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THE STUDY OF COMMUTATIVE SEMIGROUPS WITH GREATEST GROUP-HOMOMORPHISM(1)

BY

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ABSTRACT. This paper characterizes commutative semigroups which admit a greatest group-homomorphism in various ways. One of the important theorems is that a commutative semigroup S has a greatest group-homomorphic image if and only if for every $a \in S$ there are $b, c \in S$ such that $abc = c$. Further the authors study a relationship between S and a certain cofinal subsemigroup and discuss the structure of commutative separative semigroups which have a greatest group-homomorphic image.

1. **Introduction.** In the study of semigroups it would be natural to find a relationship between semigroups and groups by some means. This is our intention in studying group-homomorphisms or group-congruences. Many mathematicians have already studied characterizations of group-congruences on a semigroup in terms of certain subsemigroups ([5], [7], [13], [14], [16]). We are mainly interested in the greatest group-homomorphism on a commutative semigroup. However it does not exist in general. Accordingly the following questions are raised: Under what condition on a commutative semigroup S does there exist a greatest group-homomorphism on S? What is the structure of S admitting a greatest group-homomorphism? The "external" characterization of those semigroups, i.e. in terms of their homomorphisms to other semigroups, is easy, but the "internal" characterization, i.e. the determination of the structure and construction, seems difficult. We have not seen any paper treating the last problem except some special case treated by Head [10], and McAlister and O'Caroll [15]. This is probably due to the lack of a foundation from which to start. The purpose of this paper is to contribute to this point in a manner which may be useful in the future,

In §3 we first characterize the existence of a greatest group-homomorphism in terms of a few homomorphisms including the so-called Grothendieck homomorphism

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(1) A part of these results were presented by H. B. Hamilton at the meetings of the American Mathematical Society at Davis, April 1970, and at Pasadena, November 1970.

and as a consequence we derive a multiplicative condition which is necessary and sufficient. Theorem 3.5 is most important and fundamental for the future development of this topic. We treat special cases in §§4 and 5: a semilattice of groups in §4 and the separative case in §5. The concept of cofinal cluster in §6, together with Theorem 3.5, gives us a foundation on which to begin the study of the internal structure of semigroups having a greatest group-homomorphism. In §7 we suggest a direction for this study. In particular, we treat the separative case whose greatest semilattice homomorphic image is a chain. §2 gives the basic definitions and concepts.

A part of the results of this paper is reported in [21] without proof.

2. **Basic concepts.** Let S be a commutative semigroup. If f is a homomorphism of S *onto* a group G, f is called a group-homomorphism and G is called a group-homomorphic image of S. A congruence ρ on S is called a group-congruence on S if ρ is induced by a group-homomorphism of S, that is, S/ρ is a group.

Assume that f_0 is a group-homomorphism of S onto G_0 and that if f is any group-homomorphism of S onto G there is a homomorphism h of G_0 onto G such that $f(x) = hf_0(x)$ for all $x \in S$. Then f_0 is called a greatest group-homomorphism of S and G_0 is called a greatest group-homomorphic image of S, while the congruence ρ_0 induced by f_0 is called the smallest group-congruence on S. Not all commutative semigroups have a smallest group-congruence.

Let f be a group-homomorphism, $S \longrightarrow G$, and ρ the group-congruence on S induced by f. The inverse image of the identity element of $G \cong S/\rho$ under f is called the kernel of f or of ρ, and is denoted by $\operatorname{Ker} f$ or $\operatorname{Ker}\rho$.

A subsemigroup H of a commutative semigroup S is called cofinal in S or a cofinal subsemigroup of S if for every $x \in S$ there is an element $y \in S$ such that $xy \in H$. A subsemigroup U of S is called unitary in S if $x \in S$, $a \in U$ and $ax \in U$ implies $x \in U$. For example the kernel of a group-congruence ρ on S is unitary and cofinal in S. In particular, a cofinal subsemigroup of a semilattice has the following sense. Let L be a semilattice. It is regarded as a join semilattice (or upper semilattice), i.e. $\alpha, \beta \in L$, $\alpha \leq \beta$, if and only if $\alpha\beta = \beta$. A cofinal subsemilattice M of L is a subsemilattice M of L which satisfies: for each $\alpha \in L$ there is $\beta \in M$ such that $\alpha \leq \beta$. A cofinal subsemilattice M is unitary in L if and only if $M = L$.

Dubreil [7] and others ([5], [13], [14], [16]) studied group-congruences and Theorem 2.1 is a consequence of known theorems [4, Chapter 10] but we can obtain them directly because of commutativity.

Let S be a commutative semigroup and A a cofinal subsemigroup of S. Define a relation ρ_A on S by

$$x \rho_A y \text{ if and only if } ax = by \quad \text{for some } a, b \in A.$$

Theorem 2.1. *The following statements hold.*

(2.1.1) *ρ_A is a group-congruence on S and $A \subseteq \operatorname{Ker}\rho_A$.*

(2.1.2) *If $U = \operatorname{Ker}\rho_A$, then $\rho_A = \rho_U$.*

(2.1.3) *If σ is any group-congruence on S and if we let $V = \operatorname{Ker}\sigma$, then $\sigma = \rho_V$.*

(2.1.4) *The kernel U of the homomorphism $S \longrightarrow S/\rho_A$ is equal to A if and only if A is unitary.*

Let X be a subsemigroup of S and let $\tilde{X}$ be the unitary subsemigroup generated by X, i.e. the smallest unitary subsemigroup of S containing X.

Proposition 2.2. *$\tilde{X} = \{x \in S\colon ax \in X$ for some $a \in X\}$ and the following are satisfied.*

(2.2.1) *$X \subseteq \tilde{X}$.*

(2.2.2) *$X \subseteq Y$ implies $\tilde{X} \subseteq \tilde{Y}$.*

(2.2.3) *$\tilde{\tilde{X}} = \tilde{X}$.*

Proposition 2.3. (2.3.1) *Let h be a homomorphism of S onto S'. If A is cofinal [unitary] in S then $h(A)$ is cofinal [unitary] in S'.*

(2.3.2) *If A is cofinal [unitary] in B and if B is cofinal [unitary] in C then A is cofinal [unitary] in C.*

Thus the join semilattice of all unitary cofinal subsemigroups of S is isomorphic to the join semilattice of all group-congruences on S under the map $A \longrightarrow \rho_A$.

Let $\mathcal{P}$ denote an implicational property, i.e. a property expressed by a system of implications. A homomorphism h of S onto S' is called a $\mathcal{P}$-homomorphism of S if S' satisfies $\mathcal{P}$, and the congruence on S induced by h is called a $\mathcal{P}$-congruence on S. It goes without saying that there always exists a smallest $\mathcal{P}$-congruence on S. The terminology "greatest $\mathcal{P}$-homomorphism" and "greatest $\mathcal{P}$-homomorphic image" are defined as usual. The partition of all elements of S induced by the greatest $\mathcal{P}$-homomorphism of S is called the greatest $\mathcal{P}$-decomposition of S. In this paper "$\mathcal{P}$" is replaced by one of "semilattice", "cancellative" and "separative".

We define a *gr*-homomorphism to be a homomorphism g_0 of S into an abelian group Gr having the universal repelling property with respect to homomorphisms of S into abelian groups (see [1], [4, §12.1], [12, p. 43]). That is, if g is a homomorphism of S into an abelian group G there is a unique homomorphism h of Gr into G such that $g = hg_0$. Gr is called a *gr*-group (Grothendieck group) or free abelian group on S.

The smallest cancellative congruence σ on a commutative semigroup S is given by $x\ \sigma\ y$ iff $ax = ay$ for some $a \in S$. The *gr*-homomorphism g_0 is obtained as the natural homomorphism $S \longrightarrow S/\sigma$ followed by the embedding of S/σ into its group Q of quotients. Clearly all commutative semigroups have a *gr*-homomorphism. We have the following immediately.

