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SMOOTH EMBEDDINGS OF HOMOLOGICALLY SIMILAR MANIFOLDS

BY

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ABSTRACT. We consider the situation where we have two smooth n-manifolds $N \subseteq M$ with $H_*(M, N) = 0$ and show that given a smooth embedding of N into some manifold Q we may, under suitable conditions, extend this to embeddings of M into Q, $Q \times I$, or $Q \times I^2$ (where I is the unit interval). We can apply these results to obtain smooth embeddings of homologically k-connected manifolds into $(2n - k + 1)$-dimensional euclidian space.

0. **Preliminaries.** We will be primarily interested in smooth manifolds with (perhaps empty) boundary, by which we will mean a manifold with a C^∞ structure. If M is a smooth n-dimensional manifold we will sometimes use the notation M^n if we need to emphasize the dimension. If $f: M^k \to N^n$ is a smooth embedding of M into N, then we will frequently without explicit mention make the natural identification of M and $f(M)$ in order to avoid excessive notation. All manifolds and maps will be smooth unless otherwise stated. If we are given an embedding of a manifold N in a manifold M, we can always change the embedding slightly so that N will be contained in the interior of M by shrinking N away from the boundary via the collar neighborhood of ∂M in M.

1. **Statement and discussion of main results.** The problem which we will be most concerned with is the following. Suppose that we have two smooth n-manifolds with boundary M^n and N^n with $N \subseteq M$ with $H_*(M, N) = 0$. Then it would seem that M and N must be quite similar in many ways. We might expect for example that if we could embed N smoothly into some smooth manifold Q that we could also embed M into Q. We will shortly give a counterexample to this conjecture, yet we will be able to prove some theorems that are very close to this. We will generally assume that M, N, ∂M, ∂N are connected.

The following three theorems are our main results along these lines. Let Hypothesis (#) denote "M^n and N^n are smooth manifolds with boundary with $N \subseteq M$, $H_*(M, N) = 0$, and $n > 4$."

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Theorem A. *Given Hypothesis* (#), *if there is a smooth* 1*-trivial embedding,* $f\colon N \to Q^q$ (*i.e., an embedding such that* $f_\#\colon \pi_1(N) \to (0) \subseteq \pi_1(Q)$) *and if* $n < q$, *then there is a smooth embedding* $M \times I \subseteq Q \times I^2$.

Theorem B. *Given Hypothesis* (#), *if there is a smooth* 1*-trivial embedding* $N \times I \subseteq Q^q$, *with* $q - n > 3$ *then we can smoothly embed* $M \times I \subseteq Q$.

An additional assumption will allow us to obtain better results. Let Hypothesis (##) denote "M^n and N^n are smooth manifolds with boundary with $N \subseteq M$, $H_*(M, N) = 0$, $n > 5$ such that the map $j_\#\colon \pi_1(\partial M) \to \pi_1(\overline{M - N})$ induced by the inclusion is onto."

Theorem A$'$. *Given Hypothesis* (##), *then if there is a smooth* 1*-trivial embedding* $N \subseteq Q^q$ *with* $n < q$, *then there is a smooth embedding* $M \subseteq Q \times I$.

Theorem B$'$. *Given Hypothesis* (##), *then if there is a smooth* 1*-trivial embedding* $N \subseteq Q^q$, *with* $q - n > 3$, *then we can smoothly embed* $M \subseteq Q$.

Example 1. In general, it is false that if $H_*(M, N) = 0$ and $N \subseteq E^q$, then $M \subseteq E^q$. For the example, let P^n be a homology sphere which is not simply connected. Let D^n be a closed n-ball in the interior of P, let $M = \overline{P - D}$; let N be a closed n-ball in the interior of M. Certainly we have $H_*(M) = 0$; since N is an n-ball, $N \subseteq E^n$. But M does not embed in E^n, for if so then we would have an embedding of $S = \partial M$ in E^n, this sphere would bound a ball in E^n, and we would be forced to conclude that M was a ball, which it is not. Thus the codimension zero version of the conjecture is false; however, we have no such counterexamples in other low codimensions.

Suppose we have $N^n \subseteq M^n$ with $H_*(M, N) = 0$ (we may assume that $N \subseteq \operatorname{Int} M$). What we really need to examine is the manifold $H = \overline{M - N}$. By excision, we will have $H_*(H, \partial N) = 0$ and by duality (Milnor [11]) we will have $H_*(H, \partial M) = 0$. Now if it happened that H were in fact an h-cobordism then the embedding problem is trivial, for if the Whitehead torsion $t(H, \partial M) = \alpha$, then as in Stallings [13] we can construct another h-cobordism H', with one boundary component ∂M such that $t(H', \partial M) = -\alpha$. We then form the manifold $M' = M + H' = N + H + H'$. But $H + H'$ is an h-cobordism with $t(H + H', \partial N) = 0$ and thus $H + H' \approx \partial N \times I$; therefore $M' \approx N$. Since we can embed N we have an embedding of M', but since $M \subseteq M'$ this gives an embedding of M.

