diagonal of $V \& V$ by $L(M)$, a *graph* is a multigraph which is 1-1 and where $L(M)$ is empty. In particular the *underlying graph* \bar{M} of a multigraph is the function $M' \circ K$ where M' is the subgraph on the set $E \setminus L(M)$ and K is the kernel of M' . Thus exactly one edge is sent to a vertex pair $[v_i, v_j]$ iff $i \neq j$ and $M^{-1}([i, j])$ is not empty. \bar{M} can be thought of as a maximal *subgraph* (submultigraph which is also a graph) of M .

A *path* of length p from a vertex v to a vertex v' is a $(p + 1)$ -tuple of vertices $(v = v_1, \dots, v_{p+1} = v')$ and p -tuple of edges (e_1, \dots, e_p) such that $M(e_i) = [v_i, v_{i+1}]$. Path connectedness or *connectedness* induces the obvious equivalence relation on V and M is *connected* if $v \sim v'$ for all $v, v' \in V$. Associated with any subset of edges E' , we have the equivalence relation on V in which $v_i \sim v_j$ iff $v_i \sim v_j$ in the submultigraph $M(E')$.

One may form a pregeometry $G(M)$ whose points are the edges of M and whose closure relation is defined for all $E' \subseteq E$ by $\bar{E}' = \{M^{-1}(v_i, v_j)\}$ for all $v_i \sim v_j$. It is easy to verify that this is a geometric closure operator and that the circuits of $G(M)$ are those edges which form a path (v_1, \dots, v_{p+1}) in which $v_1 = v_{p+1}$ but no other vertices are equal. The bases of $G(M)$ are maximal sets which contain no circuits, called the *spanning trees* of a connected multigraph. The one point circuits (loops) of G are the edges in $L(M)$ (the inverse image under M of the diagonal) and the two point circuits of G are those edge pairs which get mapped into the same pair of distinct vertices. Hence if \bar{M} is the graph underlying the multigraph M then $G(\bar{M})$ is the canonical geometry $\bar{G}(M)$ associated with the pregeometry $G(M)$. Also if $E' \subseteq E$, then $G(M(E'))$ is the subgeometry $G(M(E)) \setminus (E \setminus E')$.

M is *two-connected* if $G(M)$ is connected iff M is connected and every two edges lie in some circuit. If $|V| = n$, we define the *complete graph* K_n as a graph in which $|M^{-1}([v_i, v_j])| = 1$ for all $v_i \neq v_j$. Since closed sets of $G(M)$ are in 1-1 correspondence with partitions of the vertex set into path components, the lattice of closed sets of K_n is the lattice of partitions of an n -element set. Every geometry of a graph on n vertices is a subgeometry of K_n .

Proposition 11.2. Let M be a multigraph with associated pregeometry G . Then for all $e \in E$, the deletion $G \setminus e$ is the pregeometry of the submultigraph $G(M(E \setminus \{e\}))$; and if $M(e) = [v_i, v_j]$ then the contraction G/e is the restriction of

the composite function $I \circ M$ to $E \setminus \{e\}$ where $I: V \& V \rightarrow V \& V/v_i \sim v_j$ identifies $[v_i, v]$ and $[v_j, v]$ for all $v \in V$. (G/e can be thought of as coalescing the vertices v_i and v_j and then removing the loop e .)

If M_1 and M_2 are both one-connected multigraphs then the pregeometry $G(M_1 \wedge M_2)$ is the direct sum $G(M_1) \oplus G(M_2)$ where $M_1 \wedge M_2$ (the wedge product of two multigraphs) is the disjoint union of M_1 and M_2 followed by an identification (coalescence) of a vertex $v_1 \in M_1$ and a vertex $v_2 \in M_2$. Conversely, if M' is a one-connected multigraph such that $G(M') = G(M_1) \oplus G(M_2)$ where M_1 and M_2 are one-connected then $M' = M'_1 \wedge M'_2$ where $G(M'_1) = G(M_1)$ and $G(M'_2) = G(M_2)$.

Proof. The M -closure of a set depends only on the vertex partition associated with paths in the set. A subset of $E \setminus \{e\}$ would induce the same vertex partition in M and in $M \setminus e = M(E \setminus \{e\})$ and would thus induce the same closure except that e would not appear in $M \setminus e$. Hence $G(M \setminus e) = G \setminus e$.