Proposition 2.4. *Let $C = S/\sigma$ where σ is defined as above.*

(2.4.1) *g_0 is both injective and surjective if and only if $S \cong C \cong Q$.*

(2.4.2) *g_0 is injective but not surjective if and only if $S \cong C$ and $C \ncong Q$.*

(2.4.3) *g_0 is not injective but is surjective if and only if $S \ncong C$ and $C \cong Q$.*

(2.4.4) *g_0 is neither injective nor surjective if and only if $S \ncong C$ and $C \ncong Q$.*

One sees the definition of archimedeaness in [3] or [17]. The classification of commutative archimedean semigroups can be characterized in Proposition 2.5 by the behavior of the *gr*-homomorphism. First the following classification is known as an immediate consequence of [3, Theorem 4. 12] or [17].

Commutative archimedean semigroups are classified as follows:

I. S has a unique idempotent.

(I.1) Nil-semigroup, i.e. S has a zero and some power of every element is a zero.

(I.2) S has an idempotent which is not a zero, i.e. S is an ideal extension of a nontrivial abelian group by a nil-semigroup.

II. S has no idempotent.

(II.1) S is cancellative, i.e. $\mathfrak{N}$-semigroup.

(II.2) S is not cancellative.

Proposition 2.5. *Let S be a commutative archimedean semigroup. Then*

(2.5.1) *S is an abelian group if and only if g_0 is both injective and surjective.*

(2.5.2) *S is not a group but has an idempotent if and only if g_0 is not injective but surjective.*

(2.5.3) *S is an $\mathfrak{N}$-semigroup if and only if g_0 is injective but not surjective.*

(2.5.4) *S is neither cancellative nor does it have an idempotent if and only if g_0 is neither injective nor surjective.*

Proof. (2.5.1) is obvious.

(2.5.2) If S is not a group but has an idempotent e, S is not cancellative. We can see that $Se = \{xe\colon x \in S\}$ is the *gr*-group and $g_0(x) = xe$. Thus g_0 is not injective but surjective.

(2.5.3) If S is an $\mathfrak{N}$-semigroup, then $S \cong C$ and $C \ncong Q$. By Proposition 2.4, f is injective but not surjective.

(2.5.4) If S is not cancellative and does not have an idempotent, then $S \ncong C$ and C is an $\mathfrak{N}$-semigroup by [19, Theorem 4, p. 262]. Hence f is neither injective nor surjective.

We have thus proved the implication of (2.5.1) through (2.5.4) in one direction. The other direction is a logical consequence of the disjointness of the statements. Q. E. D.

3. Characterization by homomorphisms and its consequence. Before entering the main discussion we need some preparation. Assume that S is not a group.

Proposition 3.1. *Let S be a commutative semigroup. S has a greatest group-homomorphism if and only if a proper ideal I of S has a greatest group-homomorphism.*

Proof. Let $f: S \longrightarrow G$ be a group-homomorphism. Since I is an ideal of S, $f(I)$ is an ideal of the group G, and hence $f(I) = f(S) = G$. Thus $f \mid I$ is a group-homomorphism $I \longrightarrow G$.

Let $g: I \longrightarrow H$ be a group-homomorphism. Define $\bar{g}: S \longrightarrow H$ by $\bar{g}(x) = g(ax)$ where $a \in I$ and $g(a) = e$, e being the identity element of H. If $g(a) = g(b) = e$ for $a, b \in I$, then, for $x \in S$,

$$g(ax) = g(ax)g(b) = g(axb) = g(bxa) = g(bx)g(a) = g(bx).$$

Hence $\bar{g}$ is well defined. Also we have, for $x, y \in S$, that

$$\bar{g}(xy) = g(xya) = g(xya)g(a) = g(xyaa) = g(xaya) = g(xa)g(ya) = \bar{g}(x)\bar{g}(y),$$

therefore $\bar{g}$ is a homomorphism. Clearly $\bar{g} \mid I = g$, thus $\bar{g}$ is surjective. To prove uniqueness of $\bar{g}$ let $f: S \longrightarrow G$ be any group-homomorphism such that $f \mid I = g$. Choose $a \in \operatorname{Ker} g$. Then for all $x \in S$, $f(x) = f(a)f(x) = f(ax) = g(ax) = \bar{g}(x)$.

We have proved that there is a one-to-one correspondence from all group-homomorphisms f of S to all group-homomorphisms g of I by the map $f \longrightarrow g$, $g = f \mid I$. In other words the semilattice of all group-congruences on S is isomorphic to the semilattice of all group-congruences on I. Therefore our conclusion is easily derived. (Notice that the statement is true for all proper ideals I, equivalently for some proper ideal.) Q.E.D.

A commutative semigroup is called separative [3] if it satisfies $a^2 = ab = b^2$ implies $a = b$. It is known in [3] that a commutative semigroup is separative if and only if its archimedean components are cancellative. A smallest separative congruence τ_0 on a commutative semigroup S is given by

$a\tau_0 b$ if and only if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ for some positive integer n.

Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be the greatest semilattice decomposition of a commutative semigroup S. Each S_α is archimedean ([3], [17]).

We define a relation τ_1 on S as follows:

$a\tau_1 b$ if and only if a and b are in the same archimedean component S_α and $ax = bx$ for some $x \in S_\alpha$.

Lemma 3.2. $\tau_1 = \tau_0$.

Proof. $\tau_0 \subseteq \tau_1$ is clear. To prove $\tau_1 \subseteq \tau_0$, assume $a\tau_1 b$. Then a and b are in S_α and $ax = bx$ for $x \in S_\alpha$. Since S_α is archimedean $b^n = xy$, $a^n = xz$ for some

$y, z \in S_\alpha$ and some $n > 0$. Now $ax = bx$ implies $axy = bxy$, hence $ab^n = b^{n+1}$. Likewise $ba^n = a^{n+1}$. Q.E.D.

We have

Theorem 3.3. *Let S be a commutative semigroup. Then the following are equivalent:*

(3.3.1) *The gr-homomorphism g_0 is surjective.*

(3.3.2) *The greatest cancellative homomorphic image C of S is a group.*

(3.3.3) *S has a greatest group-homomorphism.*

(3.3.4) *The greatest separative homomorphic image of S has a greatest group-homomorphism.*

Proof. The equivalence of (3.3.1) and (3.3.2) follows from Proposition 2.4. The equivalence between (3.3.2) and (3.3.3) is obtained by Head [10], and McAlister and O'Caroll [15]. The equivalence of (3.3.3) and (3.3.4) follows from the fact that groups are separative. Q.E.D.

Corollary 3.4. *Let S be a commutative archimedean semigroup. The following are equivalent:*

(3.4.1) *The gr-homomorphism g_0 is surjective.*

(3.4.2) *S has an idempotent.*

(3.4.3) *S has a greatest group-homomorphism.*

Remark. Head [10] and McAlister and O'Caroll [15] proved that a commutative semigroup S has a greatest group-homomorphic image if and only if the greatest cancellative homomorphic image of S is a group. Head did it from a general point of view [4, §11.6]; McAlister and O'Caroll used a group theoretical result. If S is a commutative cancellative semigroup and if S is not a group, S is a semilattice of S_0 and S_1 where S_0 is a nonempty ideal without idempotent and S_1 is either a group or empty [11]. Hence Head, McAlister and O'Caroll's result is equivalent to the statement that a commutative cancellative semigroup without idempotent has no smallest group-congruence. However, the following is still true. A commutative cancellative semigroup without idempotent has no minimal group-congruence. Note that "minimal" means "having no smaller one", hence "minimal" is weaker than "smallest". To prove the above, a minimal cofinal unitary subsemigroup is used. (See details in [22].)

As a consequence of Theorem 3.3 we have

Theorem 3.5.(2) *Let S be a commutative semigroup. The following are equivalent:*

(3.5.1) *S has a greatest group-homomorphism.*

(2) This was obtained by T. Tamura.

(3.5.2) *For all* $x, y \in S$ *there exist* $z, u \in S$ *such that* $xu = yzu$.

(3.5.3) *For every* $a \in S$ *there are* $b, c \in S$ *such that* $abc = c$.

Proof. We prove (3.3.2) $\Rightarrow$ (3.5.3) $\Rightarrow$ (3.5.2) $\Rightarrow$ (3.3.2) since (3.5.1) is equivalent to (3.3.2) by Theorem 3.3.