Now if H and each component of the boundary of H were simply connected then it would easily follow that H would be an h-cobordism, but if they are not *all* simply connected then H need not be an h-cobordism. The following two examples are useful to keep in mind for relating properties of such manifolds H.

Example 2. Take $H = \overline{M - N}$ where M and N are as in the above example; M is a Poincaré sphere minus a ball; N is a subball of M. Then H is an example where both components of ∂H are simply connected (they are spheres) but H is not an h-cobordism, since H is not simply connected.

Example 3. Let M^n be a Mazur manifold [9]—that is, a contractable manifold with nonsimple connected boundary; let N be an n-ball in the interior of M. In this case, H and one of the boundaries, ∂N, are simply connected, but the other boundary component, ∂M, is not.

Example 4. We now give an example of an H-cobordism (H, M_0, M_1) such that $\pi_1(M_0) \approx \pi_1(H)$ and $\pi_1(M_1) \approx \pi_1(H)$ via inclusion and yet H is not an h-cobordism. We will use an example of Stallings found in Kervaire [7, Theorem V], of an embedding, f, of an n-sphere, $n \geq 3$, in an $(n+2)$-sphere with $\pi_1(S^{n+2} - f(S^n)) \approx Z$ and $\pi_2(S^{n+2} - f(S^n)) \neq 0$. Let γ be the generator of $\pi_1(S^{n+2} - f(S^n))$. We may represent γ by a smoothly embedded circle $g: S^1 \to S^{n+2}$. Both $g(S^1)$ and $f(S^n)$ have trivial normal disk bundles—call these T_1 and T_n, respectively; we may choose these so that $T_1 \cap T_n = \varnothing$. Note that $\partial T_1 \approx \partial T_n \approx S^1 \times S^n$. Now let $H = \overline{S^{n+2} - T_1 - T_n}$, and let $M_0 = \partial T_1$, $M_1 = \partial T_n$.

The following three theorems give some indication of what implications follow from the assumption that $M^n \supseteq N^n$ is a homotopy equivalence. The proofs are straightforward; the crossing with I^2, in 1.1, or the crossing with I together with the hypotheses about the fundamental groups in 1.2, is used to assure codimension 3 spines of the manifolds in question so as to apply Lemma 2.10 for the fundamental groups.

Theorem 1.1 (crossing with I^2). *Suppose that $M^n \supseteq N^n$, $n > 3$, is a homotopy equivalence. Let $M' = M \times I^2$; $N' = N \times I^2$. We may consider $N' \subseteq \operatorname{Int} M'$; let $H' = \overline{M' - N'}$. Then H' is an h-cobordism between $\partial M'$ and $\partial N'$.*

Theorem 1.2 (crossing with I). *Suppose that $N^n \subseteq M^n$ is a homotopy equivalence with $n > 4$. Let $M' = M \times I$; $N' = N \times I$, we can consider $N' \subseteq \operatorname{Int} M'$. Let $H' = \overline{M' - N}$, $H = \overline{M - N}$, and suppose that $\pi_1(\partial M) \to \pi_1(H)$ and $\pi_1(\partial N) \to \pi_1(H)$ induced by inclusion are all onto, then H' is an h-cobordism.*

Theorem 1.3. (a) *If $N^n \subseteq M^n$ is a homotopy equivalence, $H = \overline{M - N}$, and if N and ∂N are simply connected, then $2H$ is an h-cobordism.* (b) *If $N^n \subseteq M^n$ is a homotopy equivalence with N simply connected, $N' = N \times I$, $M' = M \times I$, $H' = \overline{M' - N'}$, then $2H'$ is an h-cobordism, where $2H'$ denotes two copies of H' identified along ∂M.*

We can now use these theorems to obtain the following smooth embedding results.

Theorem 1.4. (1) *If we have the hypothesis of Theorem* 1.1 *with* $n > 3$; *then if* $N \times I^2 \subseteq Q$, *then* $M \times I^2 \subseteq Q$.

(2) *If we have the hypothesis of Theorem* 1.2 *with* $n > 4$; *then if* $N \times I \subseteq Q$, *then* $M \times I \subseteq Q$.

(3) *If we have the hypothesis of Theorem* 1.3(a) *with* $n > 4$; *then if* $N \subseteq Q$, *then* $M \subseteq Q$. *If we have the hypothesis of Theorem* 1.3(b) *with* $n > 4$; *then if* $N \times I \subseteq Q$, *then* $M \times I \subseteq Q$.

