TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 173, November 1972

ON THE CLASSIFICATION OF SIMPLE ANTIFLEXIBLE ALGEBRAS

BY

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ABSTRACT. In this paper, we begin a classification of simple totally antiflexible algebras (finite dimensional) over splitting fields of characteristic $\neq 2, 3$. For such an algebra A let P be the largest associative ideal in A^+ and let N be the radical of P. We say that A is of type (m, n) if N is nilpotent of class m with $\dim A = n$. Define $N_i = N_{i-1} \cdot N$, $N_1 = N$, then A is said to be of type $(m, n, d_1, d_2, \cdots, d_q)$ if A is of type (m, n), $\dim(N_i - N_{i-1}) = d_i$ for $1 \leq i \leq q$ and $\dim(N_i - N_{i+1}) = 1$ for $q < i < m$. We then determine all nodal simple totally antiflexible algebras of types (n, n), $(n-k, n, k+1)$, $(n-2, n)$ (over fields of characteristic $\neq 2, 3$) and of type $(3, 6)$ (over the field of complex numbers). We also give preliminary results for nodal simple totally antiflexible algebras of type $(n-k, n, k, 2)$ and of type $(m, n, d_1, \cdots, d_q)$ in general with $m > 2$ (the case $m = 2$ has been classified by D. J. Rodabaugh).

1. **Introduction.** A totally antiflexible algebra is a nonassociative algebra (finite dimensional) satisfying

$$(x, y, z) = (z, y, x) \tag{1}$$

and

$$(x, x, x) = 0 \tag{2}$$

where $(x, y, z) = (xy)z - x(yz)$. Totally antiflexible algebras have been studied by C. Anderson and D. Outcalt [1], F. Kosier [3] and D. Rodabaugh [4], [5], [6], [7], [8]. These are known to be related to the algebras of commutative nilpotent matrices [8]. There is not much known about the algebras of commutative nilpotent matrices. In this paper we complete the classification of simple nodal totally antiflexible algebras that are related to the algebras of commutative nilpotent matrices discussed by D. A. Suprunenko and R. I. Tyškevič [10] and certain other types.

Define $x^1 = x$, $x^{k+1} = x^k x$ and $x^{\cdot 1} = x$, $x^{\cdot k+1} = x^{\cdot k} \cdot x$. It is known [6] that a totally antiflexible algebra need not be power-associative when char. $\neq 0$. However, A^+ is power-associative so $x^{\cdot m} \cdot x^{\cdot n} = x^{\cdot m+n}$ for all positive integers m and n. We will call y nilpotent or nil if $y^{\cdot n} = 0$ for some n. If x in A implies

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Presented to the Society, August 3, 1971; received by the editors August 19, 1971.
AMS 1969 subject classifications. Primary 1760.
Key words and phrases. Simple algebras, antiflexible algebras, totally antiflexible algebras.

(1) This paper contains major results from the author's doctoral dissertation written under the supervision of Dr. D. J. Rodabaugh.

$x = \alpha \cdot 1 + z$ for α in the base field and z nil, we say that A is nearly nodal. A nearly nodal algebra is nodal if the set of nil elements do not form a subalgebra.

2. **Preliminaries.** We will state some known results on the structure of simple totally antiflexible algebras. We also need (see [1], [7])

Definition 2.1. A field K is said to be a splitting field for an algebra A if every primitive idempotent of A_K is absolutely primitive and if every element in $(A_K)_{11}(e)$ for e primitive can be written as $ke + y$ with k in K and y nil or $y = 0$.

Definition 2.2. Let A be an algebra over a field F of char. $\neq 2, 3$. The mapping $\phi: A \times A \longrightarrow B$ for $B \subseteq A$ will be called an antiflexible map provided $B \subseteq \{x: xy = yx$ for all y in $A\}$ and

(3) ϕ is bilinear over F,

(4) $\phi(x, y) + \phi(y, x) = 0$,

(5) $\phi(x^2, x) = 0$,

(6) $\phi(x, y) = 0$ if y is in B,

(7) $\phi((x, y), z) = 0$.

For α, β in F and antiflexible maps ϕ_1, ϕ_2 define $\alpha\phi_1 + \beta\phi_2$ by

$$\alpha\phi_1 + \beta\phi_2(x, y) = \alpha \cdot \phi_1(x, y) + \beta\phi_2(x, y).$$

It is clear that $\alpha\phi_1 + \beta\phi_2$ is an antiflexible map.

Definition 2.3. Let A be an algebra over a field of char. $\neq 2, 3$ and ϕ be an antiflexible map. Define $A(\phi)$ as the algebra formed from A with multiplication replaced by

$$x * y = xy + \phi(x, y).$$

It is known [4] that A is totally antiflexible if and only if $A(\phi)$ is totally antiflexible. Furthermore, if ψ is an antiflexible map on $A(\phi)$, then $A(\phi)(\psi) = A(\phi + \psi)$.

We now summarize certain results in [1], [4] by the following theorem.

Theorem 2.1. *If A is a simple not associative totally antiflexible algebra, over a splitting field F of char. $\neq 2, 3$ then A^+ is associative, A has an identity element and $A = A_1 + \cdots + A_n$ where $A_i = A_{11}(e_i)$ for e_i primitive. Furthermore, $\phi(x, y) = (1/2)(x, y)$ is an antiflexible map and $A = A^+(\phi)$.*

We will then be interested in those algebras from which simple algebras can be constructed. We say that a totally antiflexible algebra A is nearly simple if there is an antiflexible map ϕ such that $A(\phi)$ is simple.

Theorem 2.2 [8]. *Let A be a totally antiflexible algebra over a field of char. $\neq 2, 3$ and assume A^+ is associative. Then A is nearly simple if and only if A^+ is nearly simple.*

As a result of Theorems 2.1 and 2.2, to find all simple algebras, we need only consider the nearly simple associative and commutative algebras. These are known to contain an identity element. For, if A is a nearly simple associative commutative algebra then there is an antiflexible map ϕ with $A(\phi)$ simple. So by Theorem 1.1, $A(\phi)$ contains an identity element. Therefore A contains an identity element. Hence, throughout this paper unless specified we will assume that A is a totally antiflexible algebra with identity element over a splitting field K of char. $\neq 2, 3$ and that A^+ is associative. Consequently, $A = A_1 + \cdots + A_n$ with $A_i = A_{11}(e_i)$ for e_i primitive and $A_iA_j = 0$ if $i \neq j$. For, since A^+ is associative, $A_{10}(e) + A_{01}(e) = 0$ for any idempotent e (see also [5], [7]). In addition, since K is a splitting field, each element in A_i has the form $\alpha e_i + z$ for α in K and z nil. Thus A has a basis consisting of primitive idempotents and nil elements. We define the following sets:

(8) $N = \{x: x \text{ in } A \text{ and } x \text{ is nil}\}$,

(9) $N_i = N_{i-1} \cdot N$ with $N_1 = N$,

(10) $\overline{N_i} = N_i - N_{i+1}$ (quotient or difference algebra),

(11) $N_i' = \{x: x \text{ is in } N_i \text{ but not in } N_{i+1}\}$,

(12) $M_i = \{x: x \cdot N \subseteq M_{i-1}\}$ with $M_0 = 0$.

M_1 $(= M)$ is called the annihilator of N.

For each x in A^+ define $T_x: y \longrightarrow y \cdot x$ and note that, since there is an identity element 1 in A and A^+ is associative, $x \longrightarrow T_x$ is an isomorphism of A^+ onto $\{T_x\}$. Thus, if $\dim A = n$, we can think of either A or one of its subalgebras as an algebra of commutative $n \times n$ matrices.

For some m, $N_m = 0$ with $N_{m-1} \neq 0$ and so $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_m = 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = N$ and $N_i \subseteq M_{m-i}$ for all i. We say that A (or N) is of type (m, n) if A^+ (or N^+) is isomorphic to an algebra of commutative $n \times n$ matrices for $n = \dim A$ with $N_m = 0 \neq N_{m-1}$. The algebra A (or N) is said to be of class m. An algebra A (or N, the radical of A^+) is of type $(m, n, d_1, \cdots, d_q)$ if A (or N) is of type (m, n), $\dim \overline{N_i} = d_i$ for $1 \leq i \leq q$ and $\dim \overline{N_i} = 1$ for $q < i \leq m-1$. Note that if $N_i = N_{i+1}$ then $N_i = N_j$ for all $j \geq i$. Hence either $\overline{N_i} = 0$ or $\dim \overline{N_i} \geq 1$. It is known [8] that if $\dim \overline{N_i} = 1$ for some i then $\dim \overline{N_{i+k}} = 1$ for $k = 0, \cdots, m-i-1$. Hence we can assume that $d_i > 1$ for $1 \leq i \leq q$.

By Theorems 2.1 and 2.2 the problem of classifying all simple totally antiflexible algebras is reduced to finding

(i) a characterization of all nearly simple associative commutative algebras,

(ii) all possible antiflexible maps ϕ that give rise to simple antiflexible algebras.

The following two theorems summarize results from [8].

Theorem 2.3. *Let P be an associative, commutative algebra over a field of char. $\neq 2, 3$ and let ϕ be a bilinear map from $P \times P \longrightarrow B \subseteq P$ such that $\phi(P, B) = 0$.*

Then ϕ *is an antiflexible map if and only if, for every* n; $y_1, \cdots, y_n$,

$$\sum_{j=1}^{n} \phi(\pi_{i\neq j} y_i, y_j) = 0. \tag{13}$$

If x is in M_i, y in N_j, z in N_{j+1}, $j \geq i \geq 1$, and ϕ an antiflexible map, then $x \cdot y = 0$ and $\phi(x, z) = 0$ [8]. Now we state necessary and sufficient conditions for a totally antiflexible algebra to be simple.

Theorem 2.4. *Let* A *be a totally antiflexible algebra over a splitting field of char.* $\neq 2, 3$ *with* A^+ *associative. Then* A *is simple if and only if*

(14) *for every nonzero* x *in* M_1 *there is a* y *in* N *with* $(x, y) \neq 0$,

(15) *no element of* $\{e(x, y)\}$ *generates a proper ideal where* e *is a primitive idempotent,*

(16) *for each primitive idempotent* e *in* A, $\{e(x, y)\}$ *is not nil.*

The proof of the following theorem is similar to that of Theorem 2.4.

Theorem 2.5. *Let* A *be a totally antiflexible nodal algebra, over a field of char.* $\neq 2, 3$ *with* A^+ *associative. Then* A *is simple if and only if*

(17) *for every nonzero* x *in* M_1 *there is a* y *in* N *with* $(x, y) \neq 0$,

(18) *for each* x *in* M_1 *and* y *in* N_1', (x, y) *does not generate a proper ideal of* A.

Proof. If A is simple, then the conclusion follows from Theorem 2.4.

Conversely, suppose A satisfies (17) and (18) and J is a proper ideal of A. Let x be in J, $x \neq 0$; then $x = \alpha \cdot 1 + z$ for some α in F and z nil. If $\alpha \neq 0$, write $u = (-1/\alpha)$. Then for some m, $u^{\cdot m} = 0 \neq u^{\cdot (m-1)}$ and so $1 = (1 - u) \cdot (1 + u + \cdots + u^{\cdot (m-1)})$ is in J. Hence $J = A$ which is impossible. Therefore suppose $\alpha = 0$. If z is in M, let $u = z$ and if z is not in M_1, then there is a y in N with $u = z \cdot y$ in $M_1 \cap J$. Hence by (17) there is a v in N such that $(u, v) \neq 0$ and (u, v) is in J. Therefore by (18) it follows that $J = A$, which is a contradiction. Consequently A is simple.

In a similar way we can prove the following.

Lemma 2.1. *If* A *is a nodal totally antiflexible algebra over a field of char.* $\neq 2, 3$ *and if for* $\phi(x, y) = (1/2)(x, y)$, $N \cap \{\phi(x, y)\} = 0$, *then* A *is simple if and only if for every nonzero* x *in* M_1 *there exists a* y *in* N *such that* $\phi(x, y) \neq 0$.