(3.3.2) $\Rightarrow$ (3.5.3). Let C be the greatest cancellative homomorphic image of S and $f: S \to C$ be the homomorphism of S onto C. Let $a \in S$. Since C is a group by (3.3.2), C contains an identity element denoted by $f(p)$ and there is $f(b) \in C$ such that $f(a)f(b) = f(p)$. Multiplying both sides by $f(p)$, we have $f(a)f(b)f(p) = f(p)^2 = f(p)$ and then $f(abp) = f(p)$. Recall that the smallest cancellative congruence σ on S is given by $x\sigma y$ if and only if $xz = yz$ for some $z \in S$. Accordingly $f(abp) = f(p)$ implies $abpz = pz$ for some $z \in S$. Let $c = pz$. Then we have $abc = c$.

(3.5.3) $\Rightarrow$ (3.5.2). Let $x, y \in S$. By (3.5.3) there are $u, v \in S$ such that $u = yvu$, hence $xu = xyvu = yxvu$. Let $z = xv$. Then we have $xu = yzu$.

(3.5.2) $\Rightarrow$ (3.3.2). Let σ be the smallest cancellative congruence on S. By the assumption for $x, y \in S$ there are $z, u \in S$ such that $xu = yzu$, which implies $x\sigma yz$. This tells us that S/σ is a group. Thus (3.3.2) is derived. Q.E.D.

Remark. The fact that (3.5.1) is equivalent to (3.5.3) is obtained by M. S. Putcha independently of the authors. See Corollary 2.2, p. 52, Semigroup Forum 3 (1971).

Also, we can easily prove that (3.5.2) implies (3.5.3): Let a be an arbitrary element of S and take $x = a^2$, $y = a$. By (3.5.2) there are $z, u \in S$ such that $a^2zu = au$. Take $b = z$, $c = au$. Then we have $abc = c$.

An element p of a commutative semigroup S is called a local identity element if $pa = a$ for some $a \in S$. Let I be the set of all local identity elements of S.

Lemma 3.6. *If I is not empty, I is a unitary subsemigroup of S.*

Proof. Let $p, q \in I$. By definition $pa = a$ and $qb = b$ for some $a, b \in S$. Then $(pq)(ab) = (pq)(ba) = p(qb)a = pba = (pa)b = ab$. Hence $pq \in I$. Assume $p \in I$, $x \in S$ and $px \in I$. There are a and b such that $pa = a$ and $pxb = b$. Then $xba = xbpa = ba$, whence $x \in I$. Thus I is a unitary subsemigroup of S. Q.E.D.

I is called the local identity subsemigroup of S.

Theorem 3.5′. *A commutative semigroup S has a greatest group-homomorphism if and only if the local identity subsemigroup of S is not empty and it is cofinal in S.*

Proof. This is the restatement of Theorem 3.5. The element ab in (3.5.3) is a local identity element of S. Cofinality is obvious. Q.E.D.

The alternate proof of the "if" part of Theorem 3.5′. Let p be a local identity element of S. If ϕ is any group-homomorphism of S, then $pa = a$ implies

$(p\phi)(a\phi) = a\phi$, whence $p\phi \in \operatorname{Ker}\phi$. On the other hand, by Lemma 3.6 and the assumption, the local identity subsemigroup of S is unitary cofinal and hence I is the smallest unitary cofinal subsemigroup. Accordingly S has a greatest group-homomorphism. Q.E.D.

Corollary 3.7. *If the subsemigroup of all idempotents of S is not empty and cofinal in S, then S has a greatest group-homomorphism.*

Thus the existence of the cofinal local identity subsemigroup completely determines the greatest group-homomorphism. Of course the condition "cofinality" is important. Only the existence of local identities is not sufficient, for example, the infinite cyclic semigroup with identity element adjoined does not have a greatest group-homomorphism.

4. **Group-congruences on a semilattice of groups.** As an application of Theorem 3.5 we have

Proposition 4.1. *A semilattice of abelian groups has a greatest group-homomorphism.*

Proof. Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be a semilattice Γ of abelian groups S_α. Let x, $y \in S$ and $x \in S_\alpha$, $y \in S_\beta$. Let $u \in S_{\alpha\beta}$ be taken arbitrarily. Then xu, $yu \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is a group, $xu = yuz$ for some $z \in S_{\alpha\beta}$. By Theorem 3.5, S has a greatest group-homomorphism. Q.E.D.

A semilattice of abelian groups is determined by a semilattice Γ (assumed to be an upper semilattice here), a system of abelian groups $\{S_\alpha : \alpha\in\Gamma\}$ and a transitive system $\{\phi_\alpha^\beta : \alpha \le \beta,\ \alpha,\ \beta\in\Gamma\}$ of homomorphisms ϕ_α^β of S_α into S_β, $\alpha\le\beta$. In detail a transitive system $\{\phi_\alpha^\beta : \alpha \le \beta,\ \alpha,\ \beta\in\Gamma\}$ is a system of homomorphisms $\phi_\alpha^\beta : S_\alpha \to S_\beta$, $\alpha\le\beta$, α, $\beta\in\Gamma$, satisfying

(1) ϕ_α^α is the identity mapping of S_α for all $\alpha\in\Gamma$.

(2) $\phi_\beta^\gamma \phi_\alpha^\beta = \phi_\alpha^\gamma$ for all $\alpha\le\beta\le\gamma$.

The operation is defined in the disjoint union $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ as follows: for $x_\alpha \in S_\alpha$, $y_\beta \in S_\beta$,

$$x_\alpha y_\beta = \phi_\alpha^{\alpha\beta}(x_\alpha)\cdot\phi_\beta^{\alpha\beta}(y_\beta).$$

S is, of course, commutative. (See details in [3, p. 128].)

Define a relation ρ_0 on S by

$$x\rho_0 y \text{ iff } \phi_\alpha^\xi(x) = \phi_\beta^\xi(y),\quad x\in S_\alpha,\ y\in S_\beta \quad \text{for some } \xi \ge \alpha\beta.$$

It is easy to see ρ_0 is an equivalence. To show compatibility, let $x\in S_\alpha$, $y\in S_\beta$, $z \in S_\gamma$ and assume $\phi_\alpha^\xi(x) = \phi_\beta^\xi(y)$. Then

$$\begin{aligned}\phi_{\alpha\gamma}^{\xi\gamma}(zx) &= \phi_{\alpha\gamma}^{\xi\gamma}(\phi_\gamma^{\alpha\gamma}(z)\phi_\alpha^{\alpha\gamma}(x)) = \phi_\gamma^{\xi\gamma}(z)\phi_\alpha^{\xi\gamma}(x) = \phi_\gamma^{\xi\gamma}(z)\phi_\beta^{\xi\gamma}(y)\\ &= \phi_{\beta\gamma}^{\xi\gamma}(\phi_\gamma^{\beta\gamma}(z)\phi_\beta^{\beta\gamma}(y)) = \phi_{\beta\gamma}^{\xi\gamma}(zy).\end{aligned}$$

Therefore ρ_0 is a congruence on S.

Proposition 4.2. *$\rho_0 = \{(x, y) \in S \times S: zx = zy$ for some $z \in S\}$, and ρ_0 is the smallest group-congruence on S.*

Proof. First we prove $x\rho_0 y$ implies $zx = zy$ for some $z \in S$. Let $x \in S_\alpha$, $y \in S_\beta$. By assumption $\phi_\alpha^\xi(x) = \phi_\beta^\xi(y)$ for some $\xi \geq \alpha\beta$. Let $z \in S_\xi$. From the above proof we have $\phi_\xi^\xi(zx) = \phi_\xi^\xi(zy)$ whence $zx = zy$. Conversely assume $zx = zy$, $x \in S_\alpha$, $y \in S_\beta$, $z \in S_\gamma$. Then $\gamma\alpha = \gamma\beta$. By definition,

$$\phi_\gamma^{\alpha\gamma}(z)\phi_\alpha^{\alpha\gamma}(x) = \phi_\gamma^{\beta\gamma}(z)\phi_\beta^{\beta\gamma}(y),$$

and $\phi_\alpha^{\alpha\gamma}(x) = \phi_\beta^{\beta\gamma}(y)$ by cancellation in $S_{\alpha\gamma}$. Therefore, $x\rho_0 y$. Thus we have proved that ρ_0 is the smallest cancellative congruence on S. By Proposition 4.1 and Theorem 3.3, ρ_0 is the smallest group-congruence. Q.E.D.