Definition. $(H; M_0, M_1)$ will be called an *H-cobordism*, or homology cobordism if $\partial H = M_0 \cup M_1$ with $M_0 \cap M_1 = \varnothing$, and such that $H_*(H, M_0) = 0$. (Note that is follows from duality that $H_*(H, M_0) = 0$ implies that $H_*(H, M_1) = 0$.)

2. **Handlebody theorems.** Suppose we write D^n as $D^r \times D^{n-r}$, and consider $\partial D^r \times D^{n-r}$ to be a subset of ∂D^n via the formula: $\partial D^n = \partial D^r \times D^{n-r} + D^r \times \partial D^{n-r-1}$ where the identifications are on $\partial D^r \times \partial D^{n-r-1}$. Suppose also that N^n is a given smooth n-manifold and that A is a smooth submanifold of ∂N which is diffeomorphic to $\partial D^r \times D^{n-r}$ via a diffeomorphism h, $h: \partial D^r \times D^{n-r} \to A$. The map h is called the *a-map* or *attaching map*. The space $N + D^n$ with identifications via h will be a smooth manifold called N *plus the r-handle* h; this will usually be denoted more simply by $N + h$. If the value of r is to be emphasized, we will also call $N + h$, N *plus a handle of type* r.

The subset A will be called the *attaching set of* h, or the a-set of h. The subset of A corresponding to $\partial D^r \times (0)$ will be called the *attaching sphere of* h, or the a-sphere. The subset of $\partial(N + h)$ corresponding to $D^r \times \partial D^{n-r}$ will be called the *boundary of* h, and denoted ∂h. A subset corresponding to a subset of ∂h of the form $x \times \partial D^{n-r}$, where $x \in D^r$, will be called a *b-sphere of* h.

Definition. *A handle decomposition of* M *relative to* N, where M and N are both n-manifolds will be a diffeomorphism of M with N plus some handles; we will write this as $M = N + h_1 + h_2 + \cdots + h_s$. If we wish to emphasize the types of the handles in the decomposition, we will use superscripts so that, for example, $N + h_1^i + h_2^j$ will denote N plus a particular handle of type i plus a particular handle of type j. M^n *is a handle decomposition relative to* N^{n-1} will mean that M has a handle decomposition on $N \times I$ with all handles attached on $N \times (1)$.

Lemma 2.1. *Given a handle decomposition* $M = N + h_1 + \cdots + h_s$, *then there is a natural handle decomposition on* $M \times I$ *that has handles of the same type as those in the decomposition of* M. *That is, we may write* $M \times I = N \times I + g_1 + \cdots + g_s$ *where if* A_i *and* S_i, $i = 1, \cdots, s$, *denote the a-set and the a-sphere of* h_i, *then* $A_i \times I$ *and* $S_i \times \frac{1}{2}$ *will correspond to the a-set and a-sphere of* g_i. $\square$

Proposition 2.2 (changing the order of the handles if the a-sets are disjoint). *Suppose that $M = N + h + h'$; with A and A' denoting the a-sets of h and h', respectively. If A and A' are disjoint subsets of ∂N, then $N + h + h' \approx N + h' + h$.* □

The following is a standard theorem about handlebodies [6].

Lemma 2.3 (moving an a-sphere by an isotopy). *Suppose that $M = N + h_1 + \cdots + h_s$ is a handle decomposition and that S denotes the a-sphere of h_1. Suppose that we are given an isotopy H_t of S in ∂N. Then we can obtain an equivalent handle decomposition $M = N + g_1 + \cdots + g_s$ such that the a-sphere of g_1 is $H_1(S)$.* □

The following lemma is a version of the product neighborhood theorem.

Lemma 2.4. *If P and Q are smooth manifolds and h': $P \times \{0\} \to Q \times \{0\}$ is a diffeomorphism then there is a diffeomorphism, h, unique up to isotopy, h: $P \times I \to Q \times I$ such that $h|P \times \{0\} = h'$.* □

Definition. Suppose what we are given a handle decomposition $M^n = N^n + h_1 + h_2 + \cdots + h_s$. Then we will say that the decomposition is *nicely handled* if

(1) The handles are added in order of increasing type—that is, if $i \le j$ then the type of h_i is less than or equal to the type of h_j.

(2) Let $N(k)$ denote N plus all those handles of type less than or equal to k ($N(k)$ is analogous to the k-skeleton). We will require that the a-sets of all the $(k+1)$-handles in the decomposition are disjoint subsets of $\partial N(k)$, for all values of k.