Both G/e and $G(M/e)$ are defined on the same edge set, $E \setminus \{e\}$. We will show that the closure operator J in G/e is the same as the closure operator K in $G(M/e)$.

Let $E' \subseteq E \setminus \{e\}$ and $e' \in E \setminus \{e\}$ be arbitrary such that $M(e') = [v_k, v_m]$ (either or both vertices could be v_i or v_j). Then $e' \in J(E')$ iff there is a path in M connecting v_k and v_m with edges in $E' \cup \{e\}$; and $e' \in K(E')$ iff there is a path in M/e connecting v_k and v_m with edges in E' . Assume P is a path linking v_k and v_m in M with edges in $E' \cup \{e\}$. Then deleting e wherever it appears in P , we get a path in M/e since if $P = (\dots, e_n, e, e_{n+1}, \dots)$ then e_n and e_{n+1} share the common vertex $\overline{v_i v_j}$ and $P' = (\dots, e_n, e_{n+1}, \dots)$ is part of a path in M/e .

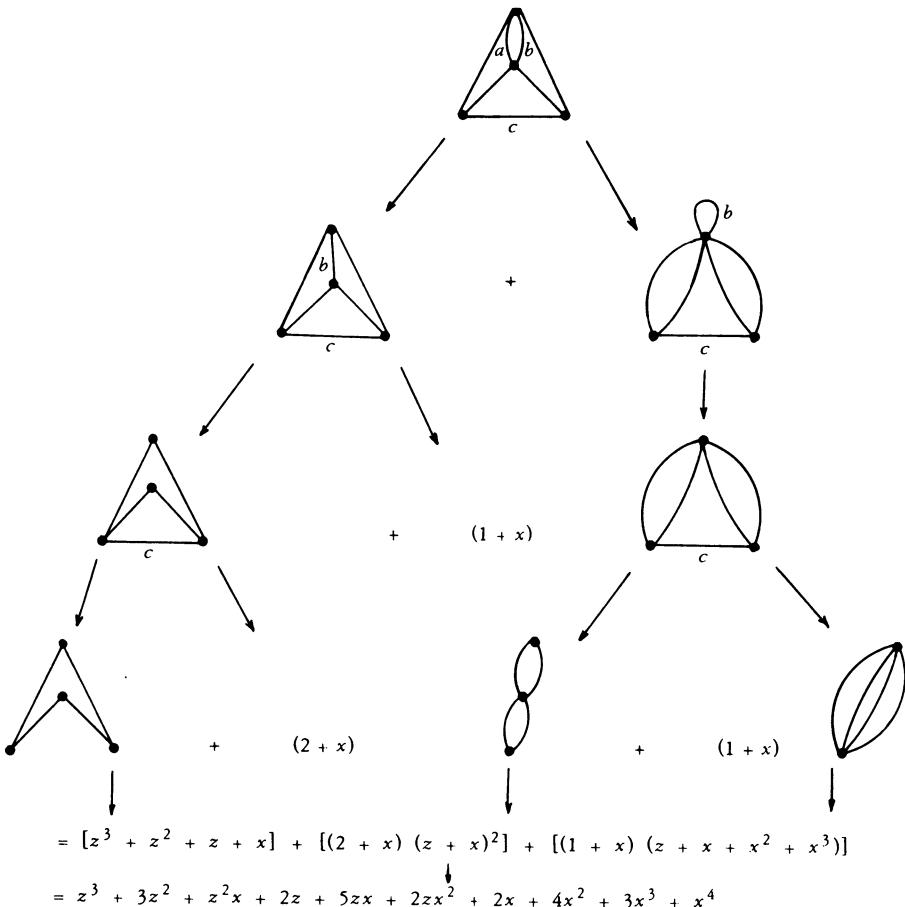
Conversely, assume $(\dots, e_n, e_{n+1}, \dots)$ is a path in M/e in which e_n and e_{n+1} share the common vertex $\overline{v_i v_j}$. Then if e_n and e_{n+1} share a common vertex in M (e.g. v_i), $(\dots, e_n, e_{n+1}, \dots)$ would be part of a path in M while if for example $M(e_n) = [v', v_j]$ and $M(e_{n+1}) = [v_k, v'']$ then $(\dots, e_n, e, e_{n+1}, \dots)$ would be part of a path in M .