S/ρ_0 coincides with the so-called direct limit $\varinjlim\{S_\alpha, \phi_\alpha^\beta, \Gamma\}$ of a system $\{S_\alpha: \alpha \in \Gamma\}$ with respect to ϕ_α^β in case Γ is a semilattice. It is usually defined as a factor group of the direct sum of S_α's without considering the semigroup S. (See, for example, [8], [12].)

Let $S = \bigcup\{S_\alpha, \phi_\alpha^\beta, \Gamma\}$ be a semilattice of abelian groups. For each $\alpha \in \Gamma$, let

$$K_\alpha = \bigcup\{\operatorname{Ker} \phi_\alpha^\xi : \alpha \leq \xi,\ \xi \in \Gamma\}$$

where $\bigcup$ denotes the set union. K_α is, however, a subgroup of S_α. Let $G = S/\rho_0$ and $G_\alpha = S_\alpha/K_\alpha$.

Proposition 4.3. $\operatorname{Ker}\rho_0 = \bigcup\{K_\alpha, \tilde{\phi}_\alpha^\beta, \Gamma\}$ *and* $G \cong \varinjlim\{G_\alpha, \bar{\phi}_\alpha^\beta, \Gamma\}$ *where $\tilde{\phi}_\alpha^\beta$ is the restriction of ϕ_α^β to K_α and $\bar{\phi}_\alpha^\beta$ is an injective homomorphism of G_α into G_β.*

5. **Separative case.** By Theorem 3.3 our problem is reduced to that in commutative separative semigroups. Let S be a commutative separative semigroup and $S = \bigcup_{\alpha \in \Gamma} S_\alpha$ be the greatest semilattice decomposition of S. Each S_α is cancellative archimedean and hence either a group or an $\mathfrak{N}$-semigroup. S can be embedded into a semigroup $C(S)$ which is a semilattice of abelian groups, $C(S) = \bigcup_{\alpha\in\Gamma} C_\alpha$, such that C_α is the quotient group of S_α, $C_\alpha = \{x_\alpha y_\alpha^{-1}: x_\alpha, y_\alpha \in S_\alpha\}$. (See [3].) The operation in $C(S)$ is defined by

$$(x_\alpha y_\alpha^{-1})(z_\beta u_\beta^{-1}) = (x_\alpha z_\beta)(y_\alpha u_\beta)^{-1}.$$

Let ρ_0 be the smallest group-congruence on $C(S)$. The kernel $\operatorname{Ker}\rho_0$ will be denoted by $\mathbf{K}(C(S))$.

Proposition 5.1. *Let S be a commutative separative semigroup. Then the following are equivalent.*

(5.1.1) *S has a greatest group-homomorphism.*

(5.1.2) *$S \cap \mathbf{K}(C(S)) \neq \emptyset$ and it is cofinal in S.*

Proof. (5.1.1) $\to$ (5.1.2). Let f_0 be the greatest group-homomorphism of $C(S)$ and let $f_1 = f_0 | S$. Then $f_1(S)$ is a cancellative homomorphic image of S but it is a subgroup of $f_0(S)$ because S has a greatest group-homomorphism and hence its greatest cancellative homomorphic image is a group by Theorem 3.3. Clearly $\operatorname{Ker} f_1 = S \cap \mathbf{K}(C(S)) \neq \varnothing$. Its cofinality is obvious.

(5.1.2) $\to$ (5.1.1). We are to prove (5.1.2) $\to$ (3.5.3). Then we will get (5.1.1) by Theorem 3.5. Let $\Gamma_1 = \{\alpha \in \Gamma: S_\alpha \cap \mathbf{K}(C(S)) \neq \varnothing\}$. Since Γ_1 is a homomorphic image of $S \cap \mathbf{K}(C(S))$, Γ_1 is cofinal in Γ. Let $a \in S$, say $a \in S_\xi$. There is $\alpha \in \Gamma_1$ such that $\xi \leq \alpha$. Take $u \in S_\alpha$ arbitrarily. Then $au \in S_\alpha$. Let f_0 be the same as defined just above. By assumption there is $d \in S_\alpha$ such that $f_0(d) = f_0(e)$ where e is the identity element of C_α. Since S_α is archimedean $auv = d^m$ for some $v \in S_\alpha$ and some $m > 0$; and $f_0(auv) = f_0(d^m) = f_0(e)$. Recalling that f_0 is the greatest cancellative homomorphic image of $C(S)$ by Proposition 4.2, there is an element $c'p^{-1} \in S$, c', $p \in S_\gamma$ for some γ.

$$auv(c'p^{-1}) = e(c'p^{-1}), \text{ hence } auvc' = ec'.$$

Let $b = uv$, $c = ec'$. Then $abc = auvec' = auvc' = ec' = c$. Q.E.D.

In the proof of (5.1.1) $\to$ (5.1.2), we have stated that $f_1(S)$ is a subgroup of $f_0(C(S))$ but in fact $f_1(S) = f_0(C(S))$. This is due to the following proposition which describes a relation between group-congruences of S and $C(S)$.

Proposition 5.2. *Let S be a commutative separative semigroup. A group-homomorphism $f: S \to G$ can be uniquely extended to a group-homomorphism $\bar{f}$ of $C(S)$ in the following sense:*

$\bar{f} | S = f$, $f(S) = \bar{f}(C(S))$ and if g is a group-homomorphism of $C(S)$ such that $f(S) \subseteq g(C(S))$ and $g | S = f$, then $g = \bar{f}$ on $C(S)$.

Proof. Let $x \in C(S)$, $x = ab^{-1}$ for some a, $b \in S_\alpha$. Define $\bar{f}$ by

$$\bar{f}(x) = f(a)(f(b))^{-1}.$$

If $x = a_1 b_1^{-1} = a_2 b_2^{-1}$, then $a_1 b_2 = a_2 b_1$ and $f(a_1) f(b_2) = f(a_2) f(b_1)$ whence $f(a_1)(f(b_1))^{-1} = f(a_2)(f(b_2))^{-1}$. Thus $\bar{f}$ is well defined. To prove $\bar{f}$ is a homomorphism,

$$\begin{aligned} \bar{f}(ab^{-1} \cdot cd^{-1}) &= \bar{f}(ac \cdot (bd)^{-1}) = f(ac)(f(bd))^{-1} \\ &= f(a)(f(b))^{-1} f(c)(f(d))^{-1} = \bar{f}(ab^{-1}) \bar{f}(cd^{-1}). \end{aligned}$$

It is easy to show $\bar{f}(x) = f(x)$ for $x \in S$ and $\bar{f}(C(S)) = G$. Hence $\bar{f}$ is a group-homomorphism $C(S) \to G$. To prove uniqueness let $g: C(S) \to G'$ be a group-homomorphism such that $g(x) = f(x)$ for all $x \in S$ and $G \subseteq G'$. Then for $x = ab^{-1} \in C(S)$,

$$g(x) = g(ab^{-1}) = g(a) g(b)^{-1} = f(a) f(b)^{-1} = \bar{f}(ab^{-1}) = \bar{f}(x). \quad \text{Q.E.D.}$$

Remark. Theorem 3.5 is not used in the proof of Proposition 5.1. As a consequence of Proposition 5.1 and Proposition 5.2, we get an alternate proof of (3.5.3).