(3) If a $(k+1)$-handle intersects a k-handle we will require that the a-set of the $(k+1)$-handle goes right around the k-handle. By this we mean the following. Suppose that h is a k-handle in the decomposition, that A' is the a-set of h', and that $A \cap \partial h \neq \varnothing$. We have $A' \approx \partial D^{k+1} \times D^{n-k-1}$. We will require that $A' \cap \partial h$ consists of a disjoint collection of $(n-1)$-disks denoted by D_i, each corresponding to a subset of the form $B_i^k \times D^{n-k-1}$ where the B_i are subdisks of ∂D^{k+1}. Furthermore, this cartesian structure must be compatible with the cartesian structure of $\partial h \approx D^k \times \partial D^{n-k}$. That is, there is a collection of $(n-k-1)$-disks in ∂D^{n-k}, $\{C_i\}$, such that if f_i: $D^k \to \partial D^{k+1}$ is the inclusion map of B_i and if g_i: $D^{n-k-1} \to \partial D^{n-k}$ is the inclusion map of C_i, then the inclusion of D_i into ∂h is given by $f_i \times g_i$.

Remarks. Conditions (1) and (2) are essentially the requirements of a "nice" handle decomposition in the sense of Smale [14]—i.e., one corresponding to a "nice" or self-indexing Morse function.

Condition (3) essentially says that the set A' does not double back on the handle h, nor does it twist around h; furthermore the "fibers of A' line up with the fibers of ∂h."

The proof of the following theorem is an easy generalization of the proof that every handle decomposition is equivalent to a nice handle decomposition, Barden [1].

Theorem 2.5. *Let $M = N + h_1 + \cdots + h_s$ be a handle decomposition. Then there is an equivalent handle decomposition of M relative to N which is nicely handled.* $\square$

In the constructions which are to follow, we will need the following concept of relative transversality.

Definition. Suppose that A^a, B^b and C^c are submanifolds of Q^q, with $C \subseteq A \cap B$. We will consider the tangent manifolds of A and B to be contained in the tangent manifold of Q, thus $T(A)_x$ will denote the tangent plane of A at x which we will consider as a hyperplane in $T(Q)_x$, the tangent plane of Q at x. We will say that A is transverse to B relative to C if the following hold:

(1) If $x \in C$, then $T(A)_x$ and $T(B)_x$ span an $(a + b - c)$-dimensional hyperplane in $T(Q)_x$.

(2) If $x \notin C$, then we require that A and B are transverse in the usual sense; that is, that $T(A)_x$ and $T(B)_x$ span $T(Q)_x$, it being understood that this condition is vacuous if $x \notin A \cap B$, and that if $a + b < q$, then transversality at x will mean that $x \notin A \cap B$.

Lemma 2.6. *Suppose that $M^n = N^n + h$, where h is a $(k+1)$-handle; and suppose we have $N^n \subseteq Q^q$. Let A and S be the a-set and a-sphere, respectively, of the handle h. Then we can extend the embedding of N to an embedding of M if and only if*

(1) *there is a $(k+1)$-disk B in Q with $B \cap N = S = \partial B$ and B is transverse to A (and therefore to N) relative to S, and*

(2) *a certain obstruction α is zero. This obstruction is an element in $\pi_k(V(q-k-1, n-k-1))$.*

Proof. The proof of the lemma follows easily after we define α. We first define a map $F\colon \partial B \to V(q-k-1, n-k-1)$ as follows. By our transversality, we may assume B is orthogonal to A. For each $x \in \partial B$, let $f(x)$ be the $(n-k-1)$-frame at x, normal to B, corresponding to the standard frame of D^{n-k-1} via $\{x\} \times D^{n-k-1} \subseteq A \approx \partial B \times D^{n-k-1}$; this may be considered a frame in R^{q-k-1} by projection on the $(q-k-1)$-dimensional fiber of the trivial normal bundle of B in Q. Then α will be the homotopy class of f in $\pi_k(V(q-k-1, n-k-1))$. $\square$

Lemma 2.7. *If $q - n > k$, then $\pi_k(V(q-k-1, n-k-1)) = 0$.*

Proof. By Steenrod [15, 25.6, p. 132], we find that $\pi_k(V(x, y)) = 0$ if $x - y > k$.

Combining the above two lemmas we have

Lemma 2.8. *If we have $M^n = N^n + h$, where h is a k-handle, and suppose that $N^n \subseteq Q^q$, and suppose there is a k-disk B, in Q which spans S, the set corresponding to the a-sphere of h, then we may find a smooth subdisk B' in Q which spans S and which is transverse to N relative to S if $q > n + k$.* $\square$

Definition. Suppose that S^a and S^b are transverse subspheres of Q^{a+b}; and suppose that we choose orientations of each of these manifolds. An orientation will give a specific orientation to each tangent plane, given say by a preferred ordered basis. Then if $x \in S^a \cap S^b$, the intersection number will be defined to be plus one if we take a basis of $T(Q)_x$ by taking *first* the basis vectors which correspond to the chosen basis of $T(S^a)_x$ in $T(Q)_x$ and *then* the preferred basis vectors corresponding to the chosen basis of $T(S^b)_x$, and if this ordered basis gives the same orientation to $T(Q)_x$ as the chosen one. If the orientation is not the chosen one, then we will say the intersection number of the point x is a minus one.