$G(M_1) \oplus G(M_2)$ is the pregeometry formed from the disjoint union of edges and circuits of M_1 and M_2 . If $M_1 \wedge M_2$ is the wedge product of M_1 and M_2 in which $v_1 \in M_1$ and $v_2 \in M_2$ are identified then $M_1 \wedge M_2$ is one-connected since every vertex in M_1 is joined to v_1 and every vertex in M_2 is joined to v_2 . But any closed path containing edges in both M_1 and M_2 must list the identified vertex $\overline{v_1 v_2}$ twice. Since M_1 and M_2 are subgraphs of $M_1 \wedge M_2$, $G(M_1)$ and $G(M_2)$ are subgeometries and the circuit set $\underline{C}(M_1 \wedge M_2)$ is the disjoint union $\underline{C}(M_1) \cup \underline{C}(M_2)$. Also $E(M_1 \wedge M_2) = E(M_1) \cup E(M_2)$ and hence $G(M_1 \wedge M_2) = G(M_1) \oplus G(M_2)$.

Conversely, assume M' , M_1 and M_2 are one-connected and $G(M') = G(M_1) \oplus G(M_2)$. Then M' must have subgraphs M'_1 and M'_2 such that $G(M'_i) = G(M_i)$ ($i = 1, 2$). If these subgraphs have no vertices in common M' is not one-connected (no path can join $v_1 \in M'_1$ with $v_2 \in M'_2$). If they have two vertices v and v' in common then since M'_i is connected there is a minimal path P_i in M_i joining v and v' ($i = 1, 2$).

But then $P_1 P_2$ would contain a circuit intersecting both M'_1 and M'_2 which would contradict the definition of direct sum. Since $E(M') = E(M'_1) \cup E(M'_2)$ and $V(M') = V(M'_1) \wedge V(M'_2)$ we are done.

Example 11.3. In analogy with the decomposition of P'_4 (the pregeometry of contractions of K'_4 the complete four graph with a multiple edge) in (3.4) and as an illustration of the techniques in (11.2) we will decompose K'_4 in the Tutte subdecomposition associated with all multigraphs.



Definition 11.4. A *coloring* (or n -coloring) of a multigraph M is a function c from the vertex set of M to a set (of n elements) such that for all $e \in E$, $M(e) = [v, v']$ implies $c(v) \neq c(v')$. A multigraph is n -colorable if it admits an n -coloring.

An *orientation* \vec{M} of M is a function $\vec{M} = O \circ M$ where $O: V \times V \rightarrow V \times V$ assigns to $[v_i, v_j]$ either of the ordered pairs (v_i, v_j) or (v_j, v_i) . (We may think of an orientation as an assignment of a direction to each edge such that, for simplicity, we assume each edge joining the same two vertices has the same direction.)

A *chain* (or *zero-chain*) (with values in A_n) on \vec{M} is a function f from $V(\vec{M})$ into an abelian group A_n of order n . A *coboundary* (*one-coboundary*) is a function \bar{f} from $E(\vec{M})$ to A_n associated with a chain f such that if $\vec{M}(e) = (v_i, v_j)$ then $\bar{f}(e) = f(v_i) - f(v_j)$. It is well known that \bar{f} is a coboundary for some chain iff there exists a subset $\bar{C} \subseteq C$ for every circuit C of M such that $\sum_{e \in \bar{C}} \bar{f}(e) = \sum_{e \in C \setminus \bar{C}} \bar{f}(e)$. Note also that a coboundary \bar{f} is nowhere zero iff f is a coloring.

Theorem 11.5. *Let M be a multigraph with exactly k one-connected components and let G be its associated pregeometry. Then the rank of the pregeometry $r(G)$ equals $|V(M)| - k$; and the number of λ -colorings of M , $c(M, \lambda)$, equals $\lambda^k \chi(G)$.*

Proof. The rank of $G(M)$ is the cardinality of a basis (maximal subset of $E(M)$ containing no circuit) or spanning forest F . But the intersection of F with any connected component M_i of M has cardinality $|V(M_i)| - 1$. Hence $|F| = \sum_{i=1}^k (|V(M_i)| - 1) = |V(M)| - k$.