First assuming separativity we can prove that (5.1.2) implies (3.5.3), and (3.5.3) implies (5.1.1). Let S be a commutative semigroup and $\bar{S}$ be the greatest separative homomorphic image of S. For $x \in S$, $\bar{x}$ denotes the image of x under $S \to \bar{S}$. By Theorem 3.3 it is sufficient to prove that "$\overline{abc} = \bar{c}$" holds in $\bar{S}$ if and only if it holds in S. Assume that for $\bar{a} \in \bar{S}$, there are $\bar{y}$, $\bar{z} \in \bar{S}$ such that $\overline{ayz} = \bar{a}\bar{y}\bar{z} = \bar{z}$. Let $\bar{S} = \bigcup_{\alpha\in\Gamma} \bar{S}_\alpha$ be the greatest semilattice decomposition of S. Then $\bar{S} = \bigcup_{\alpha\in\Gamma} \bar{S}_\alpha$ where $\bar{S}_\alpha$ is cancellative archimedean. Let $a \in S_\alpha$, $y \in S_\beta$, $z \in S_\gamma$, $\alpha\beta \leq \gamma$. Since $ayz\tau_1 z$ in the sense of Lemma 3.2, there is $u \in S_\gamma$ such that $ayzu = zu$. Setting $b = y$ and $c = zu$, we have $abc = c$. Conversely, let $\bar{a} \in \bar{S}$. By the assumption for $a \in S$ there are b, $c \in S$ such that $abc = c$ which implies $\bar{a}\bar{b}\bar{c} = \bar{c}$.

6. **Cofinal clusters and group-homomorphisms.** Let S be a commutative semigroup and $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be the greatest semilattice decomposition of S. Each S_α, $\alpha\in\Gamma$, is archimedean. If Λ is a subsemilattice of Γ, then $W = \bigcup_{\alpha\in\Lambda} S_\alpha$ is called a cluster of S. W is the inverse image of Λ under the homomorphism $S \to \Gamma$. W is cofinal in S if and only if Λ is cofinal in Γ.

Proposition 6.1.(3) *Let S be a commutative semigroup and W a cofinal cluster in S. A group-homomorphism f of W onto G can be uniquely extended to a homomorphism f^* of S onto G such that $f = f^*|W$.*

Proof. Let $a \in S$. We are to prove first

(1) There are $b \in S$ and $c \in W$ such that $ab \in W$, $bc \in W$ and $abc \in \operatorname{Ker} f$.

Since W is cofinal in S and $f(W) = G$ is a group, there are $b \in S$ and $c' \in W$ such that $ab \in W$ and $abc' \in \operatorname{Ker} f$. Let $\bar{x}$ denote the element $\alpha\in\Gamma$ such that $x \in S_\alpha$. Now let $c = c'abc'$. Then

$$\overline{abc} = \overline{(abc')^2} = \overline{abc'}, \quad \overline{bc} = \overline{a(bc')^2} = \overline{abc'} \quad \text{and} \quad \bar{c} \leq \overline{abc} = \overline{abc'} \leq \bar{c}.$$

It follows that abc', abc, bc and c are in a same archimedean component, say S_δ. Since $abc' \in \operatorname{Ker} f \subseteq W$, we see $c \in W$, $bc \in W$ and $abc \in \operatorname{Ker} f$, completing the proof of (1).

Define $f^*\colon S \to G$ by $f^*(a) = (f(bc))^{-1}$ where b and c are found by (1). Suppose that b and c satisfy (1) and also b' and c' satisfy (1). Then $f(ab'c') = f(abc) = e^*$ where e^* is the identity element of G. Let $abc \in S_\delta$, $ab'c' \in S_{\delta'}$, and let $e \in S_\xi \cap \operatorname{Ker} f$ for $\xi \geq \delta\delta'$. Then $f(aeb'c') = f(aebc) = e^*$ implies $f(ae)f(b'c') = f(ae)f(bc)$ and $(f(b'c'))^{-1} = (f(bc))^{-1}$ where we notice that since $\bar{a} \leq \delta \leq \xi$, $ae \in S_\xi$ and hence $ae \in W$. Therefore f^* is well defined. It is easy to see that $f^*|W$ coincides with f, and $f^*(S) = G$. Let $a_i \in S$, $i = 1, 2$. Let b_i and c_i be the elements of W satisfying (1) for a_i ($i = 1, 2$). Then b_1b_2 and c_1c_2 satisfy (1) for a_1a_2. We can easily prove $f^*(a_1a_2) = f^*(a_1)f^*(a_2)$. To prove uniqueness of f^*, suppose $g\colon S \to G$ is any other extension of f. Let $a \in S$ and b, c be as above (1).

(3) This proposition was obtained by H. B. Hamilton.

Since $abc \in W$, $g(abc) = f(abc) = e^*$ which implies $g(a) = (g(bc))^{-1}$ but $bc \in W$ and so $g(bc) = f(bc)$, therefore $g(a) = (f(bc))^{-1} = f(a)$. Q.E.D.

Let $f: W \longrightarrow G$ and $g: W \longrightarrow H$ be group-homomorphisms. Let σ and ρ be the congruences on W associated with f and g respectively. Also let σ^* and ρ^* be the congruences on S associated with f^* and g^* respectively.

Proposition 6.2. *$\sigma \subset \rho$ if and only if $\sigma^* \subset \rho^*$.*

Proof. Note that $a\ \sigma\ b$ if and only if $f(a) = f(b)$; $a\ \rho\ b$ if and only if $g(a) = g(b)$; $a\ \sigma^*\ b$ if and only if $f^*(a) = f^*(b)$; $a\ \rho^*\ b$ if and only if $g^*(a) = g^*(b)$. Also $\sigma \subseteq \rho$ if and only if $f(x) = f(y)$ implies $g(x) = g(y)$ if and only if $\operatorname{Ker} f \subseteq \operatorname{Ker} g$. Using this information and definition of f^* and g^* we can prove in a routine manner that $\sigma \subseteq \rho$ implies $\sigma^* \subseteq \rho^*$. Assume $\sigma^* = \rho^*$. Then $\sigma = \sigma^* | W = \rho^* | W = \rho$. Thus we have the "only if" part. Conversely, if $\sigma^* \subset \rho^*$ it is easy to see $\sigma \subseteq \rho$, but we have $\sigma \subset \rho$ by Proposition 6.1. Q.E.D.

Let $\mathfrak{L}(D)$ denote the set of all group-congruences on a semigroup D. $\mathfrak{L}(D)$ is a join semilattice. A cluster W of S is called proper if $W \neq S$.

Theorem 6.3. *We assume that S is not archimedean. A commutative semigroup S has a greatest group-homomorphism if and only if there is a proper cofinal cluster W of S such that W has a greatest group-homomorphism. Further $\mathfrak{L}(S)$ is isomorphic to $\mathfrak{L}(W)$.*

Proof. Assume that a cofinal cluster W of S has a greatest group-homomorphism. Let $S = \bigcup_{\alpha \in \Gamma} S_\alpha$, $W = \bigcup_{\lambda \in \Lambda} S_\lambda$ be defined as above. Let $x \in S$, and assume $x \in S_\alpha$, $\alpha \in \Gamma$. Since Λ is cofinal in Γ, there is $\gamma \in \Gamma$ such that $\alpha \leq \gamma$. Take $a \in S_\gamma$ arbitrarily. Then $xa \in S_\gamma \subseteq W$. By Theorem 3.5 there are b and c in W such that

$$(xa)bc = c \quad \text{or} \quad x(ab)c = c.$$

This shows that S has a greatest group-homomorphism again by Theorem 3.5.

Conversely assume that $S = \bigcup_{\alpha \in \Gamma} S_\alpha$ has a greatest group-homomorphism f_0. Let $\Gamma_1 = \{\alpha \in \Gamma: S_\alpha \cap \operatorname{Ker} f_0 \neq \varnothing\}$. Then Γ_1 is cofinal in Γ. Let Λ be a non-empty cofinal proper subset of Γ_1. Clearly Λ is cofinal in Γ. Let $W = \bigcup_{\lambda \in \Lambda} S_\lambda$. W is then a proper cofinal cluster of S. Let f be a group-homomorphism of S. We are to prove $f(W) = f(S)$. Let $x \in S$, say $x \in S_\alpha$ for some $\alpha \in \Gamma$. Since Λ is cofinal in Γ, there is $\lambda \in \Lambda$ such that $\alpha \leq \lambda$. Let $y \in S_\lambda \cap \operatorname{Ker} f_0$. Since $\operatorname{Ker} f_0 \subseteq \operatorname{Ker} f$ we have

$$f(x) = f(x)f(y) = f(xy) \in f(S_\lambda) \subseteq f(W).$$

This proves $f(W) = f(S)$ and hence $f | W$ is a group-homomorphism. The isomorphism of $\mathfrak{L}(S)$ and $\mathfrak{L}(W)$ follows from Propositions 6.1 and 6.2. Accordingly W has a greatest group-homomorphism. Q.E.D.