Proof. We only need to prove that the condition is sufficient. Hence suppose that for every nonzero x in A there is a y in N such that $\phi(x, y) \neq 0$ and J is a proper ideal of A. Let x be in A, $x \neq 0$. Then $x = \alpha \cdot 1 + z$ with α in F and z nil. If $\alpha \neq 0$, define $u = (-1/\alpha)z$, then for some m, $(1/\alpha)x \cdot (1 + u + \cdots + u^{\cdot m}) = 1$

is in J which is impossible. Therefore, suppose that $\alpha = 0$, for every x in J and $J \subseteq N$. Let x be in J, $x \neq 0$. If x is not in M, then for some x, $x \cdot z$ is in M, and so there is a y with $\phi(x \cdot z, y) \neq 0$. But then $\phi(x \cdot z, y)$ is not nil with $\phi(x \cdot z, y)$ in J, which contradicts the fact that $J \subseteq N$. Hence A is simple.

For an antiflexible map ϕ on an algebra A we define

$$H(\phi) = \{\phi(x, y): x, y \text{ are in } A\} \quad \text{and} \quad (x\phi y) = x * y - y * x$$

where $x * y = xy + \phi(x, y)$. Thus $(x\phi y) = (x, y) + 2\phi(x, y)$.

If $H(\phi) = Z(A)$ (the center of A) and if $Z(A) \neq \{0\}$ then $Z(A)$ is a field [9]. Hence $H(\phi) \cap N = Z(A) \cap N = \{0\}$ and we have proved the following lemma.

Lemma 2.2. *Let A be a nodal totally antiflexible algebra over a splitting field F of char. $\neq 2, 3$. If $H(\phi) = Z(A)$ then $H(\phi) \cap N = \{0\}$.*

The following two theorems summarize the results on algebras of class 2 [8].

Theorem 2.6. *Let P be a nearly nodal associative commutative algebra over a field of char. $\neq 2, 3$ with $N \cdot N = 0$. Let $\{x_i\}_{i=1}^n$ be a basis for N. If ϕ is an antiflexible map them $P(\phi)$ is simple if and only if there is a nonsingular matrix $X = (x_{i,j})$ with $\phi(x_i, x_j) = x_{i,j}$.*

Theorem 2.7. *Let P be an associative commutative algebra over a splitting field F with $N \cdot N = 0$. Then P is nearly simple if and only if*

(19) *there is an identity element in P,*

(20) *for every primitive idempotent e, $\dim P_{11}(e) \geq 3$,*

(21) *either* 1 *is not primitive or* $\dim P$ *is odd.*

Next, three theorems summarize results on algebras of types (n, n) and $(n - k, n, k + 1)$ from [8].

Theorem 2.8. *Let N be an associative commutative nilalgebra of dimension $n - 1$ over a field F. If N is of class m and if* $\operatorname{char} F = 0$ *or* $\operatorname{char} F \geq m$ *or* $\operatorname{char} F \geq n - m + 2$ *or* $\dim \overline{N_k} = 1$ *with* $\operatorname{char} F > k$ *then there is an x in N with $x^{m-1} \neq 0$.*

Theorem 2.9. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative nilalgebra of type (n, n) over a field F of char. $\neq 2, 3$. Then P is nearly simple if and only if* $\operatorname{char} F$ *divides n.*

Theorem 2.10. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative nilalgebra of type $(n - k, n, k + 1)$ with $n - k > 2$ over a field F of char. $\neq 2, 3$. The algebra P is nearly simple if and only if the following holds:*

(22) *N is spanned by $a, \cdots, a^{n-k-1}, b_1, \cdots, b_k$ where $ab_i = b_i b_j = 0$; $i, j = 1, 2, \cdots, k$,*

(23) *either* $n - k = \text{char } F$ *with* k *even or* $n - k = m \text{ char } F$ *for* $m > 1$.

3. **Nodal algebras of type** (n, n), $(n - k, n, k + 1)$ **and** $(n - 2, n)$. We now focus attention on nodal algebras. If A is such an algebra then $\dim A = 1 + \dim N$. Since all simple antiflexible algebras of class 2 have been determined [8], in classifying simple antiflexible algebras of type (m, n), we can assume $m > 2$. The following theorem gives an answer to the second question for algebras of type (n, n). Notationally we use a^0 for 1.

Theorem 3.1. *Let* $P = F \cdot 1 \oplus N$ *be a nearly simple, associative, commutative, nearly nodal algebra of type* (n, n) *over a field of* F *of char.* $\neq 2, 3$. *If* ϕ *is an antiflexible map on* P *and if* a *is chosen in* N *so that* $a^{n-1} \neq 0$, *then* $P(\phi)$ *is simple if and only if*

(24) $H(\phi)$ *is a subset of the algebra generated by* 1 *and* a^p, *and* $n = mp$,

(25) $\phi(a^{n-1}, a) = \sum_{i=0}^{m-1} \beta_{ip} a^{ip}$, β_{ip} *in* F *with* $\beta_0 \neq 0$.

Proof. By Theorems 2.8 and 2.9 (see also Theorem 5.3 of [8]) there is an element a in P such that P is generated by 1 and a and $n = mp$ for some positive integer m. Also using Theorem 2.3 we have $\phi(a^i, a^j) = j\phi(a^{i+j+1}, a) = -i\phi(a^{i+j-1}, a)$. Hence, $\phi(a^i, a^j) = 0$ if either $p | i$ (p divides i) or $p | j$ or $p \nmid (i + j)$ or $i + j > n$.

Now suppose $P(\phi)$ is simple. Then $\phi(a^{n-1}, a) \neq 0$. Let x, y be in P. Then

$$\phi(x, y) = \sum_{i=0}^{n-1} \alpha_i a^i, \qquad \alpha_i \text{ in } F,\ i = 0, \cdots, n - 1.$$

Since ϕ is an antiflexible map on P, we have $0 = \phi(\phi(x, y), a^{n-1}) = \alpha_1 \phi(a, a^{n-1})$. But $\phi(a^{n-1}, a) \neq 0$, so $\alpha_1 = 0$. Suppose $\alpha_i = 0$ for all $i < k$ with $p \nmid i$ and $p \nmid k$. Then

$$0 = \phi(\phi(x, y), a^{n-k}) = \alpha_k \phi(a^k, a^{n-k}) = -k\alpha_k \phi(a^{n-1}, a),$$

which implies that $\alpha_k = 0$. Hence by mathematical induction, $\alpha_i = 0$ for all i with $p \nmid i$ and so $\phi(x, y) = \sum_{i=0}^{m-1} \alpha_{ip} a^{ip}$. This proves (24).

If $x = a^{n-1}$ and $y = a$, we get $\phi(a^{n-1}, a) = \sum_{i=0}^{m-1} \beta_{ip} a^{ip}$, β_{ip} in F. If $\beta_0 = 0$, then a generates a proper ideal of $P(\phi)$. Hence $\beta_0 \neq 0$ and so (25) is satisfied.

Conversely, suppose ϕ satisfies (24) and (25) and J is an ideal of $P(\phi)$. Let x be in J, $x \neq 0$, then $x = \sum_{i=1}^{n-1} \alpha_i a^i$, α_i in F, $i = 1, \cdots, n - 1$. Let j be the least integer such that $\alpha_j \neq 0$. If $p | j$ then

$$\phi(x * a, a^{n-j-1}) = \phi(xa, a^{n-j-1}) = -(j + 1)\alpha_j \left(\sum_{i=0}^{m-1} \beta_{ip} a^{ip} \right).$$

Therefore $(-1/(j + 1)\alpha_j \beta_0)\phi(xa, a^{n-j-1}) = 1 - z$ is in J, z is nil.

On the other hand, if $p \nmid j$ then

$$(-1/j\alpha_j\beta_0)\phi(x, a^{n-j}) = (-1/j\beta_0)\phi(a^j, a^{n-j}) = 1 - z$$

is in J for some z; z is nil. Hence in either case for some positive integer r, $(1-z)(1+z+\cdots+z^r) = 1$ is in J. Therefore $J = P(\phi)$ and so $P(\phi)$ is simple.

In a similar way we prove the following theorem.

Theorem 3.2. *Let $P = F \cdot 1 \oplus N$ be a nearly simple, associative commutative nearly nodal algebra of type $(n-k, n, k+1)$, $n-k > 2$, over a field F of characteristic $p \neq 2, 3$. Let ϕ be an antiflexible map and let $a, b_1, \cdots, b_k$ be as in Theorem* 2.10. *Then $P(\phi)$ is simple if and only if*

(26) *$H(\phi)$ is contained in the algebra generated by* 1 *and a^p, $n-k = mp$,*

(27) $\phi(a^{n-k-1}, a) = \sum_{i=0}^{m-1} \alpha_{ip}a^{ip}$, α_{ip} *in F, $i = 0, \cdots, m-1$, with $\alpha_0 \neq 0$.*

(28) *$\phi(b_i, b_j) = x_{i,j}$, $i, j = 1, \cdots, l$, where $X = (x_{i,j})$ is the matrix of Theorem* 2.6, *$l = k$ if k is even and $l = k-1$ if k is odd. Furthermore, if k is odd and $\phi(b_k, b_j) = 0$ for all j, then for any α, β in F, $\phi(\alpha a^{n-k-1} + \beta b_k, a) \neq 0$.*

Proof. As in the last theorem we note that $\phi(a^i, a^j) = 0$ if either $p|i$ or $p|j$ or $p \nmid (i+j)$ or $i+j > n$. Also, since $a^2 b_i = 0$, using Theorem 2.3 it follows that $\phi(a^s, b_i^r) = 0$ if either $s > 1$ or $r > 1$, $i = 1, \cdots, k$. Now suppose $P(\phi)$ is simple. Then $\phi(a^{n-k-1}, a) \neq 0$. Let x, y be in P. Then

$$\phi(x, y) = \sum_{i=0}^{n-k-1} \gamma_i a^i + \sum_{j=1}^{k} \delta_j b_j; \qquad \gamma_i, \delta_j \text{ are in } F.$$

Since $\phi(\phi(x, y), a^{n-k-1})$ is an antiflexible map, we have $0 = \phi(\phi(x, y), a^{n-k-1}) = \gamma_1\phi(a, a^{n-k-1})$ which implies that $\gamma_1 = 0$. Suppose that $\gamma_i = 0$ for all $i < l$ with $p \nmid i$, $p \nmid l$ and $n-k-l > 1$. Then

$$0 = \phi(\phi(x, y), a^{n-k-l}) = -l\gamma_l\phi(a^{n-k-l}, a).$$

Therefore, $\gamma_l = 0$. Thus by mathematical induction $\gamma_i = 0$ for $1 \leq i \leq n-k-2$ with $p \nmid i$ and we have $\phi(x, y) = \sum_{i=0}^{m-1} \gamma_{ip}a^{ip} + Y_1$ where $Y_1 = \gamma_{n-k-1}a^{n-k-1} + \sum_{j=1}^{k} \delta_j b_j$ is in M. For any z in P, $0 = \phi(\phi(x, y), z) = \phi(Y_1, z)$, so $Y_1 = 0$. Consequently, (26) is satisfied.

On taking $x = a^{n-k-1}$, $y = a$, we get

$$\phi(a^{n-k-1}, a) = \sum_{i=0}^{m-1} \alpha_{ip}a^{ip}; \qquad \alpha_{ip} \text{ is in } F.$$

If $\alpha_0 = 0$ then a generates a proper ideal of $P(\phi)$. So $\alpha_0 \neq 0$ and we have (27). The first part of (28) follows from Theorem 2.7. Now if k is odd, $\phi(b_k, b_j) = 0$ for $1 \leq j \leq k$, and if α and β are arbitrary elements of F, then $\phi(\alpha a^{n-k-1} + \beta b_k, b_i) = 0$

for $1 \leq i \leq k$. Since $\alpha a^{n-k-1} + \beta b_k$ is in M, $\phi(\alpha a^{n-k-1} + \beta b_k, a) \neq 0$.