We will show that $(-1)^{r(M)} c(M, \lambda)/\lambda^k$ is an invariant of the decomposition of the Tutte-Grothendieck group and then apply (3.7). If $e \in G$ is neither an isthmus nor a loop then $M \setminus e$ and M/e have the same number of components as M . Assume $M(e) = [v, v']$. The λ -colorings of $M \setminus e$ may be partitioned into two sets: those in which $f(v) \neq f(v')$ and those which assign to v and v' the same "color". The former set represents all the colorings of M and the latter gives a well defined way to describe all the colorings of M/e ($f(vv') = f(v)$). Hence $c(M, \lambda) = c(M \setminus e, \lambda) - c(M/e, \lambda)$ and dividing that equation by $(-1)^{r(M)} \lambda^k$ we see that $(-1)^{r(M)} c(M, \lambda)/\lambda^k$ is a group invariant. We must evaluate it on all pre-Boolean algebras B_{nm} . But if $m > 0$, B_{nm} contains a loop and for all λ the vertex incident with the loop and hence the pregeometry cannot be colored. If B_{n_i} is a one-connected component of B_n then we may choose a vertex $v_1 \in B_{n_i}$ and color it any one of λ colors. Choose another vertex v_2 adjacent to it (such that $M^{-1}([v_1, v_2])$ is nonempty). Then v_2 may be colored with any color except $f(v_1)$. Choosing recursively any vertex which is incident with vertex chosen before we will have counted $\lambda(\lambda - 1)^{|V|-1} = \lambda(\lambda - 1)^{|E|}$ colorings for B_{n_i} . Note that no vertex will be incident to two distinct vertices chosen before since we would then have a circuit. If B_n has one-connected components of (edge) cardinality n_1, \dots, n_k then since each component may be colored independently we see that $c(B_n, \lambda) = (\lambda(\lambda - 1)^{n_1}) \dots (\lambda(\lambda - 1)^{n_k}) = \lambda^k (\lambda - 1)^{r(B_n)}$. Hence $(-1)^{r(M)} c(M, \lambda)/\lambda^k$ is a group invariant with boundary values $(-1)^n c(B_{nm}, \lambda)/\lambda^k = (1 - \lambda)^n \delta(m, 0)$. But $(-1)^{r(G)} \chi(G)$ is a group invariant (in fact it is a ring invariant) with the same values on B_{nm} . Hence $\chi(G) = c(M, \lambda)/\lambda^k$.

We will apply some of the above techniques to invariants of K_n , the complete graph on n vertices.

Proposition 11.6. Let $M = K_n(a_1, \dots, a_{n-1})$ be the multigraph on the vertex set v_1, \dots, v_n where $|M^{-1}([v_i, v_i])| = 0$ for all i , $|M^{-1}([v_i, v_j])| = 1$ for all $1 \leq i < j \leq n-1$, and $|M^{-1}([v_i, v_n])| = a_i$ for all $1 \leq i \leq n-1$. Hence $K_n(1, 1, \dots, 1) = K_{n+1}(0, \dots, 0) = K_n$. Let $t_n(\dots)$ be the Tutte polynomial of $K_n(\dots)$. Then t_n may be computed from the following recursions:

$$t_n(a_1, \dots, a_i, \dots, a_{n-1}) = t_n(\dots, a_i - 1, \dots)$$

$$+ x^{a_i-1} t_{n-1}(a_1 + 1, \dots, a_{i-1} + 1, a_{i+1} + 1, \dots, a_{n-1} + 1)$$

for all $a_i > 0$, $\sum_{j=1}^{n-1} a_j > 1$; $t_n(0, \dots, 0, 1, 0, \dots, 0) = z t_{n-1}(1, 1, \dots, 1)$; and $t_1 = 1$.