A cluster $W = \bigcup_{\lambda \in \Lambda} S_\lambda$ in $S = \bigcup_{\alpha \in \Gamma} S_\alpha$ is called a group-cluster of S if S_λ is a group for all $\lambda \in \Lambda$.

Corollary 6.4. *If S contains a cofinal group-cluster then S has a greatest group-homomorphism.*

As a consequence of Theorem 6.3 the problem of group-homomorphism of $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ can be reduced to that of a cofinal cluster $W = \bigcup_{\lambda\in\Lambda} S_\lambda$ if Λ has a special type. For example,

(i) If Γ contains a cofinal chain, Λ can be chosen as a well ordered cofinal chain. (See [2], [20].)

(ii) If $|\Gamma|$ is countable, Λ can be chosen as a cofinal ω-chain, i.e. chain of ordinal ω. (See [20].)

(iii) If Γ has a greatest element as an upper semilattice, Λ can be chosen as $|\Lambda| = 1$.

For case (iii) we have

Corollary 6.5. *In case* (iii), *for example, in case Γ is finite, S contains an archimedean component S_0 which is an ideal of S. S has a greatest group-homomorphism if and only if S_0 contains an idempotent.*

This is a consequence of Proposition 3.1 or Corollary 3.4 or Theorem 3.5.

7. Study of structure of $\mathcal{G}$-semigroups. For our convenience a commutative semigroup admitting a greatest group-homomorphism will be called a $\mathcal{G}$-semigroup. So far we have obtained various characterizations of $\mathcal{G}$-semigroups, but it is far from a complete construction of $\mathcal{G}$-semigroups. The problem is related to semilattice compositions of archimedean semigroups. We will not enter deeply into this problem at this moment except for a special case.

Let us return to separative semigroups again. Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be the greatest semilattice decomposition of a commutative separative semigroup S. Again note that each S_α is cancellative archimedean.

Proposition 7.1. *Let $a \in S_\alpha$ and $\alpha \le \beta$. If $ax = x$ for some $x \in S_\beta$, then $ay = y$ for all $y \in S_\xi$, all $\xi \ge \beta$.*

Proof. Let $u \in S_\xi$. Then since $\xi \ge \beta$, $xu \in S_\xi$, and $ax = x$ implies $axu = xu$. We are now to prove that if $az = z$ for some $z \in S_\xi$ then $ay = y$ for all $y \in S_\xi$. Since S_ξ is archimedean, $z^n = yv$ for some $v \in S_\xi$ and $n > 0$. Then $az = z$ implies $az^n = z^n$ and $(ay)v = yv$, whence $ay = y$ since $ay \in S_\xi$ and S_ξ is cancellative. Q.E.D.

Assume that S is a $\mathcal{G}$-semigroup and $\alpha < \beta$, $ab = b$ for some $a \in S_\alpha$, $b \in S_\beta$. Then $S_\alpha S_\beta \subseteq S_\beta$. For each $a_\alpha \in S_\alpha$ we define a transformation $\phi^\beta_{a_\alpha}: S_\beta \to S_\beta$ by

$$x\phi^\beta_{a_\alpha} = xa_\alpha, \qquad x \in S_\beta.$$

$\phi^\beta_{a_\alpha}$ is a translation of S_β. Let $\Phi^\beta_\alpha = \{\phi^\beta_{a_\alpha} : a_\alpha \in S_\alpha\}$. Φ^β_α is a subsemigroup of the translation semigroup of S_β. Then we have

Proposition 7.2. *ϕ_α^β is a permutation group.*

Proof. Clearly Φ_α^β is a homomorphic image of S_α under $a_\alpha \to \phi_{a_\alpha}^\beta$. Since S_α is commutative archimedean, Φ_α^β is commutative archimedean but it contains the identity element ϕ_α^β. (It follows from Proposition 7.1 that ϕ_α^β is the identity mapping.) Hence Φ_α^β is a group. For each $\phi_1 \in \Phi_\alpha^\beta$, $\phi_1\phi_2 = \phi_2\phi_1 = \phi_a^\beta$ for some $\phi_2 \in \Phi_\alpha^\beta$. This implies that ϕ_1 is injective and surjective. Therefore Φ_α^β is a permutation group. Q.E.D.

When $\alpha < \beta$ and Φ_α^β is a permutation group we say that S_α is $\mathfrak{S}$-composed with S_β.

A cluster $W = \bigcup_{\lambda\in\Lambda} S_\lambda$ is called an $\mathfrak{N}$-cluster of S if all S_λ, $\lambda \in \Lambda$, are $\mathfrak{N}$-semigroups.

Let S be a commutative separative semigroup, and let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ as usual. Let $\Delta = \{\xi \in \Gamma : S_\xi$ is a group$\}$. Of course Δ is a subsemilattice of Γ. Suppose Δ is not cofinal and $\Delta \neq \Gamma$. Then there is $\alpha \in \Gamma - \Delta$ such that there is no element $\xi \in \Delta$ satisfying $\alpha \leq \xi$, in other words, the principal ideal $\alpha\Gamma$ is contained in $\Gamma - \Delta$. Therefore $W = \bigcup_{\lambda\in\Gamma-\Delta} S_\lambda$ is an $\mathfrak{N}$-cluster of S, precisely it is an ideal of S. We call such W an ideal $\mathfrak{N}$-cluster.

(7.3) *A commutative separative semigroup contains either a cofinal group-cluster or an ideal $\mathfrak{N}$-cluster.*

Thus the study of separative $\mathfrak{G}$-semigroups has been reduced to that of $\mathfrak{G}$-semigroups whose archimedean components are all $\mathfrak{N}$-semigroups. For simplicity $\mathfrak{G}$-semigroups of the latter type are called $\mathfrak{N}$-$\mathfrak{G}$-semigroups.

Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be an $\mathfrak{N}$-$\mathfrak{G}$-semigroup. By Proposition 7.2 there is a pair (α, β) such that S_α is $\mathfrak{S}$-composed with S_β. Such a pair is called a $\mathfrak{S}$-pair. Let $\mathbf{P}$ be the set of all $\mathfrak{S}$-pairs for S. We do not know at the present time how $\mathfrak{S}$-pairs are distributed in $\Gamma \times \Gamma$ and what part $\mathbf{P}$ plays in general, except the following property:

Proposition 7.4. (7.4.1) *$\mathbf{P}$ is cofinal in $\Gamma \times \Gamma$ where the order of elements of $\Gamma \times \Gamma$ is defined by the component-wise order. In particular the set of α for which $(\alpha, \beta) \in \mathbf{P}$ is cofinal in Γ.*

(7.4.2) *If $(\alpha, \beta) \in \mathbf{P}$, then $(\alpha, \gamma) \in \mathbf{P}$ for all $\gamma \geq \beta$, $\gamma \in \Gamma$.*

(7.4.3) *If (α, β) and (γ, β) are in $\mathbf{P}$, then $\beta \neq \alpha\gamma$ in Γ.*

Proof. (7.4.1), (7.4.2) are obvious.

Proof of (7.4.3). Suppose $\beta = \alpha\gamma$. Then $W = S_\beta \cup S_\alpha \cup S_\gamma$ is a cluster of S. Consider a homomorphism ϕ of W into the translation semigroup of S_β in the natural way. Under ϕ, the elements x_α (x_γ) of S_α (S_γ) are mapped into Φ_α^β (Φ_γ^β), and $\phi(S_\beta)$ is the inner translation semigroup of S_β being isomorphic to S_β since S_β is an $\mathfrak{N}$-semigroup ([6], [9]). Now $x_\alpha x_\gamma$ is in S_β by assumption but $\phi(x_\alpha)\phi(x_\gamma)$ is not in

$\phi(S_\beta)$ since $\phi(S_\beta)$ contains no permutation. Q.E.D.

Immediately we have

Proposition 7.5. *Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be an $\mathfrak{N}$-$\mathcal{G}$-semigroup. If, for all α, $\beta \in \Gamma$ such that $\alpha < \beta$, S_α is $\mathfrak{S}$-composed with S_β, then Γ has to be a chain.*

If, for all α, $\beta\in\Gamma$ such that $\alpha<\beta$, S_α is $\mathfrak{S}$-composed with S_β, we say that S is $\mathfrak{S}$-composed.