Conversely, suppose ϕ satisfies (26), (27) and (28) and J is an ideal of $P(\phi)$. Let x be in J, $x \neq 0$. Then

$$x = \sum_{i=0}^{n-k-1} \gamma_i a^i + \sum_{j=1}^{k} \delta_j b_j; \qquad \gamma_i, \delta_j \text{ are in } F.$$

Since $x \neq 0$, at least one of the γ_i or one of the δ_j is different from zero. Thus we have two cases:

Case 1. Suppose $\gamma_i \neq 0$ for some i, $1 \leq i \leq n-k-2$. Let j be the least integer such that $\gamma_i \neq 0$. If $p \nmid j$ then since $n-k-j > 1$, $\phi(x, a^{n-k-j}) = \gamma_j \phi(a^j, a^{n-k-j}) = -j\gamma_j(\Sigma_{i=0}^{m-1} \alpha_{ip} a^{ip})$. Therefore $(-1/j\gamma_j \alpha_0)\phi(x, a^{n-k-j}) = 1 - z$ is in J with z nil. Similarly if $p|j$ then $(-1/(j+1)\gamma_j \alpha_0)\phi(x * a, a^{n-k-j-1}) = 1 - z$ is in J with z nil. Thus, in either case, for some positive integer r we have $1 = (1-z)(1 + z + \cdots + z^{r-1})$ is in J. Hence $J = P(\phi)$ and so $P(\phi)$ is simple.

Case 2. Suppose $\gamma_i = 0$ for $0 \leq i \leq n-k-2$ and $\delta_j \neq 0$ for some j, say $j = l$. If $l < k$, then by Theorem 2.7 there is a j such that $\phi(b_l, b_j) = x_{l,j}$ is in $J \cap F$ and $x_{l,j} \neq 0$. Hence 1 is in J and so $J = P(\phi)$. On the other hand, if $\delta_k \neq 0$ then if k is even then there is a j such that $\phi(b_k, b_j) = x_{l,j}$ is in $J \cap F$, $x_{l,j} \neq 0$ and so $J = P(\phi)$. If k is odd and if $\phi(b_k, b_j) \neq 0$ for some j then as in Case 1 it can be shown that $J = P(\phi)$. Therefore, let us now assume that k is odd and $\phi(b_k, b_j) = 0$ for all $j = 1, \cdots, k$. Then since $x = \gamma_{n-k-1} a^{n-k-1} + \delta_k b_k$, by (28), $\phi(x, a) \neq 0$ and once again it can be shown that $J = P(\phi)$. Hence $P(\phi)$ is simple and the proof is complete.

Having determined all nodal simple antiflexible algebras of types (n, n) and $(n-k, n, k+1)$, our next interest is those of type $(n-2, n)$. If $\dim \overline{N_1} = 2$ and $\dim \overline{N_2} = 1$ then $\dim \overline{N_i} = 1$ for all $2 \leq i \leq n-3$ so that $\dim \overline{N} = n-2$. Since $\dim \overline{N} = n-1$ we conclude that either $\dim \overline{N_1} = 3$ and $\dim \overline{N_i} = 1$ for $2 \leq i \leq n-3$ or $\dim \overline{N_1} = 2$, $\dim \overline{N_2} = 2$ and $\dim N_i = 1$ for $3 \leq i \leq n-3$. We have proved the following lemma.

Lemma 3.1. *If N is of type $(n-2, n)$ then N is either of the type $(n-2, n, 3)$ or of the type $(n-2, 2, 2)$.*

We have determined all simple nodal totally antiflexible algebras of type $(n-2, n, 3)$. So now we will be interested in algebras of type $(n-2, n, 2, 2)$ and in general of type $(n-k, n, k, 2)$.

Theorem 3.3. *Let N be an associative, commutative nilalgebra of type $(n-k, n, k, 2)$ over a field of char. $\neq 2, 3$. Then there exist a, b_i, c, $i = 1, \cdots, k-1$, such that b_i is in N_1', c is in N_2' and $\{a, \cdots, a^{n-k-1}, b_1, \cdots, b_{k-1}, c\}$ is a basis of N with $a^2 b_i = 0$, $ab_ib_j = \beta_{i,j} a^{n-k-1}$, $b_ib_jb_l = \gamma_{i,j,l} a^{n-k-1}$, $i, j, l = 1, \cdots, k-1$. Furthermore, c can be chosen to be either ab_1 or b_1^2 or $b_1 b_2$.*

Proof. By Theorem 2.8, since $\dim \overline{N_3} = 1$, there is an element a in N with $a^{n-k-1} \neq 0$. Let $c_1, \cdots, c_{k-1}, c$ be chosen so that $c_1, \cdots, c_{k-1}$ are in N_1', c in N_2' and $\{a, \cdots, a^{n-k-1}, c_1, \cdots, c_{k-1}, c\}$ is a basis for N. (This is possible since $\dim \overline{N_1} = k$ and $\dim \overline{N_2} = 2$.) Then N_3 is spanned by $a^3, \cdots, a^{n-k-1}$, so

$$a^2 c_i = \sum_{j=3}^{n-k-1} \alpha_{i,j} a^j = a^2 \sum_{j=3}^{n-k-1} \alpha_{i,j} a^{j-2}; \qquad i = 1, \cdots, k-1.$$

Define $b_i = c_i - \sum_{j=3}^{n-k-1} \alpha_{i,j} a^{j-2}$; $i = 1, \cdots, k-1$. Clearly $a^2 b_i = 0$, $i = 1, \cdots, k-1$, and $\{a, \cdots, a^{n-k-1}, b_1, \cdots, b_{k-1}, c\}$ is a basis of N with b_i in N_1'. Since $ab_i b_j$ is in N_3, $ab_i b_j = \sum_{l=3}^{n-k-1} \beta_{i,j,l} a^l$. Then we have $0 = a^2 b_i b_j = \sum_{l=3}^{n-k-2} \beta_{i,j,l} a^{l+1}$ and so $\beta_{i,j,l} = 0$ for $l = 3, \cdots, n-k-2$; $i, j = 1, \cdots, k-1$. Defining $\beta_{i,j} = \beta_{i,j,n-k-1}$ one gets $ab_i b_j = \beta_{i,j} a^{n-k-1}$; $i, j = 1, \cdots, k-1$.

Also, $ab_i b_j b_l = \beta_{i,j} a^{n-k-1} b_l = 0$ (since $n - k \geq 3$); $i, j, l = 1, \cdots, k-1$. Now $b_i b_j b_l$ is in N_3 so $b_i b_j b_l = \sum_{t=3}^{n-k-1} a^t$ and $0 = ab_i b_j b_l = \sum_{t=3}^{n-k-2} \gamma_{i,j,l,t} a^{t+1}$. Hence $\gamma_{i,j,l,t} = 0$ for $i, j, l = 1, \cdots, k-1$ and $t = 3, \cdots, n-k-2$. Defining $\gamma_{i,j,l} = \gamma_{i,j,l,n-k-1}$, we have $b_i b_j b_l = \gamma_{i,j,l} a^{n-k-1}$.

To prove the second part of the theorem, we only need to show that at least one of ab_1, b_1^2, $b_1 b_2$ is not in N_3. If either ab_i or b_j^2 or $b_i b_j$ is not in N_3 for some i or j then one can rearrange the b_i's and the proof will be complete. Hence assume that ab_i, $b_i b_j$ are in N_3 for all $i, j = 1, \cdots, k-1$. Since c is in N_2', we have

$$c = \left(\alpha a + \sum_{i=1}^{k-1} \alpha_i b_i\right)\left(\beta a + \sum_{j=1}^{k-1} \beta_j b_j\right) + z; \qquad z \text{ in } N_3;$$

$$= \alpha\beta a^2 + z'; \qquad z' \text{ in } N_3.$$

Therefore, $c, a^2, a^3, \cdots, a^{n-k-1}$ are linearly dependent which is impossible and we are done.

Lemma 3.2. *Let ϕ be an antiflexible map on an associative commutative algebra P of char. $\neq 2, 3$ with $a^2 b = 0$. Then*

(29) $\phi(ab^2, a^s) = 0$ *for* $s \geq 0$,

(30) $\phi(a^s, b) = 0$ *for* $s > 2$,

(31) $\phi(ab, a^s) = 0$ *for* $s > 1$,

(32) $\phi(a^r, b^s) = 0$ *for* $r > 1$ *and* $s > 1$,

(33) $2\phi(ab, a) + \phi(a^2, b) = 0$,

(34) $2\phi(ab, b) + \phi(b^2, a) = 0$.

Proof. (29) is obvious for $s = 0$. If $s > 0$, consider ab, b, a^s, then, by Theorem 2.3, $\phi(ab^2, a^s) + \phi(a^s b, ab) + \phi(a^{s+1} b, b) = 0$. Since $a^s b = 0$ for $s > 1$

and for $s = 1$, $\phi(ab, ab) = 0$, $\phi(a^s b, ab) = 0 = \phi(a^{s+1}b, b)$. Therefore $\phi(ab^2, a^s) = 0$. To prove (30) let $s > 2$ and consider $a_1, \cdots, a_s, b$ where $a_i = a$ for $i = 1, \cdots, s$. Then using Theorem 2.3 we have $\phi(a^s, b) + s\phi(a^{s-1}b, a) = 0$ which implies that $\phi(a^s, b) = 0$. Similarly, to prove (31) consider a, b, a^s and apply Theorem 2.3.

Now suppose $r = 2$ and $s > 1$. Consider $a^2, b_1, \cdots, b_s$ with $b_i = b$ for $i = 1, \cdots, s$. Then by Theorem 2.3 we have $\phi(b^s, a^2) + s\phi(a^2b^{s-1}, b) = 0$. Since $a^2b^{s-1} = 0$, $\phi(a^2, b^s) = 0$. If $r > 2$, consider $a_1, \cdots, a_r, b^s$ where $a_i = a$ for all $i = 1, \cdots, r$, then using Theorem 2.3, $\phi(a^r, b^s) + r\phi(a^{r-1}b^s, a) = 0$ which implies $\phi(a^r, b^s) = 0$. Considering a, a, b and applying Theorem 2.3 we get (33), and (34) follows immediately by using Theorem 2.3 for a, b, b.

We are now ready to prove the following theorem.

Theorem 3.4. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative algebra of type $(n - k, n, k, 2)$ with $n - k > 3$ over a field F of char. $\neq 2, 3$. If P is nearly simple then N is spanned by $a, \cdots, a^{n-k-1}, b_1, \cdots, b_{k-1}, c$ where $a^2b_i = ab_ib_j = b_ib_jb_l = 0$, $i, j, l = 1, \cdots, k-1$, c, is either ab_1 or b_1^2 or b_1b_2 and $n - k = m$* char F *for $m > 0$.*

Proof. By Theorem 3.3, there are elements $a, b_1, \cdots, b_{k-1}, c$ with N spanned by $a, \cdots, a^{n-k-1}, b_1, \cdots, b_{k-1}, c$. Furthermore, $a^2b_i = 0$, $ab_ib_j = \beta_{i,j}a^{n-k-1}$, $b_ib_jb_l = \gamma_{i,j,l}a^{n-k-1}$ for $i, j, l = 1, \cdots, k-1$ and c is either ab_1 or b_1^2 or b_1b_2. From this it is clear that M is a subspace of the space spanned by $\{a^{n-k-1}, ab_i, b_ib_j;\ i, j = 1, \cdots, k-1\}$.