Proof. We contract and delete edges incident with v_n . In the first case if e is not an isthmus and e connects v_n with v_i (where $|M^{-1}([v_i, v_n])| = a_i$) its deletion leaves $a_i - 1$ edges joining v_n and v_i while its contraction identifies v_i and v_n and hence creates $a_i - 1$ loops. For any vertex v_j ($j \neq i, n$), $|M^{-1}([v_i, v_j])| = 1$ and $|M^{-1}([v_n, v_j])| = a_j$, hence $|M^{-1}([v_i v_n, v_j])| = a_j + 1$. The edge e joining v_i and v_n is an isthmus iff it is the only edge incident with v_n in which case $M = e \oplus K_{n-1}$. Finally, $t_1 = 1$ is a boundary condition for the empty graph.

Corollary 11.7.

$$\chi(K_n) = (\lambda - 1) \dots (\lambda - n + 1),$$

$$\mu(K_n) = (-1)^{n-1} (n-1)!,$$

$$\beta(K_n) = (n-2)!.$$

Proof. All these invariants vanish whenever there is a loop so that $f(K_n(a_1, \dots, a_{n-1})) = f(K_n(a_1, \dots, a_i - 1, \dots, a_{n-1}))$ whenever $a_i > 1$. Thus for all the invariants above:

$$f(K_n(a_1, \dots, a_{n-1})) = (k-1)f(K_{n-1}(1, \dots, 1)) + f(K_n(0, \dots, 1, \dots, 0))$$

where k is the number of positive a_i . This recursion for various evaluations of z gives:

$$(-1)^{n-1} \chi(K_n) = (n-2 + (1-\lambda))(-1)^{n-2} \chi(K_{n-1})$$

with initial condition $\chi(K_2) = \lambda - 1$ and evaluating this expression at $\lambda = 0$ we get the recursion for μ . $\beta(K_n(0, \dots, 0, 1, 0, \dots, 0)) = 0$ for $n > 2$ so that $\beta(K_n) = (n-2)\beta(K_{n-1})$ and $\beta(K_2) = 1$.

In [5] Cayley showed that there are n^{n-2} spanning (maximal) trees in the complete graph K_n . The following corollary suggests a new proof of Cayley's formula.

Corollary 11.8. If $s_m = s_m(a_1, \dots, a_n) = \sum_{i_1 < \dots < i_m} a_{i_1} \cdots a_{i_m}$ is the m th degree symmetric polynomial in the variables a_1, \dots, a_n and if $\sum a_i > 0$, then the number $c_{n+1}(a_1, \dots, a_n)$ of spanning trees of $K_{n+1}(a_1, \dots, a_n)$ equals $\sum_{i=1}^n i^{n-1-i} s_i$. In particular, $c_{n+1}(1, 0, \dots, 0) = n^{n-2}$ is the number of spanning trees of K_n .

Proof. One checks that c_i satisfies the recursions of (11.6) when $z = x = 1$.

12. Orientable pregeometries. Minty [13] has generalized the notion of orientability (11.4) to unimodular pregeometries (those which may be coordinatized over any field). Crapo [8] has counted the number of nowhere zero coboundaries for such pregeometries.

This enumeration is shown to be a problem properly addressed to an appropriate Tutte subbidecomposition which lends some insight into the structure of orientable pregeometries.

Definition 12.1. The *Bond incidence matrix* M_B of a pregeometry G has its rows indexed by bonds $B_i \subseteq G$ and columns indexed by points $p_j \in G$ such that

$$M_B(B_i, p_j) = \begin{cases} 1 & \text{if } p_j \in B_i, \\ 0 & \text{otherwise.} \end{cases}$$

We may likewise define the *circuit incidence matrix* M_C of G . An *orientation* of G is a pair of matrices (\vec{M}_B, \vec{M}_C) with entries $+1, -1$, and 0 such that $|\vec{M}_B| = M_B$, $|\vec{M}_C| = M_C$, and $(\vec{M}_B)(\vec{M}_C)^T = 0$ (i.e. an assignment of either 1 or -1 to the nonzero entries of M_B and M_C such that their rows become orthogonal). If such an orientation exists, G is called *orientable*.

Proposition 12.2. Orientability is hereditary. In particular, if G has orientation (\vec{M}_B, \vec{M}_C) then for all nonfactors $p \in G$, G/p is orientable with orientations \vec{M}_B'' and \vec{M}_C'' where \vec{M}_B'' and \vec{M}_C'' are derivable from the matrices \vec{M}_B and \vec{M}_C by deleting the p th column and some of the rows. A similar result holds for $G \setminus p$.