If h is a k-handle, and h' is a $(k+1)$-handle in some handle decomposition, then the intersection number of these two handles will be the algebraic sum of all the intersection numbers of the a-sphere of the $(k+1)$-handle and the b-sphere of the k-sphere. The sign of the intersection number will depend on the arbitrary choices of orientations of the manifolds involved.

Definition. A handle decomposition will be called an h-decomposition if $M = N + h_1 + \cdots + h_s + k_1 + \cdots + k_s$ where the h_i are all k-handles and the k_i are all $(k+1)$-handles, with the decomposition nicely handled, where we require that the intersection number of h_i and k_i is one, and that the intersection number of h_i and k_j for $i \neq j$ is zero. We will say that such a handle decomposition is an *h-decomposition of type* $(k, k+1)$.

Theorem 2.9 (the h-decomposition theorem). *Suppose that $M^n \supseteq N^n$ with $H_*(M, N) = 0$, with ∂N and ∂M connected and $n > 3$, then M can be written as "N plus a sum of h-decompositions"; that is, we may write*

$$\begin{aligned} M = N &+ h_1^1 + \cdots + h_{s_1}^1 + k_1^2 + \cdots + k_{s_1}^2 \\ &+ h_1^2 + \cdots + h_{s_2}^2 + k_1^3 + \cdots + k_{s_2}^2 + h_1^{n-2} + \cdots + h_{s_{n-2}}^{n-2} + k_1^{n-1} + \cdots + k_{s_{n-2}}^{n-1} \end{aligned}$$

where, if we let $N(j-1) = N + h_1^1 + \cdots + k_{s_{j-1}}^j$ (i.e., $N(j-1)$ is N plus all the handles of type $j-1$ or less, plus all the j-handles of the type denoted by k_1^j)

then for each j, we have $\sum_{i=1}^{s_j} h_i^j + \sum_{i=1}^{s_j} k_i^{j+1}$ is an h-decomposition on $N(j-1)$.

Proof. (M, N) has some relative handle decomposition, say, $M = N + g_1^1 + \cdots + g_r^{n-1}$; we will refer to this decomposition as $\mathfrak{D}$. We will show that we can find an equivalent handle decomposition with the desired properties.

Let $C_*(M, N)$ be an associated algebraic relative CW complex associated with $\mathfrak{D}$. This will have one r-cell for each r-handle. Let ∂_r denote the boundary operator, $\partial_r: C_r \to C_{r-1}$. Let $Z_r = \ker \partial_r$, and let $B_{r-1} = \partial_r(C_r)$. Then we have an exact sequence:

$$0 \longrightarrow Z_r \xrightarrow{\text{inc}} C_r \xrightarrow{\partial_r} B_{r-1} \longrightarrow 0.$$

Thus we may write $C_r = Z_r \oplus D_r$ where if we let $\partial'_r = \partial_r | D_r$, then $\partial'_r: D_r \to B_{r-1}$ is an isomorphism. However, since we have $H_*(C_*) = 0$, $B_{r-1} = Z_{r-1}$, we may also think of this as $\partial'_r: D_r \approx Z_{r-1}$.

We will prove the following statement by induction, on m.

Statement S_m. There is a handle decomposition, equivalent to $\mathfrak{D}$, such that $M = N + h_1^1 + \cdots + k_{s_{m-1}}^m$ plus some additional handles $\{g_i^j\}$ such that if z_i^j is the generator of C_j corresponding to the handle h_i^j, and if d_i^j is the generator of C_j corresponding to the k_i^j, *then for all* $k \leq m$ we have $\{z_i^k\}$ generates Z_k and $\{d_i^k\}$ generates D_k and $\partial d_i^k = \partial'_r(d_i^k) = z_i^{k-1}$. (Of course, $\partial z_i^k = 0$; the z_i^k's are cycles.)

The theorem we wish to prove is S_{n-1}.

Proof of S_1. We have no zero handles in our decomposition, thus no zero cells in the relative CW complex, and so $C_1 = Z_1$; and we simply choose the $\{z_i^1\}$ to correspond to the generators of the handles $\{g_i^1\}$ in the decomposition $\mathfrak{D}$; there will be no $\{d_i^1\}$.

Proof of S_m for $m \leq n-2$, assuming S_{m-1}. We have two bases for C_i. One will be the basis determined by the i-handles of the handle decomposition obtained in S_{i-1}, this will be denoted by $\{c_j^i\}$, these will be the $\{g_j^i\}$'s. The second basis for C_i will be denoted by $\{e_j^i\}$ where $\{e_j^i\} = \{d_j^i\} \cup \{z_j^i\}$. Here $\{z_j^i\}$ is an arbitrarily chosen basis for Z_i, and we define d_j^i by $d_j^i = (\partial'_i)^{-1}(z_j^{i-1})$.