Assume P is nearly simple. Then there is a ϕ with $P(\phi)$ simple. We first show that for every i and j, ab_i and b_ib_j are in M. To do this it is necessary and sufficient to prove that each $\beta_{i,j} = 0$ and each $\gamma_{i,j,l} = 0$. If $x \neq 0$ is in M, Theorem 2.5 assures the existence of a y in N with $\phi(x, y) \neq 0$. Thus, if x in M has the property that $\phi(x, y) = 0$ for all y in N then $x = 0$. Since $n - k - 1 > 2$ by (28), $\phi(a^{n-k-1}, b_i) = 0$ for all i. Hence $\phi(ab_ib_j, b_l) = \beta_{i,j}\phi(a^{n-k-1}, b_l) = 0$ and $\phi(b_ib_jb_l, b_h) = \gamma_{i,j,l}\phi(a^{n-k-1}, b_h) = 0$ for all $h, i, j, l = 1, \cdots, k-1$. Also by Theorem 2.3, $\phi(ab_ib_j, a) = -\phi(a^2b_i, b_j) - \phi(a^2b_j, b_i) = 0$ and $\phi(b_ib_jb_l, a) = -\phi(ab_ib_j, b_l) - \phi(ab_jb_l, b_i) - \phi(ab_ib_l, b_j) = 0$. Since ab_ib_j, $b_ib_jb_l$ are in M, $ab_ib_j = b_ib_jb_l = 0$ for $i, j, l = 1, \cdots, k-1$. Thus we have shown that M is the space spanned by $\{a^{n-k-1}, ab_i, b_ib_j;\ i, j = 1, \cdots, k-1\}$.

Since $a^{n-k-1} \neq 0$ is in M there is a y in N with $\phi(a^{n-k-1}, y) \neq 0$. For $q > 1$, a^q is in N_2 and $\phi(a^{n-k-1}, b_i) = 0$ for $i = 1, \cdots, k-1$; so we conclude $\phi(a^{n-k-1}, a) \neq 0$. But by Theorem 2.3, $(n-k)\phi(a^{n-k-1}, a) = 0$, so $n - k = m$ char F for some $m > 0$.

As an immediate corollary, we have

Corollary 3.1. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative nil-algebra of type $(n-2, n, 2, 2)$ with $n-2 > 2$ over a field F of char. $\neq 2, 3$. Then there are elements a, b, c such that b is in N_1', c is in N_2', and N is spanned by $a, \cdots, a^{n-3}, b, c$ with $a^2b = 0$, $ab^2 = \beta a^{n-3}$, $b^3 = \gamma a^{n-3}$ and c is either ab or b^2. Furthermore, if P is nearly simple and $n-2 > 3$, then $\beta = 0 = \gamma$ and $n - 2 = m$* char *F for some $m > 1$.*

Theorem 3.5. *Let $P = F \cdot 1 \oplus N$ be an associative commutative nearly nodal algebra of type $(n-2, n, 2, 2)$ with $n-2 > 3$ over a field F of char. $\neq 2, 3$. Then P is nearly simple if and only if*

(35) *N is spanned by $\{a, \cdots, a^{n-3}, b, ab\}$ with $a^2b = ab^2 = b^3 = 0$ and $b^2 = \alpha a^{n-3}$ for some α in F,*

(36) *$n-2 = m$* char *F for some positive integer m and if $b^2 = 0$ then $m > 1$.*

Proof. Suppose P is nearly simple. Then there is an antiflexible map ϕ with $P(\phi)$ simple. To prove (35) we only need to show that $c = ab$ and $b^2 = \alpha a^{n-3}$ in Corollary 3.1. If $c = b^2$ then $ab = \sum_{i=2}^{n-3} \alpha_i a^i + \beta b^2$ and so $0 = a^2b = \sum_{i=2}^{n-4} \alpha_i a^{i+1}$. Therefore, $\alpha_i = 0$ for $i = 2, \cdots, n-4$ and $ab = \alpha_{n-3} a^{n-3} + \beta b^2$. Thus $\phi(ab, b) = \alpha_{n-3}\phi(a^{n-3}, b) + \beta\phi(b^2, b) = 0$. Consequently, by Lemma 3.2(34), $\phi(b^2, a) = 0$. Since b^2 is in M, $b^2 = 0$ which is impossible. Hence $c = ab$ and $b^2 = \sum_{i=2}^{n-3}\beta_i a^i + \gamma ab$. Now $0 = ab^2 = \sum_{i=2}^{n-4}\beta_i a^{i+1}$, so $\beta_i = 0$ for $i = 2, \cdots, n-4$. Thus $b^2 = \beta_{n-3}a^{n-3} + \gamma ab$ and $0 = \phi(b^2, b) = \beta_{n-3}\phi(a^{n-3}, b) + \gamma\phi(ab, b) = \gamma\phi(ab, b)$. If $\phi(ab, b) = 0$ then, by Lemma 3.2(34), $\phi(b^2, a) = 0$ and consequently $b^2 = 0 = 0 \cdot a^{n-3}$. On the other hand, if $\phi(ab, b) \neq 0$, then $\gamma = 0$ and defining $\alpha = \beta_{n-3}$ we have $b^2 = \alpha \cdot a^{n-3}$.

By Corollary 3.1, $n - 2 = m$ char F for some $m \geq 1$. Suppose that $b^2 = 0$ and $m = 1$. Then it is easy to show that $H(\phi) \subseteq F \cdot 1$. Since $b^2 = 0$, by Lemma 3.2(34), $\phi(ab, b) = 0$. Therefore, since $ab \neq 0$ is in M, $\phi(ab, a) = \beta \neq 0$. Let $\phi(a^{n-3}, a) = \delta$, $\delta \neq 0$. Define $x = \beta a^{n-3} - \delta ab$, then $x \neq 0$, x is in M and $\phi(x, z) = 0$ for all z in N. This is a contradiction. Hence $m > 1$.

Conversely, suppose P satisfies (35) and (36) and $b^2 \neq 0$. Define ϕ on the basis of P as follows:

$$\phi(a^i, a^j) = \begin{cases} 0 & \text{if } i + j \neq n-2, \\ j & \text{if } i + j = n-2; \end{cases}$$

$$\phi(b^2, a) = 2 = -\phi(a, b^2);$$

$$\phi(ab, b) = -1 = -\phi(b, ab);$$

$$\phi(x, y) = 0 \quad \text{for any other pair of basis elements } x \text{ and } y.$$

Extend ϕ bilinearly to $P \times P$. Then it is easy to verify that ϕ is an antiflexible map and that (17) and (18) are satisfied. Hence, by Theorem 2.5, $P(\phi)$ is simple.

If $b^2 = 0$, then by (36), $m > 1$ so $a^p \neq 0$. Define ϕ on the basis of P as follows:

$$\phi(a^i, a^j) = \begin{cases} 0 & \text{if } i+j \neq n-2, \\ j & \text{if } i+j = n-2; \end{cases}$$

$$\phi(ab, a) = a^p = -\phi(a, ab);$$

$$\phi(a^2, b) = -2a^p = -\phi(b, a^2);$$

$$\phi(x, y) = 0 \quad \text{for any other pair of basis elements } x \text{ and } y.$$

Extend ϕ bilinearly to $P \times P$, then it is routine to verify that ϕ is an antiflexible map and (17) and (18) hold. Therefore, by Theorem 2.5, $P(\phi)$ is simple.

Having determined nearly simple algebras of type $(n-2, n)$ our next interest is to find all possible antiflexible maps that give rise to simple antiflexible algebras of type $(n-2, n)$.

Theorem 3.6. *Let $P = F \cdot 1 \oplus N$ be an associative commutative nearly simple, nearly nodal algebra of type $(n-2, n, 2, 2)$ with $n-2 > 3$ over a field F of characteristic $p \neq 2, 3$ and let ϕ be an antiflexible map on P. If a and b are as in Theorem 3.5, then $P(\phi)$ is simple if and only if*

(37) *$H(\phi)$ is a subset of the algebra generated by 1 and a^p, $n-2 = mp$,*

(38) *$\phi(a^{n-3}, a) = \sum_{i=0}^{m-1} \alpha_{ip} a^{ip}$, α_{ip} in F for $i = 0, \cdots, m-1$ with $\alpha_0 \neq 0$,*

(39) *if $b^2 = 0$, then $\phi(ab, a) = \sum_{i=1}^{m-1} \gamma_{ip} a^{ip} \neq 0$ and, for any δ in F, $\phi(ab, a) \neq \delta\phi(a^{n-1}, a)$.*

Proof. Since P is nearly simple (35) and (36) of Theorem 3.5 hold.

Assume $P(\phi)$ is simple. By Theorem 2.3, $\phi(a^i, a^j) = -i\phi(a^{i+j-1}, a) = j\phi(a^{i+j-1}, a)$, so $\phi(a^i, a^j) = 0$ if $p \mid i$ or $p \mid j$ or $p \nmid (i+j)$ or $i + j > n-2$. Since a^{n-3} is in M and $\phi(a^{n-3}, b) = 0$ (by Lemma 3.2), $\phi(a^{n-3}, a) \neq 0$. Let x, y be in P, then

$$\phi(x, y) = \sum_{j=0}^{n-3} \beta_i a^i + \beta b + \gamma ab, \qquad \beta, \gamma, \beta_i \text{ in } F \text{ for } i = 1, \cdots, n-3.$$

Now, $0 = \phi(\phi(x, y), a^{n-3}) = \beta_1 \phi(a, a^{n-3})$ which implies $\beta_1 = 0$. Suppose $\beta_i = 0$ for all $i < k$ with $p \nmid i$, $p \nmid k$ and $k < n-3$. Then we have

$$0 = \phi(\phi(x, y), a^{n-k-2}) = \alpha_k \phi(a^k, a^{n-k-2}) = -k\alpha_k \phi(a^{n-3}, a).$$

Therefore, $\alpha_k = 0$ and hence, by mathematical induction, $\alpha_i = 0$ for all $i < n-3$ with $p \nmid i$. Thus $\phi(x, y) = \sum_{i=0}^{m-1} \beta_{ip} a^{ip} + \beta b + Y$ where $Y = \beta_{n-3} a^{n-3} + \gamma ab$ is in M. If $b^2 \neq 0$, then by Lemma 3.2, $\phi(ab, b) \neq 0$ and $0 = \phi(\phi(x, y), ab) = \beta\phi(b, ab)$. Consequently, $\beta = 0$. On the other hand, if $b^2 = 0$ then $\phi(ab, a) \neq 0$ and so $\phi(a^2, b) \neq 0$ (by Lemma 3.2). Since a^2 is in N_2, we have $0 = \phi(\phi(x, y), a^2) = \beta\phi(b, a^2)$ which implies $\beta = 0$. Now for any z in P, $0 = \phi(\phi(x, y), z) = \phi(Y, z)$, so $Y = 0$ and we have (37).

Let $x = a^{n-3}$, $y = a$, then we have $\phi(a^{n-3}, a) = \sum_{i=0}^{m-1} \alpha_{ip} a^{ip}$. If $\alpha_0 = 0$ then a generates a proper ideal of $P(\phi)$, so $\alpha_0 \neq 0$. Now suppose $b^2 = 0$, then $\phi(b^2, a) = 0 = \phi(ab, b)$. Since ab is in M, $\phi(ab, a) \neq 0$. If there is a δ in F with $\phi(ab, a) = \delta\phi(a^{n-3}, a)$, define $x = ab - \delta a^{n-3}$. Then x is in M and $\phi(x, z) = 0$ for all z in P which is a contradiction. Hence for any δ in F, $\phi(ab, a) \neq \delta\phi(a^{n-3}, a)$.

Conversely, suppose ϕ satisfies (37), (38), and (39). We first note that M is spanned by $\{a^{n-3}, ab\}$. Let x be a nonzero element in M. Then $x = \beta a^{n-3} + \gamma ab$, β, γ in F, such that either $\beta \neq 0$ or $\gamma \neq 0$. By Theorem 3.5, $b^2 = \alpha a^{n-3}$ for some α in F. Now suppose $\gamma \neq 0$. If $\alpha \neq 0$, then $\phi(x, b) = \beta\phi(a^{n-3}, b) + \gamma\phi(ab, b) = (-1/2)\gamma\alpha\phi(a^{n-3}, a) \neq 0$, and if $\alpha = 0$ then, by (39), $\phi(x, a) = \beta\phi(a^{n-3}, a) + \gamma\phi(ab, a) \neq 0$. On the other hand, if $\gamma = 0$, then $x = \beta a^{n-3}$, so $\phi(x, a) = \beta \cdot \phi(a^{n-3}, a) \neq 0$. Thus (17) is satisfied. Also, it is easy to verify that (18) holds. Hence by Theorem 2.5, $P(\phi)$ is simple.