If G is the direct sum of two orientable pregeometries, $G_1 \oplus G_2$, then the pair of matrix direct sums $(\vec{M}_B(G_1) \oplus \vec{M}_B(G_2), \vec{M}_C(G_1) \oplus \vec{M}_C(G_2))$ is an orientation for G .

Finally, if G has orientation (\vec{M}, \vec{M}') , then \tilde{G} has orientation (\vec{M}', \vec{M}) .

Proof. The bonds of G/p are those bonds of G which do not contain p . Hence M_B'' is M_B with all the rows having a one in the p th position deleted and the p th column deleted.

Let C be a circuit of G . If $p \in C$ then $C \setminus \{p\}$ is a circuit of G/p . If $p \notin C$ then C is dependent in G/p . Conversely, if C' is a circuit of G/p then $C' \cup \{p\}$ is dependent in G . Hence M_C'' is M_C with the p th column deleted restricted to rows of minimal support.

If one restricts \vec{M}_B and \vec{M}_C to exactly those rows and columns which are retained by \vec{M}_B'' and \vec{M}_C'' , then the rows of \vec{M}_B'' are orthogonal to R , all rows of \vec{M}_C'' with the p th column deleted (since the rows of \vec{M}_B'' are derived from those rows of \vec{M}_B with a 0 in the p th entry). Hence the rows of \vec{M}_B'' are orthogonal to the rows of \vec{M}_C'' , a subset of R . By duality, identical statements hold for (\vec{M}_B', \vec{M}_C') , an orientation for $G \setminus p$.

If $G = G_1 \oplus G_2$, then the points, circuits, and bonds of G are the disjoint union of the points, circuits, and bonds of G_1 and G_2 respectively. Thus the bond and circuit incidence matrices of G are the respective direct sums and if oriented consistently with the orientations of each factor, the matrix product of the direct sums is the direct sum of the products both of which are zero matrices.

The final statement is a consequence of the fact that a subset A is a circuit of G iff it is a bond of \tilde{G} .

Definition 12.3. Let G be an orientable pregeometry with orientation (\vec{M}_B, \vec{M}_C) and let A_n be an additively written abelian group of order n . A coboundary f of G (with values in A_n) is a function from G to A_n associated with a linear combination $\sum a_i r_i$ of the rows of \vec{M}_B with coefficients in A_n . For all $p \in G$ and $f = \sum a_i r_i$, $f(p) = \sum a_i r_i(p)$ where $a_i r_i(p) = 0, a_i$, or $-a_i$ if the p th coordinate of r_i is 0, 1, or -1 respectively. A cycle of G is defined similarly from \vec{M}_C .

Theorem 12.4. Let G be an orientable pregeometry of rank n and nullity m . Then the number c of nowhere zero coboundaries of G (with values in an abelian group of order k) is given by an evaluation of the characteristic polynomial: $c = (-1)^n \chi(G, k)$. The number of nowhere zero cycles equals $\tilde{\chi}(G, k) = (-1)^m t(G)|_{z=0, x=1-k}$.

Proof. Let $f(G)$ be the number of nowhere zero coboundaries. We will show that $(-1)^r(G)f(G)$ is a Tutte invariant. First note that $f(G_1 \oplus G_2) = f(G_1)f(G_2)$ since the matrix $\vec{M}_B(G)$ is a direct sum so that f is a (nowhere zero) coboundary of $G_1 \oplus G_2$ iff its restriction to G_i is a (nowhere zero) coboundary ($i = 1, 2$).

We will show that for any nonfactor p , $f(G) = f(G \setminus p) - f(G/p)$ by modifying a proof by Crapo [8]. Let T be a basis for G not containing the point p . Then T is a basis for $G \setminus p$. For all $p' \in T$ there is a unique row of the matrix \vec{M}_B (and \vec{M}_B') whose support (nonzero entries) includes p' and otherwise lies in the complement of T . Taking arbitrary linear combinations of such rows, we see that coboundaries in G and $G \setminus p$ can take arbitrary values on T .