Let A be the matrix relating the basis $\{c_j^i\}$ to the basis $\{e_j^i\}$; that is, the jth column of the matrix A is the coordinate of c_j^i with respect to the basis $\{e_j^i\}$.

Since A is an invertible matrix, it can be reduced to the identity matrix by elementary column operations.

We wish to show that corresponding to each elementary column matrix with matrix, say, E_k, we can find a manipulation of the handle which realizes this change. That is, we want to find a new handle decomposition, equivalent to the one obtained in S_{m-1} (in fact, it will be identical to it on $N(i-1)$) such that if $\{c_j^{i\prime}\}$ is the new basis of C_i determined by this new decomposition, that the matrix relating $\{c_j^{i\prime}\}$ to $\{e_j^i\}$ will be the matrix EA. If we can do this successively to each E_k, $k = 1, \cdots, w$, we will finally obtain a new handle decomposition whose handles correspond to a basis $\{c_j^{i\prime\prime}\}$ of C_i and such that the matrix relating the basis $\{c_j^{i\prime\prime}\}$ to $\{e_j^i\}$ is I. This means we will have found a handle decomposition such that the cells of C_i corresponding to the handles are $\{d_j^i\} \cup \{z_j^i\}$. We will then denote the handles corresponding to the d_j^i by k_j^i and those corresponding to z_j^i by h_j^i and then $h_1^{i-1} + \cdots + h_{s_{i-1}}^{i-1} + k_1^i + \cdots + k_{s_{i-1}}^i$ will be an h-decomposition since we have $\partial d_j^i = z_j^{i-1}$.

We consider the two types of elementary column operations:

Type I. Adding one column to another.

Type II. Multiplying one column by a nonzero integer.

Operations of Type I are done by using Lemma 1.4 of Barden [1] or the corresponding operations in the proof of Lemma 2 of Kervaire [6]; here, however, we need $i < n - 1$.

Operations of Type II are also done by the Lemma 1.4 of Barden except for multiplication by -1. But this simply amounts to changing one's mind on how to pick an orientation for the cells in the associated CW complex.

The argument for S_m with $m = n - 1$ involves the same sort of argument as above using the dual decomposition. S_{n-1} will not be needed in our application of this theorem, only S_{n-2}.

Lemma 2.10. *If* $N^n \supseteq \operatorname{Int} M^n$ *and* $H = \overline{M - N}$ *and if the handle decomposition of* M rel N *has handles of type* k *or less, then the inclusion map induces isomorphisms* $\pi_i(\partial M) \approx \pi_i(H)$ *for* $i < n - k - 1$.

Proof. This is essentially Corollary 12.3 of Mazur [10]. □

3. **Proofs of main theorems.** The following construction of the space N^* is fundamental in the theorems which are to follow. The conditions we need for this construction will be denoted by "Hypotheses (*)"; and are as follows:

Hypotheses (*). Suppose $M^n \supseteq N^n$ with $H_*(M, N) = 0$, and that M has a handle decomposition on N with handles of type less than or equal to $n - 2$, where $n = \dim M$, $n > 5$.

First we use Theorem 2.10 and write M as N plus the sum of h-decompositions of type $(n-3, n-2)$ or less. That is, we may write

$$M = N + h_1^1 + \cdots + h_{s_1}^1 + k_1^2 + \cdots + k_{s_1}^2 + \cdots$$
$$+ h_1^{n-3} + \cdots + h_{s_{n-3}}^{n-3} + k_1^{n-2} + \cdots + k_{s_{n-3}}^{n-2}.$$

Let us consider $N[1] = N + h_1^1 + \cdots + h_{s_1}^1 + k_1^2 + \cdots + k_{s_1}^2$. Now $\pi_1(\partial N[1])$ has finitely many generators, and since $\dim \partial N[1] \geq 4$, these may be represented by disjointly embedded circles, $S_1, \cdots, S_m$; and these circles will have disjoint product neighborhoods $T_1, \cdots, T_m$, with the following two properties:

Property I. The T_i do not contain any points of the 2-handles $k_1^2, \cdots, k_{s_1}^2$.

Property II. None of the a-sets from the handles $h_1^2, \cdots, k_{s_{n-3}}^{n-2}$ have points in common with any of the sets T_i.

Now we will use these T_i to attach 2-handles to $N[1]$; denote these handles by $g_1^2, \cdots, g_m^2$. Define $N^* = N[1] + g_1^2 + \cdots + g_m^2$. Note that we have $\pi_1(\partial N^*) = 0$, since we have killed off the fundamental group of $\partial N[1]$ (Lemma 5.2 of Kervaire and Milnor [8]). By Property I we can consider the handles $g_1^2, \cdots, g_m^2$ to be attached to the manifold $N + h_1^1 + \cdots + h_{s_1}^1$. By Property II and Proposition 2.7 we may define $M^* = N^* + h_1^2 + \cdots + k_{s_{n-3}}^{n-2}$. Note also that we may consider $M \subseteq M^*$.