Thus we have determined all nodal simple totally antiflexible algebras of type $(n-2, n)$ with $n - 2 > 3$. The case $n - 2 = 3$ will be solved in §IV of this paper. In the rest of this section we will try to generalize some of the results obtained so far.

Theorem 3.7. *Let N be an associative commutative nilalgebra of type (m, n, d_1, d_2) over a field of char. $\neq 2, 3$. Then there exists a, b_i, c_j $(i = 1, \cdots, d_1 - 1;\ j = 1, \cdots, d_2 - 1)$ such that b_i is in N_1', c_j in N_2' and $\{a, \cdots, a^{m-1}, b_i, c_j;\ i = 1, \cdots, d_1 - 1;\ j = 1, \cdots, d_2 - 1\}$ is a basis of N with $a^2 b_i = 0$, $ab_ib_j = \alpha_{i,j}a^{m-1}$, $b_ib_jb_l = \beta_{i,j,l}a^{m-1}$ for all $i, j, l = 1, \cdots, d_1 - 1$. Furthermore, $d_2 \leq (1/2)d_1(d_1 + 1)$, c_i is in A for all i where $A = \{ab_i, b_ib_j : i, j = 1, \cdots, d_1 - 1\}$, and if $d_2 = (1/2)d_1(d_1 + 1)$ then $A \cap N_3$ is a null set.*

Proof. Since $\dim \overline{N_3} = 1$, by Theorem 2.8 there is an a in N with $a^{m-1} \neq 0$. Also since N is of type (m, n, d_1, d_2) there exists a, g_i, c_j $(i = 1, \cdots, d_1 - 1; j = 1, \cdots, d_2 - 1)$ such that g_i is in N_1', c_j is in N_2' and $\{a, \cdots, a^{m-1}, g_i, c_j : i = 1, \cdots, d_1 - 1; j = 1, \cdots, d_2 - 1\}$ is a basis of N. Then N_3 is spanned by $a^3, \cdots, a^{m-1}$ and so

$$a^2 g_i = \sum_{j=3}^{m-1} \gamma_{i,j} a^j = a^2 \sum_{j=3}^{m-1} \gamma_{i,j} a^{j-2}, \qquad \gamma_{i,j} \text{ in } F.$$

Defining $b_i = g_i - \sum_{j=3}^{m-1} \gamma_{i,j} a^{j-2}$ we get that $\{a, \cdots, a^{m-1}, b_i, c_j : i = 1, \cdots, d_1 - 1; j = 1, \cdots, d_2 - 1\}$ is a basis of N with $a^2 b_i = 0$ for all i. Since ab_ib_j is in N_3, $ab_ib_j = \sum_{t=3}^{m-1} \alpha_{i,j,t} a^t$. Then $0 = a^2 b_i b_j = \sum_{t=3}^{m-2} \alpha_{i,j,t} a^{t+1}$ which implies that $\alpha_{i,j,t} = 0$ for $i, j = 1, \cdots, d_1 - 1$; $t = 3, \cdots, m-2$. Hence defining $\alpha_{i,j} = \alpha_{i,j,m-1}$ one gets $ab_ib_j = \alpha_{i,j} a^{m-1}$ for $i, j = 1, \cdots, d_1 - 1$. Since $m \geq 3$, $ab_ib_jb_l = \alpha_{i,j} a^{m-1} b_l = 0$. Since $b_ib_jb_l$ is in N_3, $b_ib_jb_l = \sum_{t=3}^{m-1} \beta_{i,j,l,t} a^t$. Then $0 = ab_ib_jb_l = \sum_{t=3}^{m-2} \beta_{i,j,l,t} a^{t+1}$ which implies $\beta_{i,j,l,t} = 0$ for $i, j, l = 1, \cdots, d_1 - 1$; $t = 3, \cdots, m-2$. Therefore, defining $\beta_{i,j,l} = \beta_{i,j,l,m-1}$, we get $b_i b_j b_l = \beta_{i,j,l} a^{m-1}$ for $i, j, l = 1, \cdots, d_1 - 1$.

To prove the second part of this theorem we note that the number of ways of picking two distinct elements out of d_1 elements is $(1/2)d_1(d_1 - 1)$ and $b_1^2, \cdots, b_{d_1-1}^2$ are in A. Therefore, the cardinality of $A = (1/2)d_1(d_1 - 1) + d_1 + 1 = (1/2)d_1(d_1 + 1) - 1$. Since N_2 is contained in the space spanned by $A \cup \{a^2, \cdots, a^{m-1}\}$, $\dim N_2 \leq (1/2)d_1(d_1 + 1) - 1 + m - 2$. But $\dim N_2 = d_2 + m - 3$, so $d_2 \leq (1/2)d_1(d_1 + 1)$. Also, since $\dim N_2 = d_2 + m - 3$, A has $d_2 - 1$ linearly independent elements. Assume, there are only t linearly independent elements in A that are not in N_3, $t < d_2 - 1$. We can choose $c_1, \cdots, c_t$ to be these elements. Since c_{t+1} is in N'_2,

$$c_{t+1} = \left(\alpha a + \sum \alpha_i b_i\right)\left(\beta a + \sum \beta_i b_i\right) + z, \qquad z \text{ in } N_3,$$

$$= \alpha\beta a^2 + \sum_{i=1}^{t} \gamma_i c_i + z', \qquad z' \text{ in } N_3.$$

Therefore, $c_1, \cdots, c_{t+1}, a^2, \cdots, a^{m-1}$ are linearly dependent, which is impossible. Hence A has $d_2 - 1$ linearly independent elements that are not in N_3 and so c_i can be chosen in A for $i = 1, \cdots, d_2 - 1$. If $d_2 = (1/2)d_1(d_1 + 1)$ then $\dim N_2 = d_2 + m - 3 = (1/2)d_1(d_1 + 1) - 1 + m - 2 =$ cardinality of $A + m - 2$. Therefore $A \cup \{a^2, \cdots, a^{m-1}\}$ is a basis of N_2 and hence $A \cap N_3$ is a null set.

Theorem 3.8. *Let N be an associative commutative nilalgebra of type (m, n, d_1, d_2, d_3) over a field F of char. $\neq 2, 3$. Then there exists a, b_i, c_i, f_k $(i = 1, \cdots, d_1 - 1;\ j = 1, \cdots, d_2 - 1;\ k = 1, \cdots, d_3 - 1)$ such that b_i is in N'_1, c_j in N'_2, f_k in N'_3 and $\{a, \cdots, a^{m-1}, b_i, c_j, f_k;\ i = 1, \cdots, d_1 - 1;\ j = 1, \cdots, d_2 - 1;\ k = 1, \cdots, d_3 - 1\}$ is a basis of N with $a^3 b_i = 0$, $a^2 b_i b_j = \alpha_{i,j} a^{m-1}$, $a b_i b_j b_k = \beta_{i,j,k} a^{m-1}$, $b_i b_j b_k b_l = \gamma_{i,j,k,l} a^{m-1}$ for all $i, j, k, l = 1, \cdots, d_1 - 1$. Furthermore, c_i is in $A = \{ab_j, b_j b_k: j, k = 1, \cdots, d_1 - 1\}$; f_k is in $B = \{a^2 b_i, ab_i b_j, b_i b_j b_l: i, j, l = 1, \cdots, d_1 - 1\}$; $d_2 \leq (1/2)d_1(d_1 + 1)$, $d_3 \leq (1/6)d_1(d_1 + 1)(d_2 + 2)$; if $d_2 = (1/2)d_1(d_1 + 1)$, $d_3 = (1/6)d_1(d_1 + 1)(d_1 + 2)$ then $A \cap N_3$ and $B \cap N_4$ are null sets respectively.*

Proof. Since $\dim \overline{N_4} = 1$ and char $F \neq 2, 3$, by Theorem 2.8 there is an a in N with $a^{m-1} \neq 0$. Also, since N is of type (m, n, d_1, d_2, d_3) there exists g_i, c_j, f_k $(i = 1, \cdots, d_1 - 1;\ j = 1, \cdots, d_2 - 1;\ k = 1, \cdots, d_3 - 1)$ such that g_i is in N'_1, c_j in N'_2, f_k in N'_3 and N is spanned by $\{a, \cdots, a^{m-1}, g_i, c_j, f_k: i = 1, \cdots, d_1 - 1;\ j = 1, \cdots, d_2 - 1;\ k = 1, \cdots, d_3 - 1\}$. Then N_4 is spanned by $a^4, \cdots, a^{m-1}$ and so $a^3 g_i = \sum_{j=4}^{m-1} \gamma_{i,j} a^j = a^3 \sum_{j=4}^{m-1} \gamma_{i,j} a^{j-3}$. Defining $b_i = g_i - \sum_{j=4}^{m-1} \gamma_{i,j} a^{j-3}$ we get that $\{a, \cdots, a^{m-1}, b_i, c_j, f_k: i = 1, \cdots, d_1 - 1;\ j = 1, \cdots, d_2 - 1;\ k = 1, \cdots, d_3 - 1\}$ is a basis of N with $a^3 b_i = 0$ for $i = 1, \cdots, d_1 - 1$. Since $a^2 b_i b_j$ is in N_4, $a^2b_ib_j = \Sigma_{t=4}^{m-1}\alpha_{i,j,t}a^t$. Then $0 = a^3b_ib_j = \Sigma_{t=4}^{m-2}\alpha_{i,j,t}a^{t+1}$ which implies $\alpha_{i,j,t} = 0$ for $i, j = 1, \cdots, d_1 - 1$; $t = 3, \cdots, m-2$. Hence defining $\alpha_{i,j} = \alpha_{i,j,m-1}$ one gets $a^2b_ib_j = \alpha_{i,j}a^{m-1}$ for $i, j = 1, \cdots, d_1 - 1$. Since $m \geq 4$, $a^2b_ib_jb_l = \alpha_{i,j}a^{m-1}b_l = 0$. Also, since $ab_ib_jb_l$ is in N_4, $ab_ib_jb_l = \Sigma_{t=4}^{m-1}\beta_{i,j,l,t}a^t$. Then $0 = a^2b_ib_jb_l = \Sigma_{t=4}^{m-2}\beta_{i,j,l,t}a^{t+1}$ which implies $\beta_{i,j,l,t} = 0$ for $i, j, l = 1, \cdots, d_1 - 1$; $t = 1, \cdots, m-2$. Defining $\beta_{i,j,l} = \beta_{i,j,l,m-1}$ one gets $ab_ib_jb_l = \beta_{i,j,l}a^{m-1}$. Now $ab_ib_jb_kb_l = \beta_{i,j,k}a^{m-1}b_l = 0$ and $b_ib_jb_kb_l = \Sigma_{t=4}^{m-1}\gamma_{i,j,k,l,t}a^t$. So $0 = ab_ib_jb_kb_l = \Sigma_{t=4}^{m-2}\gamma_{i,j,k,l,t}a^{t+1}$ which implies $\gamma_{i,j,k,l,t} = 0$ for $i, j, k, l = 1, \cdots, d_1 - 1$; $t = 4, \cdots, m-2$. Defining $\gamma_{i,j,k,l} = \gamma_{i,j,k,l,m-1}$ we get $b_ib_jb_kb_l = \gamma_{i,j,k,l}a^{m-1}$, $i, j, k, l = 1, \cdots, d_1 - 1$.