We mean by an $\mathfrak{N}$-separative semigroup a commutative semigroup whose archimedean components are all $\mathfrak{N}$-semigroups and we assume that Γ has no greatest element.

Theorem 7.6. *Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ be an $\mathfrak{N}$-separative semigroup with chain Γ. S is a $\mathcal{G}$-semigroup if and only if S contains a cofinal cluster $W = \bigcup_{\lambda\in\Lambda} S_\lambda$ of S such that*

(1) *Λ is a well ordered cofinal chain of Γ, and*

(2) *W is a $\mathfrak{S}$-composed $\mathfrak{N}$-cluster.*

Proof. Assume that $W = \bigcup_{\lambda\in\Lambda} S_\lambda$ be a cofinal $\mathfrak{S}$-composed $\mathfrak{N}$-cluster in S such that Λ is a cofinal chain in Γ. Let $x\in W$ and $y \in W$, say, $x\in S_\alpha$ and $y\in S_\beta$, with $\alpha \leq \beta$. Let $u\in S_\gamma \subseteq W$ with $\alpha\leq\beta<\gamma$. Now S_α is $\mathfrak{S}$-composed with S_γ. Hence there exists $z\in S_\alpha$ such that $(xu)z = yu$. Therefore by Theorem 3.5, W is a $\mathcal{G}$-semigroup, and hence S is a $\mathcal{G}$-semigroup by Theorem 6.3.

Conversely assume that $S = \bigcup_{\alpha\in\Gamma} S_\alpha$ is a $\mathcal{G}$-semigroup. Let Γ be well ordered by $\preceq$. (Be aware of "$\preceq$" distinguished from "$\leq$" originally given in the chain Γ.) Say

$$\Gamma = \{\alpha_0, \alpha_1, \cdots, \alpha_\omega, \alpha_{\omega+1}, \cdots, \alpha_\pi\}$$

where $\preceq$ denotes the order for not only elements of Γ but their suffixes: $\alpha_\xi \preceq \alpha_\eta$ if and only if $\xi\preceq\eta$. Since S is a $\mathcal{G}$-semigroup, by Theorem 3.5, for each $a\in S$ there exists $b\in S$ and $c\in S$ such that $abc = c$. Thus for each $\alpha\in\Gamma$ there exists $\beta\in\Gamma$ and $\gamma\in\Gamma$ such that $S_{\alpha\beta}$ is $\mathfrak{S}$-composed with S_γ. For each $\alpha\in\Gamma$ consider the set

$$\Omega_a = \{(\alpha\beta, \gamma)\text{: there is } a \in S_\alpha, b \in S_\beta, c\in S_\gamma \text{ such that } abc = c\}.$$

We define a lexicographical order $\preceq$ in Ω_α, i.e.

$$(\alpha_\xi, \alpha_\eta) \prec (\alpha_{\xi'}, \alpha_{\eta'}) \text{ if and only if}$$
$$\text{either } \alpha_\xi \prec \alpha_{\xi'}, \text{ or } \alpha_\xi = \alpha_{\xi'}, \text{ and } \alpha_\eta \quad \alpha_{\eta'}.$$

Let (α^1, α^2) denote the $\preceq$ smallest element of Ω_α. Now define a partial transformation θ of Γ by

$$\theta(\alpha_0) = \alpha_0^1,$$

$\theta(\alpha_\xi) = (\theta(\alpha_{\xi-1})^2)^1$ if ξ is isolated and $\alpha_\xi \le (\theta(\alpha_{\xi-1}))^2$,

$= \alpha_\xi^1$ if ξ is isolated and $\alpha_\xi > (\theta(\alpha_{\xi-1}))^2$,

$= \alpha_\xi^1$ if ξ is a limit ordinal and if $\alpha_\xi > \theta(\alpha_\eta)$ for all $\eta \prec \xi$,

$= (\min\{\alpha_\zeta \in \Gamma: \alpha_\zeta > \theta(\alpha_\eta)$ for all $\eta \prec \xi\})^1$ $\left\{\begin{array}{l}\text{if } \xi \text{ is a limit ordinal and if this} \\ \text{set is not empty and } \alpha_\xi \le \theta(\alpha_\eta) \\ \text{for some } \eta \prec \xi,\end{array}\right.$

$=$ undefined if ξ is a limit ordinal and if this set is empty,

$=$ undefined if ξ is isolated and if $\theta(\alpha_{\xi-1})$ is undefined.

Let $W = \bigcup_{\xi \prec \pi+1} S_{\theta(\alpha_\xi)}$. It is cofinal because $\theta(\alpha) \ge \alpha$ whenever it is defined and if $\theta(\alpha)$ is undefined, there exists β such that $\theta(\beta) \ge \alpha$. To see that W is $\mathfrak{G}$-composed we will induct on ξ and show that $S_{\theta(\alpha_\eta)}$ is $\mathfrak{G}$-composed with $S_{\theta(\alpha_\xi)}$ for all $\eta \prec \xi$. The following can easily be used to show this. For all ξ and all $\eta \prec \xi$,

(3) $(\alpha_\xi^1)^1 = \alpha_\xi^1$,

(4) $\theta(\alpha_\xi)^1 = \theta(\alpha_\xi)$,

(5) $\theta(\alpha_{\xi+1}) \ge \theta(\alpha_\xi)^2$,

(6) $\theta(\alpha_\xi) \ge \theta(\alpha_\eta)^2$.

Each of these is an immediate consequence of the definition of θ, α^1 and α^2. Then since $\theta(\alpha_\xi) \ge \theta(\alpha_\eta)^2$, $S_{\theta(\alpha_\eta)^1}$ is $\mathfrak{G}$-composed with $S_{\theta(\alpha_\xi)}$ and hence $S_{\theta(\alpha_\eta)}$ is $\mathfrak{G}$-composed with $S_{\theta(\alpha_\xi)}$ for all $\eta \prec \xi$. From the definition of θ it follows that

$$\text{if } \theta(\alpha_\eta) \text{ and } \theta(\alpha_\xi) \text{ are defined and if } \eta \prec \xi \text{ then } \theta(\alpha_\eta) \le \theta(\alpha_\xi).$$

Let Δ be the set consisting of elements of Γ on which θ is defined. Then θ is an order-homomorphism of the well-ordered set $(\Delta, \preceq)$ into $(\Gamma, \le)$. Thus $\theta(\Delta)$ has been proved to be Λ in the assertion (1). Q.E.D.

In the following theorem we describe the construction of a chain of $\mathfrak{N}$-semigroups in order that S be an $\mathfrak{N}$-$\mathfrak{G}$-semigroup. By the group of translations of an $\mathfrak{N}$-semigroup D we mean the group of permutations of D which are translations of D. Since the translation semigroup $\mathfrak{T}$ is commutative cancellative, the group is a unique maximal subgroup of $\mathfrak{T}(D)$. See the details in [6], [9].

Theorem 7.7. *Let $(\Gamma, \le)$ be a chain without greatest element, $\{S_\alpha: \alpha \in \Gamma\}$ a family of $\mathfrak{N}$-semigroups and $\mathbf{G}_\alpha$ the group of translations of S_α. If $\alpha < \beta$, a homomorphism of S_α into $\mathbf{G}_\beta$ is denoted by F_α^β, and the image of $x_\alpha \in S_\alpha$ under F_α^β is denoted by $x_\alpha F_\alpha^\beta$ and the image of $x_\beta \in S_\beta$ under the permutation $x_\alpha F_\alpha^\beta$ is denoted* *by* $(x_\alpha F_\alpha^\beta)x_\beta$. *Assume that a system of homomorphisms* $\{F_\alpha^\beta$: *for all* α, β *with* $\alpha<\beta\}$ *is given such that whenever* $\alpha<\beta<\gamma$

$$(x_\alpha F_\alpha^\gamma)(x_\beta F_\beta^\gamma)=((x_\alpha F_\alpha^\beta)x_\beta)F_\beta^\gamma \quad \text{for all } x_\alpha\in S_\alpha,\ x_\beta\in S_\beta.$$

Let $S=\bigcup\{S_\alpha:\alpha\in\Gamma\}$. *Define an operation on* S *by*

$$x_\alpha y_\beta = y_\beta x_\alpha = \begin{cases} x_\alpha y_\alpha & \text{the product in } S_\alpha \text{ if } \alpha=\beta,\\ (x_\alpha F_\alpha^\beta)y_\beta & \text{if } \alpha<\beta.\end{cases}$$

Then S *is a* $\mathfrak{G}$-*composed* $\mathfrak{N}$-$\mathcal{G}$-*semigroup. Every* $\mathfrak{G}$-*composed* $\mathfrak{N}$-$\mathcal{G}$-*semigroup with chain* Γ *can be obtained in this manner.* S *is denoted by* $S=\bigcup\{S_\alpha, F_\eta^\xi, \Gamma\}$.