The corollary of the following lemma is a key step in our embedding theorems.

Lemma 3.1. *If Hypotheses* (*) *hold and if also* $\pi_1(\partial M^*) \to \pi_1(H^*)$ *is an isomorphism* (*this is equivalent to demanding that* $\pi_1(\partial M^*) = 0$), *where we let* $H^* = M^* - N^*$; *then* H^* *is an* h-*cobordism.* (*This will be a trivial* h-*cobordism since* $\pi_1(\partial N^*) = 0$.)

Proof. First we note that $\pi_1(H^*) = 0$. This is true since ∂N^* is simply connected, and thus the handles $h_1^2, \cdots, h_{s_2}^2$ are homotopically trivially attached as subsets of ∂N^*. Since the rest of the handles are of type 3 or greater, none of the other handles composing H^* change the fundamental group.

Now since H^* is built from the sum of h-decompositions, it is clear that $H_*(H^*, \partial N^*) = H_*(H^*, \partial M^*) = 0$. Since we have everything simply connected, H^* deforms to either boundary component. Thus H^* is a (trivial) h-cobordism. □

Corollary 3.2. *If we assume Hypotheses* (*) *with* $\pi_1(\partial M^*) = 0$ (*or equivalently* $\pi_1(\partial M^*) \to \pi_1(H^*)$ *is an isomorphism onto*) *and we somehow find a smooth embedding of* N^* *in some manifold* Q, *then* M^*, *and therefore the submanifold* $M \subseteq M^*$, *will smoothly embed in* Q. □

Lemma 3.3. *If* $N' = N^n + h_1^1 + \cdots + h_s^1$ *is* N^n *plus some* 1-*handles,* $n \geq 3$, *and if we are given an embedding* $N \subseteq \text{Int } Q$ *with* $q > n$, *then we can extend the embedding of* N *to an embedding of* N'. (*We assume here that the* 1-*handles are attached on a connected manifold.*)

Proof. We may easily find disjoint arcs in ∂N which span the a-spheres (0-spheres) of h_i^1, and these may be pushed out from N into Q so as to obtain a collection of 1-disks relatively transverse to ∂N. There is no problem thickening these into 1-handles since all the $V_{n,k}$'s are path connected. □

We are now ready to consider the problems of extending an embedding over the 2-handles. For the two handles, we will use the obstruction Theorem 2.6 which states that given $N + h^{k+1}$ and an embedding of $N^n \subseteq Q^q$ we can extend this embedding to an embedding of $N + h$ if

I. There is a $(k+1)$-disk in Q which spans the a-sphere of the handle and which intersects the manifold N only in that a-sphere.

II. A certain obstruction vanishes. Since we have k = dimension of the a-sphere of h, this obstruction will be a homotopy class in $\pi_k(V(q-k-1, n-k-1))$. This group will be zero if $(q-k-1)-(n-k-1) > k$; i.e., if $q - n > k$.

If we are to be concerned with 2-handles, condition II will cause no problem if we assume that $q - n \geq 2$. In Lemma 3.4, we may obtain disjoint spanning 2-disks by the codimension hypothesis; in Lemma 3.5 we may push the interiors of the 2-disks into $Q \times (0, 1]$ so that they miss $N \subseteq Q \times \{0\}$.

Lemma 3.4. *If $N^n \subseteq Q^q$ is a 1-trivial embedding with $q - n \geq 3$, then N can be extended to an embedding of $N^* \subseteq Q$.*

Lemma 3.5. *If $N^n \subseteq Q^q$ is a 1-trivial embedding with $q - n \geq 1$, then $N^* \subseteq Q \times I$, $q \geq 5$.*

The following lemma is a version of the Whitney separation of spheres lemma. The result of the isotopy is that we will have removed the points of intersection p and q from $S_a \cap S_b$.

Lemma 3.6. *Suppose that $M^n = N^n + h^{n-3} + h^{n-2}$ with $n > 5$ and $\pi_1(\partial N) = 0$—and thus $\partial(N + h^{n-3})$ will be simply connected. Let S_a be the sphere of the $(n-2)$-handle; let S_b be the b-sphere for the $(n-3)$-handle; we will choose a b-sphere lying in ∂N. We may suppose that S_a and S_b are transverse and thus $S_a \cap S_b$ consists of a finite number of points; suppose that p and q are two such points with opposite intersection numbers. Then we may perform the following construction:*

We can find an arc α in S_a from p to q and an arc β in S_b from p to q so that the circle $\alpha \cup \beta$ will lie in ∂N and such that the only points of $S_a \cap S_b$ which will lie on this curve will be p and q. Then we may find a 2-disk, B, in $\partial(N + h^{n-3})$ such that the boundary of B corresponds to $\alpha \cup \beta$. Such a disk may be found so as to enable us to construct the following isotopy. We can find a small neighborhood W' of α in S_a with α contained in the interior of W', and a small neighborhood, W, of B in $\partial(N + h^{n-3})$ and an isotopy f_t of W' in W fixed on $\partial W'$ such that $f_1(W') \cap S_b = \varnothing$.