As in the last theorem, c_i is in A, $d_2 \leq (1/2)d_1(d_1+1)$ and if $d_2 = (1/2)d_1(d_1+1)$ then $A \cap N_3$ is a null set. Since the number of ways of picking 3 distinct elements out of d_1 elements is $\binom{d_1}{3} = (1/6)d_1(d_1-1)(d_1-2)$ and $d_1^2 - 1$ elements a^2b_i, $b_i^2b_j$, $i, j = 1, \cdots, d_1 - 1$, are in B, cardinality of $B = \binom{d_1}{3} + d_1^2 - 1 = (1/6)d_1(d_1+1)(d_1+2) - 1$. Also since N_3 is contained in the space spanned by $B \cup \{a^3, \cdots, a^{m-1}\}$ and $\dim N_3 = d_3 + m - 4$, $d_3 + m - 4 \leq (1/6)d_1(d_1+1)(d_1+2) - 1 + m - 3$. Therefore, $d_3 \leq (1/6)d_1(d_1+1)(d_1+2)$ and B has $d_3 - 1$ linearly independent elements. Assume there are only t linearly independent elements in B that are not in N_4, $t < d_3 - 1$. Then we can choose $f_1, \cdots, f_t$ to be these elements. Since f_{t+1} is N_3',

$$
\begin{aligned}
f_{t+1} &= \left(\alpha a + \sum \alpha_i b_i\right)\left(\beta a + \sum \beta_i b_i\right)\left(\gamma a + \sum \gamma_i b_i\right) + z, \qquad z \text{ in } N_4, \\
&= \alpha\beta\gamma a^3 + \sum_{i=1}^{t} \delta_i f_i + z', \qquad z' \text{ in } N_4.
\end{aligned}
$$

Therefore, $a^3, a^4, \cdots, a^{m-1}, f_1, \cdots, f_{t+1}$ are linearly dependent, which is a contradiction. Hence B has $d_3 - 1$ linearly independent elements that are not in N_4. If $d_3 = (1/6)d_1(d_1+1)(d_1+2)$, then $B \cup \{a^3, \cdots, a^{m-1}\}$ is a basis of N_3 so $B \cap N_4$ is a null set.

A necessary condition for nodal nearly simple algebras of type (m, n, d_1, d_2) is given by the following theorem.

Theorem 3.9. *Let $P = F \cdot 1 \oplus N$ where N is an associative commutative nil-algebra of type (m, n, d_1, d_2) over a field F of char. $\neq 2, 3$. If P is nearly simple and $m > 3$ then $ab_ib_j = b_ib_jb_k = 0$ for $i, j, k = 1, \cdots, d_1 - 1$, where a and b_i are as in Theorem 3.7 and* char F *divides m.*

Proof. Suppose P is nearly simple then there is an antiflexible map ϕ such that $P(\phi)$ is simple. Since a^{m-1} is in M, $\phi(a^{m-1}, b_i) = 0$ (Lemma 3.2) for all i, $\phi(a^{m-1}, a) \neq 0$. Considering a, a, b_i, b_j and using Theorem 2.3 we have

$2\phi(ab_ib_j, a) + \phi(a^2b_i, b_j) + \phi(a^2b_j, b_i) = 0$. Since $a^2b_i = 0 = a^2b_j$, $\phi(ab_ib_j, a) = 0$. Thus $\phi(ab_ib_j, a) = \alpha_{i,j}\phi(a^{m-1}, a)$ (Theorem 3.7) which implies $\alpha_{i,j} = 0$ for $i, j = 1, \cdots, d_1 - 1$. Similarly it can be shown that $\phi(b_ib_jb_l, a) = 0$ and hence $b_ib_jb_l = 0$ for all $i, j, l = 1, \cdots, d_1 - 1$. Also, using Theorem 2.3 we have $m\phi(a^{m-1}, a) = 0$, so char F divides m.

The following lemma follows from Theorem 2.3.

Lemma 3.3. *Let ϕ be an antiflexible map on an associative commutative algebra P over a field F of char. $\neq 2$, 3 in which $a^3b = 0$ for some a, b in P. Then*

(40) $\phi(a^2b^2, a^s) = 0$, $s \geq 0$,

(41) $\phi(ab^2, a^s) = 0$, $s > 1$,

(42) $\phi(a^s, b) = 0$, $s > 3$,

(43) $\phi(a^2b, a^s) = 0$, $s > 1$,

(44) $\phi(ab, a^s) = 0$, $s > 2$,

(45) $\phi(a^r, b^s) = 0$, $r > 2$ *and* $s > 1$,

(46) $\phi(a^2b^3, a^s) = 0$, $s \geq 0$,

(47) $\phi(ab^3, a^s) = 0$, $s > 1$.

We are now ready to give a necessary condition for nearly simple nodal algebras of type (m, n, d_1, d_2, d_3).

Theorem 3.10. *Let $P = F \cdot 1 \oplus N$ where N is an associative commtative nilalgebra of type (m, n, d_1, d_2, d_3) with $m > 4$ over a field F of char. $\neq 2, 3$. If P is nearly simple then $a^2b_ib_j = ab_ib_jb_k = b_ib_jb_kb_l = 0$ for $i, j, k, l = 1, \cdots, d_1 - 1$ where a, b_i's are as in Theorem* 3.8 *and* char *F divides m.*

Proof. Suppose P is nearly simple. Then there is an antiflexible map ϕ with $P(\phi)$ simple. Since a^{m-1} is in M and, for each i, $\phi(a^{m-1}, b_i) = 0$ (Lemma 3.3), $\phi(a^{m-1}, a) \neq 0$. Considering a, a, a, b_i, b_j and using Theorem 2.3 we get $\phi(a^2b_ib_j, a) = 0$. But $\phi(a^2b_ib_j, a) = \alpha_{i,j}\phi(a^{m-1}, a)$, so $\alpha_{i,j} = 0$ for $i, j = 1, \cdots, d_1 - 1$. Similarly it can be shown that $ab_ib_jb_k = 0$ and $b_ib_jb_kb_l = 0$ for $i, j, k, l = 1, \cdots, d_1 - 1$. By Theorem 2.3, $m\phi(a^{m-1}, a) = 0$. Therefore, char F divides m.

4. **Algebras with** $N \cdot N \cdot N = 0$. We have determined all nodal simple antiflexible algebras of type $(n - 2, n)$ with $n - 2 > 3$. In this section we get a few preliminary results for algebras of class 3 and then determine all nodal simple totally antiflexible algebras of types (3, 5) and (3, 6). These are the only types discussed in [10].

Let N denote a commutative associative nilalgebra of class 3 over a field F and let $n - 1$ denote the dimension of N. If $v_1, \cdots, v_q$ is a basis of M (the annihilator of N) we write a basis for the algebra N in the form $\{u_1, \cdots, u_r,$

$v_1, \cdots, v_q\}$. Since N is of class 3, $N^2 \subseteq M$. Therefore $u_i u_j = \sum_{k=1}^q \alpha_{i,j}^k v_k$ where $\alpha_{i,j}^k = \alpha_{j,i}^k$ is in F.

Up to isomorphism, the algebra N is given by qr^2 elements $\alpha_{i,j}^k$ or q symmetric matrices $A^{(k)} = (\alpha_{i,j}^k)$ of degree r. We shall call the matrices $A^{(k)}$ the structural matrices of the algebra N. It is known [10] that if we change the basis of N, the new structural matrices are congruent to the one obtained previously.

Since $N^2 \subseteq M \subseteq N$, there is a basis $\{u_1, \cdots, u_s, v_1, \cdots, v_l, w_1, \cdots, w_t\}$ of N such that $\{v_1, \cdots, v_l\}$ is a basis of N^2 and $\{v_1, \cdots, v_l, w_1, \cdots, w_t\}$ is a basis of M. Let V be the space spanned by $\{u_1, \cdots, u_s, v_1, \cdots, v_l\}$ and W be the space spanned by $\{w_1, \cdots, w_t\}$. Then $N = V + W$, $V \cdot W = W^2 = 0$ and the annihilator of V is V^2. Hence $N = V \oplus W$ where V is of class 3 and W is of class 2. Since commutative nilalgebras N of class 2 have been determined [10], the description of commutative nilalgebras of class 3 reduces to the case in which $M = N^2$.

Lemma 4.1. *Let $A = V \oplus W$ be an associative commutative algebra over a field of char. $\neq 2, 3$. If ϕ_1 and ϕ_2 are antiflexible maps on V and W respectively then there exists an antiflexible map ϕ on A such that $\phi(v, v') = \phi_1(v, v')$ for all v, v' in V and $\phi(w, w') = \phi_2(w, w')$ for all w, w' in W.*

Proof. Let x, y be in A; then there exists v_1, v_2 in V and w_1, w_2 in W such that $x = v_1 + w_1$ and $y = v_2 + w_2$. Define ϕ on A as follows:

$$\phi(x, y) = \phi_1(v_1, v_2) + \phi_2(w_1, w_2),$$
$$\phi(v, w) = \phi(w, v) = 0 \quad \text{for } v \text{ in } V \text{ and } w \text{ in } W.$$

Then it is easy to verify that ϕ is an antiflexible map.

In a similar way we prove the following lemma. We need first to define $\phi|_V$. If $A = V + W$ and ϕ is a map on A, define $\phi|_V$ by $\phi|_V(v_1 + w_1, v_2 + w_2) = \phi(v_1, v_2)$ for v_1, v_2 in V and w_1, w_2 in W.

Lemma 4.2. *Let $A = V \oplus W$ be an associative commutative algebra over a field F. If ϕ is an antiflexible map on A such that $\phi(v, w) = 0$, $\phi(v_1, v_2)$ is in V and $\phi(w_1, w_2)$ is in W for v, v_1, v_2 in V and w, w_1, w_2 in W then $\phi = \phi|_V + \phi|_W$; $\phi|_V$, $\phi|_W$ are antiflexible maps on V and W respectively.*

Proof. We need to show that $\phi = \phi|_V + \phi|_W$. Let x, y be in A. Then $x = v_1 + w_1$, $y = v_2 + w_2$ for some v_1, v_2 in V and for some w_1, w_2 in W. Since ϕ is bilinear on A,

$$\begin{aligned}\phi(x, y) &= \phi(v_1, v_2) + \phi(v_1, w_2) + \phi(w_1, v_1) + \phi(w_1, w_2)\\ &= \phi|_V(x, y) + \phi|_W(x, y).\end{aligned}$$

Now suppose $M = N^2$. Then the matrices $A^{(1)}, \cdots, A^{(q)}$ are linearly independent and it follows that two commutative nilpotent algebras of class 3 over a field F are isomorphic if and only if corresponding spaces of bilinear forms determined by the matrices of the algebra are equivalent [10]. The proof of the following two lemmas are also found in [10].

Lemma 4.3. *If* $(1/2)r(r+1) = q$ *then up to isomorphism there is exactly one associative commutative nilalgebra* N *of class* 3 *over a field* F *such that the* dim N *over* F *is* $q + r$ *and* $M = N^2$.

Lemma 4.4. *If* $M = N^2$ *then* $q \leq (1/2)r(r+1)$.

Lemma 4.5. *Let* $P = F \cdot 1 \oplus N$ *where* N *is an associative commutative nilalgebra of type* $(3, n)$ *over a splitting field* F *of char.* $\neq 2, 3$ *with* $M = N^2$. *If* $v_1, \cdots, v_q, u_1, \cdots, u_r$ *is a basis of* N *such that* $v_1, \cdots, v_q$ *is a basis of* M *and, for each* i, $u_i^2 = \sum_{j \neq i} \alpha_{i,j} v_j$, *then* P *is nearly simple.*

Proof. Define ϕ on the basis of P as follows:

$$\phi(v_i, v_j) = \delta_{i,j} = -\phi(u_j, v_i) \quad \text{where } \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

$$\phi(x, y) = 0 \text{ for any other pair of basis elements } x \text{ and } y.$$

Extend ϕ bilinearly to all of $P \times P$. Then it is routine to verify that ϕ is an antiflexible map and $P(\phi)$ is simple.

Now we consider algebras of type (3, 5). Note that if P is a nodal simple totally antiflexible algebra of type (3, 5) then it is either of type (3, 5, 3) or of type (3, 5, 2, 2). By Theorem 2.10, there is no nodal simple totally antiflexible algebra of type (3, 5, 3) over a field of char. $\neq 2, 3$.