Let f be a coboundary of G and let f' be a coboundary of $G \setminus p$ which both agree on the basis $T \subseteq G \setminus p$, and let $p'' \in (G \setminus p) \setminus T$. If C is the unique circuit in $G \setminus p$ (and hence in G) which contains p'' but otherwise lies in T , let r and r' be the rows in \vec{M}_C and \vec{M}_C' respectively corresponding to C . Then $r(p_i)$ can be

thought of as an endomorphism of A_k for all $p_i \in G$ where $r(p_i)$ is the (r, p_i) entry of \vec{M}_C ; and $+1(a) = a$, $-1(a) = -a$, and $0(a) = 0$ for all $a \in A_k$. Then by orthogonality

$$0 = \sum_{p_i \in C} r(p_i)f(p_i) = r(p'')f(p'') + \sum_{p_i \in T \cap C} r(p_i)f(p_i).$$

Thus f is uniquely determined on p'' by its values on T . But $r(p'') = f'(p'') \neq 0$ and $\sum_{p_i \in T \cap C} r(p_i)f(p_i) = \sum_{p_i \in T \cap C} r'(p_i)f'(p_i)$, hence $f(p'') = f'(p'')$ for all $p'' \in \{G\} \setminus \{p\}$. But the values f takes on T also determine its value on p ; hence f' is nowhere zero on $G \setminus p$ iff f is nowhere zero on $\{G\} \setminus \{p\}$ (and has some fixed arbitrary value on p). Thus there is a 1-1 correspondence between nowhere zero coboundaries of $G \setminus p$ and coboundaries of G which are nowhere zero on $\{G\} \setminus \{p\}$.

The rows R of \vec{M}_B'' are exactly those rows R' of \vec{M}_B with a zero in the p th column (12.2). Since coboundaries of G which are zero on p are freely and uniquely determined by their values on a basis T ($p \in T$), such coboundaries are linear multiples of R since $T \setminus \{p\}$ is a basis for G/p ; and so coboundaries can be found taking arbitrary values on $T \setminus \{p\}$ in G/p using rows R which correspond to R' . Hence there is a 1-1 correspondence between nowhere zero coboundaries of G/p and coboundaries in G which are 0 on p but nonzero elsewhere. Thus $f(G \setminus p) - f(G/p)$ counts the number of coboundaries of G which are nowhere zero on $\{G\} \setminus \{p\}$ and in addition not zero on p . But this number is just $f(G)$.

A loop x has no bonds and hence no coboundaries so that $f(x) = 0$ while an isthmus has no circuits and hence no constraints so any nonzero element of A_k can serve as a coboundary; so $(-1)^{r(z)}f(z) = 1 - k$ and we are done by the fundamental theorem (3.6). We count the number of cycles by dualizing and noting that cycles of G are coboundaries of \tilde{G} (see 12.2). Then we may apply (4.3) and (4.7).

We remark that the concepts of orientable pregeometries and nowhere zero coboundaries generalize the topological notions of orientation and nowhere zero one-coboundaries (colorings) of multigraphs.

To see this, we orient our multigraph M as in (11.4). Then if $C = (v_1, e_1, v_2, e_2, \dots, v_{p+1})$ is a circuit and $\vec{M}(e_i) = (v_i, v_{i+1})$ we place an entry of 1 in the (C, e_i) coordinate of \vec{M}_C , and we place a -1 if $\vec{M}(e_i) = (v_{i+1}, v_i)$.

A bond of M consists of all the edges joining two connected blocks V_1 and V_2 of a vertex partition of a connected component of M . Then if $\vec{M}(e) = (v, v')$ we place an entry of 1 in the (B, e) coordinate of \vec{M}_B if $v \in V_1$ and $v' \in V_2$ and a -1 if $v \in V_2$, $v' \in V_1$. If r is the row of \vec{M}_C associated with C and r' the row of \vec{M}_B associated with B then $\sum r(e_i)r'(e_i) = 0$ since there are an even number of edges in common with B and C and giving those edges e_j the induced order from C , $r(e_j)r'(e_j)$ alternates in sign.

To lend some insight into the enumerations (11.5) and (12.4) one notes that every chain of M gives rise to a unique coboundary and every coboundary determines all the values of a chain once the value for one vertex has been specified for each one-connected component of M . The resulting chain is a coloring iff the coboundary is nowhere zero.

The generalization is meaningful since for example it allows one to consider coboundaries of duals of nonplanar graphs.

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