Proof is obvious.

Note that for arbitrary $\{S_\alpha:\alpha\in\Gamma\}$ with chain Γ, there always exists a composition, for example we can define $x_\alpha F_\alpha^\beta$ by the identity mapping of S_β for all $x_\alpha\in S_\alpha$.

We can describe the compositions by induction on $\alpha\in\Gamma$.

Let Γ be a well-ordered chain without greatest element. It is regarded as a set of ordinals $\Gamma=\{0,1,\cdots,\omega,\cdots,\pi,\cdots\}$. Assume a family of $\mathfrak{N}$-semigroups $\{S_\alpha:\alpha\in\Gamma\}$ is given. Let $\overline{S}_1=S_0$. A $\mathfrak{G}$-composed $\overline{S}_2=\overline{S}_1\cup S_1$ is constructed by a homomorphism $\overline{S}_1$ into the group of translations of S_1 and f_1^2 is defined to be an inclusion map of $\overline{S}_1$ into $\overline{S}_2$. Thus we have a $(\overline{S}_2, f_1^2)$. Assume that for all $\xi<\alpha$, a $\mathfrak{G}$-composed $\mathfrak{N}$-$\mathcal{G}$-semigroup $\overline{S}_\xi$ and inclusion maps $f_{\xi_1}^{\xi_2}$ of $\overline{S}_{\xi_1}$ into $\overline{S}_{\xi_2}$ for all ξ_1,ξ_2 with $\xi_1\le\xi_2\le\xi$ such that $f_{\xi_2}^{\xi_3}f_{\xi_1}^{\xi_2}=f_{\xi_1}^{\xi_3}$, and f_ξ^ξ is the identity map of $\overline{S}_\xi$, that is, assume that $(\overline{S}_\xi, f_{\xi_1}^{\xi_2};\ \xi_1\le\xi_2\le\xi)$ is obtained for all $\xi<\alpha$. Now we are to define $(\overline{S}_\alpha, f_{\xi_1}^{\xi_2};\ \xi_1\le\xi_2\le\alpha)$. If α is an isolated ordinal, $\overline{S}_\alpha$ is defined by $\overline{S}_\alpha=\overline{S}_{\alpha-1}\cup S_{\alpha-1}$ as a $\mathfrak{G}$-composed $\mathfrak{N}$-$\mathcal{G}$-semigroup, and f_ξ^α, $\xi<\alpha$, is defined by $f_\xi^\alpha=f_{\alpha-1}^\alpha f_\xi^{\alpha-1}$ where $f_{\alpha-1}^\alpha$ is the inclusion map of $\overline{S}_{\alpha-1}$ into $\overline{S}_\alpha$. If α is a limit ordinal, $\overline{S}_\alpha$ is defined by the direct limit $\overline{S}_\alpha=\varinjlim\{\overline{S}_{\xi_1}, f_{\xi_1}^{\xi_2}, \Gamma\}$ and f_ξ^α is defined to be the inclusion of $\overline{S}_\xi$ into $\overline{S}_\alpha$.

We have to add a remark. In a $\mathfrak{G}$-composed $\mathfrak{N}$-$\mathcal{G}$-semigroup $S=\bigcup\{S_\alpha, F_\eta^\xi, \Gamma\}$ with well ordered chain Γ, we get a homomorphism h^α of $\overline{S}_\alpha$ onto a subgroup Φ^α of $\mathbf{G}_\alpha$, the group of translations of S_α, for each $\alpha\in\Gamma$, $\alpha\ne 0$. Note that $h^\alpha|S_\eta=F_\eta^\alpha$ and $\Phi^\alpha=\bigcup_{\eta<\alpha}\Phi_\eta^\alpha$. For each α, $\beta\in\Gamma$ with $\alpha<\beta$, there is a unique homomorphism g_α^β of Φ^α onto Φ^β such that $h^\beta=g_\alpha^\beta h^\alpha$. To prove this it is sufficient to show if $\alpha<\beta<\gamma$,

$$\phi_{a_\alpha}^\beta=\phi_{b_\alpha}^\beta \text{ implies } \phi_{a_\alpha}^\gamma=\phi_{b_\alpha}^\gamma \quad \text{for all } \gamma\ge\beta$$

in which the notation $\phi_{a_\alpha}^\beta$ is defined before Proposition 7.2. By assumption $xa_\alpha=xb_\alpha$ for all $x\in S_\beta$. Let $y\in S_\gamma$. Then $y^2xa_\alpha=y^2xb_\alpha$ and $(yx)(ya_\alpha)=(yx)(yb_\alpha)$,

$yx \in S_\gamma$. Since S_γ is cancellative, we have $ya_\alpha = yb_\alpha$ for all $y \in S_\gamma$.

From $h^\beta = g_\alpha^\beta h^\alpha$ it follows that, whenever $\alpha < \beta < \gamma$,

$$g_\alpha^\gamma = g_\beta^\gamma g_\alpha^\beta, \quad \text{and } g_\xi^\xi \text{ is the identity} \quad \text{for all } \xi.$$

When $\eta < \alpha$, recall $F_\eta^\alpha = h^\alpha \mid S_\eta$, and let $K_\alpha = \bigcup \{\operatorname{Ker} F_\alpha^\xi : \alpha < \xi\}$. Let

$$\widetilde{F_\alpha^\xi} = F_\alpha^\xi \mid K_\alpha.$$

We have a similar result to Proposition 4.3.

Proposition 7.8. *The greatest group-homomorphic image of* $S = \bigcup \{S_\alpha, F_\eta^\xi, \Gamma\}$ *is isomorphic to* $\varinjlim \{\Phi^\alpha, g_\alpha^\beta, \Gamma\}$, *and the kernel of the homomorphism is* $\bigcup \{K_\alpha, \widetilde{F_\alpha^\xi}, \Gamma\}$.

Finally we list basic property on the class of $\mathcal{G}$-semigroups which are not necessarily separative.

(i) A homomorphic image of a $\mathcal{G}$-semigroup is a $\mathcal{G}$-semigroup.

(ii) An ideal of a $\mathcal{G}$-semigroup is a $\mathcal{G}$-semigroup.

(iii) If S is a commutative semigroup and if S is a union of $\mathcal{G}$-semigroups, then S is a $\mathcal{G}$-semigroup.

(iv) A semilattice of $\mathcal{G}$-semigroups is a $\mathcal{G}$-semigroup.

(v) If a commutative semigroup S contains a cofinal $\mathcal{G}$-subsemigroup, then S is a $\mathcal{G}$-semigroup.

(vi) If a commutative semigroup S contains a $\mathcal{G}$-semigroup as an ideal of S, then S is a $\mathcal{G}$-semigroup.

(vii) If S_ξ, $\xi \in \Xi$, are $\mathcal{G}$-semigroups, the direct product of $\Pi_{\xi \in \Xi} S_\xi$ is a $\mathcal{G}$-semigroup.

Some of the above have been proved in this paper, but we note that we can very easily prove (i) through (vii) by using (3.5.3) of Theorem 3.5.

Restating Theorem 3.5′,

(viii) S has a greatest group-homomorphism if and only if

(1) there is at least one local identity element e: $be = b$,

(2) for each $a \in S$ there is a local identity element e and an element x such that $ax = e$.

This shows that the concept of $\mathcal{G}$-semigroups is really a natural generalization of groups.

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