Theorem 4.1. *Let* $P = F \cdot 1 \oplus N$ *be an associative commutative nearly nodal algebra of type* (3, 5, 2, 2) *over a field* F *of char.* $\neq 2, 3$. *Then* P *is nearly simple if and only if*

(48) N *is spanned by* $\{a, a^2, b, c\}$ *with* b *in* N'_1, c *in* N'_2 *and* c *is either* ab *or* b^2,

(49) *if* $c = ab$, *then* $b^2 = \gamma a^2 + \delta ab$ *with* $4\gamma + \delta^2 = 0$, *and if* $c = b^2$ *then* $ab = \alpha a^2 + \beta b^2$ *with* $4\alpha\beta = 1$.

Proof. Suppose P is nearly simple. Then there is a ϕ with $P(\phi)$ simple. By Corollary 3.2, N is spanned by a, a^2, b, c with b in N'_1, c in N'_2 and c is either ab or b^2. Note that M is spanned by a^2, ab, b^2. So $\phi(a^2, b) \neq 0 \neq \phi(b^2, a)$ and, by Lemma 3.3, $\phi(ab, a) = (-1/2)\phi(a^2, b) \neq 0$, $\phi(ab, b) = (-1/2)\phi(b^2, a) \neq 0$. If $c = ab$ then since b^2 is in N_2, $b^2 = \gamma a^2 + \delta ab$, for some γ, δ in F. Therefore, $\phi(b^2, a) = \delta\phi(ab, a)$ or equivalently $-2\phi(ab, b) = (-1/2)\delta\phi(a^2, b)$. Now

$$0 = 4\phi(b^2, b) = 4\gamma\phi(a^2, b) + 4\delta\phi(ab, b) = (4\gamma + \delta^2)\phi(a^2, b).$$

Since $\phi(a^2, b) \neq 0$, we have $4\gamma + \delta^2 = 0$.

On the other hand if $c = b^2$ then $ab = \alpha a^2 + \beta b^2$; α, β are in F. So $\phi(ab, a) = \beta\phi(b^2, a)$ or equivalently $(-1/2)\phi(a^2, b) = -2\beta\phi(ab, b)$. Thus $\phi(ab, b) = \alpha\phi(a^2, b) = 4\alpha\beta\phi(ab, b)$ which implies that $4\alpha\beta = 1$.

Conversely, suppose P satisfies (48) and (49). If $c = ab$, define ϕ on the basis of P as follows:

$$\phi(a^2, b) = 4 = -\phi(b, a^2), \qquad \phi(ab, a) = -2 = -\phi(a, ab),$$

$$\phi(ab, b) = \delta = -\phi(b, ab), \qquad \phi(b^2, a) = -2\delta = -\phi(a, b^2),$$

$$\phi(x, y) = 0 \quad \text{for any other pair of basis elements } x \text{ and } y.$$

Extend ϕ bilinearly to $P \times P$. Then it is easy to verify that ϕ is antiflexible and $P(\phi)$ is simple.

If $c = b^2$, we define ϕ on the basis of P as follows:

$$\phi(a^2, b) = 2 = -\phi(b, a^2), \qquad \phi(ab, a) = -1 = -\phi(a, ab),$$

$$\phi(ab, b) = 2\alpha = -\phi(b, ab), \quad \phi(b^2, a) = -4\alpha = -\phi(a, b^2),$$

$$\phi(x, y) = 0 \quad \text{otherwise.}$$

Extend ϕ bilinearly to all of $P \times P$. Then ϕ is antiflexible and $P(\phi)$ is simple.

Theorem 4.2. *Let $P = F \cdot 1 \oplus N$ be an associative, commutative, nearly nodal, nearly simple algebra of type* (3, 5, 2, 2) *over a field F of char. $\neq 2, 3$. If ϕ is an antiflexible map on P, then $P(\phi)$ is simple if and only if $H(\phi) \subseteq F \cdot 1$ and $\phi(a^2, b) \neq 0$.*

Proof. By Theorem 4.1, N is spanned by a, a^2, b, c where c is either ab or b^2. Also, if $c = ab$ then $b^2 = \gamma a^2 + \delta ab$ with $4\gamma + \delta^2 = 0$, and if $c = b^2$ then $ab = \alpha a^2 + \beta b^2$ with $4\alpha\beta = 1$.

Now assume $P(\phi)$ is simple. Then since a^2 and b^2 are each in M we have $\phi(a^2, b) \neq 0$ and $\phi(b^2, a) \neq 0$. By Lemma 3.2, $\phi(ab, a) \neq 0 \neq \phi(ab, b)$. Let x, y be in P; then $\phi(x, y) = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 b + \alpha_4 c$. Since ϕ is an antiflexible map, we have $0 = \phi(\phi(x, y), a^2) = \alpha_3\phi(b, a^2)$. Thus $\alpha_3 = 0$. Also, $0 = \phi(\phi(x, y), b^2) = \alpha_1\phi(a, b^2)$ which implies $\alpha_1 = 0$. Therefore, $\phi(x, y) = \alpha_0 + Y$ where $Y = \alpha_2 a^2 + \alpha_4 c$ is in M and, for any z in P, $0 = \phi(\phi(x, y), z) = \phi(Y, z)$. Hence $Y = 0$ and so $\phi(x, y) = \alpha_0$ is in $F \cdot 1$.

Conversely, suppose $H(\phi) \subseteq F \cdot 1$ and $\phi(a^2, b) = 4\eta \neq 0$. Note that M is spanned by a^2, c. If $c = ab$ then since ϕ is antiflexible, $\phi(ab, a) = -2\eta$, $\phi(b^2, a) = -2\delta\eta$ and $\phi(ab, b) = \delta\eta$. Let x be a nonzero element of M; then $x = \mu a^2 + \nu ab$. If $\nu \neq 0$, then $\phi(x, a) \neq 0$, and if $\nu = 0$ then $\phi(x, b) \neq 0$. Hence (17) is satisfied. Similarly if $c = b^2$, it can be shown that for each nonzero x in M there is a y in N with $\phi(x, y) \neq 0$. Hence in either case, by Theorem 2.5, $P(\phi)$ is simple.

In the rest of this section we will restrict ourselves to the nodal algebras of type (3, 6) over the field of complex numbers. The proof of the following theorem is found in [10].

Theorem 4.3. *The set of all associative-commutative nilalgebras of type* (3, 6) *over the field of complex numbers contains only* 13 *algebras that are distinct up to isomorphism*:

$P_1 = [u_1, u_2, u_3, u_1^2, u_2^2]$ (*algebra spanned by* u_1, u_2, u_3, u_1^2 *and* u_2^2), $u_3^2 = u_1^2 + u_2^2$, $u_j u_k = 0$ *for* $j, k = 1, 2, 3$;

$P_2 = [u_1, u_2, u_3, u_1^2, u_3^2]$, $u_2^2 = 2u_3^2 - u_1^2$, $u_j u_k = 0$ *for* $j \neq k$; $j, k = 1, 2, 3$;

$P_3 = [u_1, u_2, u_3, u_1^2, u_2u_3]$, $u_2^2 = u_1^2 + iu_2u_3$, $u_3^2 = u_1^2 - iu_2u_3$, $u_1u_2 = u_1u_3 = 0$;

$P_4 = [u_1, u_2, u_3, u_1^2, u_2u_3]$, $u_2^2 = u_1^2 + (i-1)u_2u_3$, $u_3^2 = u_1^2 - (i+1)u_2u_3$, $u_1u_2 = u_1u_3 = 0$;

$P_5 = [u_1, u_2, u_3, u_1^2, u_1u_2]$, $u_2^2 = u_1^2 = u_3^2$, $u_2u_3 = -iu_1u_2$, $u_1u_3 = 0$;

$P_6 = [u_1, u_2, u_3, u_1^2, u_1u_3]$, $u_2^2 = u_1^2$, $u_2u_3 = iu_1u_3$, $u_1u_2 = u_3^2 = 0$;

$P_7 = [u_1, u_2, u_3, u_4, u_1^2]$, $u_2^2 = u_3^2 = u_4^2 = u_1^2$, $u_j u_k = 0$ *for* $j \neq k$; $j, k = 1, 2, 3, 4$;

$P_8 = [u_1, u_2, u_1^2, u_1u_2, u_2^2]$;

$P_9 = [u, u^2, v_1, v_2, v_3]$, v_j *in* M, $uv_j = v_jv_k = 0$ *for* $j, k = 1, 2, 3$;

$P_{10} = [u_1, u_2, u_1^2, v_1, v_2]$, v_j *is in* M, $u_2^2 = u_1^2$, $u_jv_k = 0$ *for* $j, k = 1, 2$, $u_1u_2 = 0$;

$P_{11} = [u_1, u_2, u_3, u_1^2, v]$, $u_1^2 = u_2^2 = u_3^2$, $u_ju_k = u_jv = 0$ *for* $j \neq k$; $j, k = 1, 2, 3$; v *in* M;

$P_{12} = [u_1, u_2, u_1^2, u_2^2, v]$, v *in* M, $u_1u_2 = u_1v = u_2v = 0$;

$P_{13} = [u_1, u_2, u_1^2, u_1u_2, v]$, $u_2^2 = u_1^2 - 2iu_1u_2$, $u_1v = u_2v = 0$.

Using this we have the following result.

Theorem 4.4. *Let* $P = C \cdot 1 \oplus N$ *where* N *is an associative, commutative, nilalgebra of type* (3, 6) *over the field of complex numbers* C. *Then* P *is nearly simple if and only if* $N = P_6$.

Proof. Suppose P is nearly simple, then there is an antiflexible map ϕ with $P(\phi)$ simple.

If $N = P_1 = [u_1, u_2, u_3, u_1^2, u_2^2]$, $u_3^2 = u_1^2 + u_2^2$, $u_1u_2 = u_2u_3 = u_3u_1 = 0$, then $\phi(u_1u_2, u_1) = \phi(u_1u_3, u_1) = 0$. So by Lemma 3.2, $\phi(u_1^2, u_2) = \phi(u_1^2, u_3) = 0$. Hence $\phi(u_1^2, z) = 0$ for all z in P. This is impossible as u_1^2 is a nonzero element of M and so $N \neq P_1$. The same reasoning also proves that N cannot be any one of P_2, P_3 and P_4.

So assume that $N = P_5 = [u_1, u_2, u_3, u_1^2, u_1u_2]$, $u_1^2 = u_2^2 = u_3^2$, $u_2u_3 = iu_1u_2$, $u_1u_3 = 0$. Then $\phi(u_1^2, u_2) = \phi(u_2^2, u_2) = 0$ and $\phi(u_1^2, u_3) = \phi(u_3^2, u_3) = 0$. Consequently, $\phi(u_1^2, z) = 0$ for all z in P. Since u_1^2 is in M, $u_1^2 = 0$. This is impossible, so $N \neq P_5$.

Now suppose $N = P_7 = [u_1, u_2, u_3, u_4, u_1^2]$, $u_1^2 = u_2^2 = u_3^2 = u_4^2$, $u_ju_k = 0$ for

$j \neq k$; $j, k = 1, 2, 3, 4$. Then P is of type $(3, 6, 4)$. But by Theorem 2.10 there is no nearly simple algebra of type $(3, 6, 4)$. Thus $N \neq P_7$.

If $N = P_8 = [u_1, u_2, u_1^2, u_1u_2, u_2^2]$, then since u_1^2, u_2^2 are in M, $\phi(u_1^2, u_2) \neq 0$ and $\phi(u_2^2, u_1) \neq 0$. Also, by Lemma 3.2, $\phi(u_1^2, u_2) + 2\phi(u_1u_2, u_1) = 0$ and $\phi(u_2^2, u_1) + 2\phi(u_1u_2, u_2) = 0$. Therefore, $\phi(u_1u_2, u_1) \neq 0 \neq \phi(u_1u_2, u_2)$. We will first show that $H(\phi) \subseteq C \cdot 1$. Let x, y be in P, then

$$\phi(x, y) = \alpha_0 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_1^2 + \alpha_4u_1u_2 + \alpha_5u_2^2,$$

α_i in C for $i = 0, 1, 2, 3, 4, 5$. Since ϕ is an antiflexible map, $0 = \phi(\phi(x, y), u_2^2) = \alpha_1\phi(u_1, u_2^2)$ which implies $\alpha_1 = 0$. Also, $0 = \phi(\phi(x, y), u_1^2) = \alpha_2\phi(u_2, u_1^2)$ implies that $\alpha_2 = 0$. Therefore, $\phi(x, y) = \alpha_0 + Y$ where $Y = \alpha_3u_1^2 + \alpha_4u_1u_2 + \alpha_5u_2^2$ is in M. Now for any z in P, $0 = \phi(\phi(x, y), z) = \phi(Y, z)$ so $Y = 0$. Hence $H(\phi) \subseteq C \cdot 1$. Suppose $\phi(u_1u_2, u_1) = \alpha \neq 0$ and $\phi(u_1u_2, u_2) = \beta \neq 0$. Then by Lemma 3.2, $\phi(u_1^2, u_2) = -2\alpha$ and $\phi(u_2^2, u_1) = -2\beta$. Let $x = 2\beta u_1u_2 + \alpha u_2^2 + \beta^2/\alpha u_1^2$. Then x is a nonzero element of M and $\phi(x, u_1) = \phi(x, u_2) = 0$. Hence for all z in P, $\phi(x, z) = 0$ and so $x = 0$. This is a contradiction and so $N \neq P_8$.

Let $N = P_9 = [u, u^2, v_1, v_2, v_3]$, v_j in M, $uv_j = v_jv_k = 0$ for $j, k = 1, 2, 3$. Since u^2 is in N_2, $\phi(u^2, v_j) = 0$ for $j = 1, 2, 3$. Hence $u^2 = 0$ which is impossible. Thus $N \neq P_9$.

Next suppose $N = P_{10} = [u_1, u_2, u_1^2, v_1, v_2]$, v_j in M, $u_1^2 = u_2^2$, $u_1u_2 = 0 = u_jv_k$ for $j, k = 1, 2$. Then $\phi(u_1^2, u_j) = \phi(u_1^2, v_j) = 0$ for $j = 1, 2$. Consequently, $u_1^2 = 0$ which is a contradiction.

Now assume $N = P_{11} = [u_1, u_2, u_3, u_1^2, v]$, v is in M, $u_1^2 = u_2^2 = u_3^2$, $u_ju_k = u_jv = 0$ for $j \neq k$; $j, k = 1, 2, 3$. By Theorem 2.1 it follows that $\phi(u_1^2, u_1) = \phi(u_1^2, v_j) = 0$ for $j = 1, 2, 3$. Since u_1^2 is in M, $u_1^2 = 0$. This is impossible, so $N \neq P_{11}$.

If $N = P_{12} = [u_1, u_2, u_1^2, u_2^2, v]$, v is in M, $u_1u_2 = u_jv = 0$ for $j = 1, 2$, then $\phi(u_1^2, u_j) = -2\phi(u_1u_j, u_1) = 0$, $\phi(u_1^2, v) = -2\phi(u_1v, u_1) = 0$ for $j = 1, 2$. Therefore $u_1^2 = 0$, a contradiction.

Finally, suppose $N = P_{13} = [u_1, u_2, u_1^2, u_1u_2, v]$, v in M, $u_2^2 = u_1^2 - 2iu_1u_2$, $u_jv = 0$ for $j = 1, 2$. Then $H(\phi) \subseteq C \cdot 1$. For, if x, y are in P, then $\phi(x, y) = \alpha_0 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_1^2 + \alpha_4u_1u_2 + \alpha_5v$, α_j in C, $j = 0, 1, 2, 3, 4, 5$. Since $P(\phi)$ is simple, $\phi(u_1^2, u_2) \neq 0$, $\phi(u_1u_2, u_1) \neq 0 \neq \phi(u_1u_2, u_2)$, and either $\phi(v, u_1) \neq 0$ or $\phi(v, u_2) \neq 0$. Now $0 = \phi(\phi(x, y), u_1^2) = \alpha_2\phi(u_2, u_1^2)$ which implies $\alpha_2 = 0$. Similarly, $0 = \phi(\phi(x, y), u_1u_2) = \alpha_1\phi(u_1, u_1u_2)$ implies $\alpha_1 = 0$. Thus we have $\phi(x, y) = \alpha_0 + Y$ where $Y = \alpha_3u_1^2 + \alpha_4u_1u_2 + \alpha_5v$ is in M. For any z in P, $0 = \phi(\phi(x, y), z) = \phi(Y, z)$ so $Y = 0$. Hence $\phi(x, y)$ is in $C \cdot 1$. Let $\phi(u_1u_2, u_1) = \alpha \neq 0$, $\phi(v, u_1) = \beta$, $\phi(v, u_2) = \gamma$. Then $\phi(u_1^2, u_2) = -2\alpha$, $\phi(u_2^2, u_1) = -2i\alpha$ and $\phi(u_1u_2, u_2) = -i\alpha$. If $\beta = 0$, define $x = \gamma u_1^2 + 2\alpha v$, $x \neq 0$, x is in M. Then

$\phi(x, u_1) = 0 = \phi(x, u_2)$. Consequently, $\phi(x, z) = 0$ for all z in P and so $x = 0$, which is impossible. On the other hand, if $\beta \neq 0$, define $x = 2\beta u_1 u_2 - 2\alpha v + (i\beta - \gamma)u_1^2$. Then x is a nonzero element in M and $\phi(x, u_1) = \phi(x, u_2) = 0$. Hence $\phi(x, z) = 0$ for all z in P and so $x = 0$. This is a contradiction.

Thus the only possibility left up to isomorphism is $N = P_6 = [u_1, u_2, u_3, u_1^2, u_1u_3]$, $u_1^2 = u_2^2$, $u_2u_3 = iu_1u_3$, $u_3^2 = u_1u_2 = 0$. In this case $\phi(u_1^2, u_2) = \phi(u_2^2, u_2) = 0$, $\phi(u_2^2, u_1) = 0$. Since $P(\phi)$ is simple, $\phi(u_1^2, u_3) = \phi(u_2^2, u_3) = -2\phi(u_1u_3, u_1) \neq 0$. By Lemma 3.2, $0 = \phi(u_2^2, u_3) + 2\phi(u_2u_3, u_2) = \phi(u_1^2, u_3) + 2i\phi(u_1u_3, u_2)$ and, by Theorem 2.3, $0 = \phi(u_1u_2, u_3) + \phi(u_1u_3, u_2) + \phi(u_2u_3, u_1) = \phi(u_1u_3, u_2) + i\phi(u_1u_3, u_1) = \phi(u_1u_3, u_2 + iu_1)$.

Conversely, suppose $P = C \cdot 1 + P_6$. Define ϕ on the basis of P as follows:

$$\phi(u_1u_3, u_1) = 1 = -\phi(u_1, u_1u_3),$$
$$\phi(u_1^2, u_3) = -2 = -\phi(u_3, u_1^2),$$
$$\phi(u_1u_3, u_2) = -i = -\phi(u_2, u_1u_3),$$
$$\phi(x, y) = 0 \quad \text{for any other pair of basis elements } x \text{ and } y.$$

Extend ϕ belinearly to all of $P \times P$. Then it is a routine to verify that ϕ is an antiflexible map and $P(\phi)$ satisfies (17). Hence, by Theorem 2.5, $P(\phi)$ is simple and the proof is complete.

Having characterized all nearly simple nodal algebras of type (3, 6) over the field of complex numbers we are now interested in finding all possible candidates for ϕ.

Theorem 4.5. *Let $P = C \cdot 1 \oplus P_6$ where C is the basefield of complex numbers and $P_6 = [u_1, u_2, u_3, u_1^2, u_1u_3]$, $u_1^2 = u_2^2$, $u_2u_3 = iu_1u_3$, $u_3^2 = u_1u_2 = 0$ and let ϕ be an antiflexible map. Then $P(\phi)$ is simple if and only if*

(50) *for each x, y in P there exists α_j, $j = 0, 1, 2, 4, 5$, such that $\alpha_1 = i\alpha_2$ and $\phi(x, y) = \alpha_0 + \alpha_1u_1 + \alpha_2u_2 + \alpha_4u_1^2 + \alpha_5u_1u_3$,*

(51) *$\phi(u_1u_3, u_1) = \beta_0 + \beta_1u_1 + \beta_2u_2 + \beta_4u_1^2 + \beta_5u_1u_3$ for some β_j in C, $j = 0, 1, 2, 4, 5$, with $\beta_1 = i\beta_2$ and $\beta_0 \neq 0$.*

Proof. Assume $P(\phi)$ is simple. Then $\phi(u_1^2, u_3) = -2\phi(u_1u_3, u_1) \neq 0$. Let x, y be in P, then $\phi(x, y) = \alpha_0 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 + \alpha_4u_1^2 + \alpha_5u_1u_3$, α_j in C for $j = 0, 1, 2, 3, 4, 5$. Since ϕ is an antiflexible map, $0 = \phi(\phi(x, y), u_1^2) = \alpha_3\phi(u_3, u_1^2)$. Therefore, $\alpha_3 = 0$. Also, $0 = \phi(\phi(x, y), u_1u_3) = \alpha_1\phi(u_1, u_1u_3) + \alpha_2\phi(u_2, u_1u_3)$ and $0 = \phi(u_1u_2, u_3) + \phi(u_1u_3, u_2) + \phi(u_2u_3, u_1) = \phi(u_1u_3, u_2) + i\phi(u_1u_3, u_1)$. Consequently, $(\alpha_1 - i\alpha_2)\phi(u_1u_3, u_1) = 0$ which implies $\alpha_1 = i\alpha_2$. If $x = u_1u_3$ and $y = u_1$ we get $\phi(u_1u_3, u_1) = \beta_0 + i\beta_2u_1 + \beta_2u_2 + \beta_4u_1^2 + \beta_5u_1u_3$ for some β_j, $j = 0, 2, 4, 5$ in C. If $\beta_0 = 0$ then $\phi(u_1u_3, u_1)$ generates a proper ideal of $P(\phi)$ so $\beta_0 \neq 0$. We observe here that for each x, y in P there is an α in C

such that $\phi(x, y) = \alpha\phi(u_1u_3, u_1)$, so if $\phi(x, y) \neq 0$ then $\alpha_0 \neq 0$.

Conversely suppose ϕ satisfies (50) and (51). Let J be an ideal of $P(\phi)$ with x in J, $x \neq 0$. Then $x = \gamma_0 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3 + \gamma_4u_1^2 + \gamma_5u_1u_3$, γ_j in C. If $\gamma_0 \neq 0$ then $(1/\gamma_0)x = 1 - z$ is in J for some z with $z^3 = 0$, and if $\gamma_0 = 0$, $\gamma_3 \neq 0$ then $(-1/2\gamma_3\beta_0)\phi(x, u_1^2) = 1 - z$ is in J. Now suppose $\gamma_0 = \gamma_3 = 0$ and $\gamma_2 \neq 0$. Then $(-1/2\gamma_2\beta_0)\phi(x * u_2, u_3) = (-1/2\beta_0)\phi(u_2^2, u_3) = 1 - z$ is in J. Let $\gamma_j = 0$ for $j = 0, 2, 3$ and $\gamma_1 \neq 0$. Then $(-1/\beta_0\gamma_1)\phi(x, u_1u_3) = (-1/\beta_0)\phi(u_1, u_1u_3) = 1 - z$. If $\gamma_j = 0$ for $j = 0, 1, 2, 3$, $\gamma_4 \neq 0$, then since $\phi(u_1u_3, u_3) = (-1/2)\phi(u_3^2, u_1) = 0$, $(-1/2\gamma_4\beta_0)\phi(x, u_3) = 1 - z$ is in J. Finally, if $\gamma_j = 0$ for $j = 0, 1, 2, 3, 4$ and $\gamma_5 \neq 0$, then $(1/\gamma_5\beta_0)\phi(x, u_1) = 1 - z$ is in J.

Hence in all cases $1 = (1 - z)(1 + z + z^2)$ is in J and so $J = P(\phi)$. Therefore, $P(\phi)$ is simple and we are done.

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