The construction of f_t is essentially the same as the Whitney isotopy as described in Milnor [12, Theorem 6.6]. However, this is not the same removal of pairs of intersection points. The reason is this: The isotopy of the disk W'

given by our lemma will not, in general, give us an isotopy of S_a, as is usually obtained in versions of Whitney's theorem—we will only obtain an isotopy of part of S_a. This is due to the particular dimensions involved. The dimension of S_a is $n-3$; and the dimension of S_b is 2; both of these spheres are contained in $\partial(N + h^{n-3})$ which is an $(n-1)$-manifold. We will obtain B by using relative general position on the disk which spans $\alpha \cup \beta$ via the hypothesis that $\pi_1(\partial N) = 0$ and $n-1 \geq 5$. S_b is a 2-sphere and, again, since $n-1 \geq 5$ it is easy to make sure that $B \cap S_b = \beta$. *But* the codimension of S_a in $\partial(N + h^{n-3})$ is 2 and thus, in general, the intersection of B and S_a will be zero-dimensional. The effect of this is that after the isotopy f_t, $f_1(W')$ may intersect $S_a - W'$ and thus we will not end up with an embedding of S_a. However, the self-intersections of S_a will be of such a nature that we will, later, be able to handle these without difficulty.

We may now apply Lemma 3.6 to homotope the a-spheres (and thus the a-sets) of the ith $(n-2)$-handles off all $(n-3)$-handles except for one disk which goes around the ith $(n-3)$-handle.

Lemma 3.7. *Suppose that $M = N$ plus an h-decomposition of type $(n-3, n-2)$ with $n > 5$, ∂N simply connected. Let A_i and S_i denote the a-set and a-sphere, respectively, of the ith $(n-2)$-handle, $k_i^{(n-2)}$. For each A_i and $(n-3)$-handle h_i, choose one component, Y_i of $A_i \cap \partial h_i$; this will be an $(n-1)$-disk and will go right around the handle h_i as in the definition of a nicely handled decomposition. We will assume that the $(n-3)$-disk $X_i = S_i \cap Y_i$ intersects the b-sphere of $k_i^{(n-3)}$ with intersection number 1; we may now pair off the rest of the intersection points of S_i with each of the b-spheres of the $(n-3)$-handles so that we may apply Lemma 3.6 to each pair.*

There is a set of subdisks W_{ij} such that each $W_{ij} \subseteq S_i$ (the j-index is an arbitrary ordering of the W_{ij} for fixed i) and subsets U_{ij} of $\underline{A_i \text{ corresponding}}$ to $W_{ij} \times D^2$, and an isotopy, G_t of ∂N such that if we let $Z_i = (A_i - Y_i - \Sigma_j U_{ij}) \cup \Sigma_j G_1(U_{ij})$, then $Z_i \cap \partial h_j = 0$ for all $i \neq j$. In other words, if we consider the homotopies, G_t^i, on A_i defined as G_t on $\Sigma_j U_{ij}$ and the identity on the rest of A_i then we will have

(1) $G_1^i(A_i) \cap \partial h_j = \emptyset$ if $i \neq j$;

(2) $G_1^i(A_i) \cap \partial h_i = Y_i$. □

The next theorem is the result which will allow us to extend a given embedding over an h-decomposition of type $(n-3, n-2)$.

Theorem 3.8. *Suppose that $M^n = N^n$ plus an h-decomposition of type $(n-3, n-2)$ with $n > 5$ with ∂N simply connected. If we have an embedding of $N^n \subseteq \operatorname{Int} Q^q$ with $n < q$, then we can extend the embedding of N to an embedding of M in Q.*

Proof. Recall the Z_i of the previous lemma; these are $(n-1)$-disks with self-intersections. Z_i is A_i minus Y_i after the homotopy which removed cancelling intersection points. Let $T_i = G_1(\overline{S_i - X_i})$, that is, T_i is the part of $G_1(S_i)$ which lies in Z_i. These are $(n-3)$-disks with self-intersections; these self-intersections are due to the isotopies of subdisks W_{ij} of S_i used in the deformation disks D_{ij}, and are confined entirely in the collection of neighborhoods W'_{ij} of D_{ij}. Also we note that we do not have any singularities on ∂W_{ij}, since the ∂W_{ij} are not moved in the isotopies.

We will now define a subset T'_i of $\partial N \times I$ which will correspond to T_i pushed into this set from the boundary, $\partial N \times (0)$. We will also define a similar set Z'_i for Z_i. Then we will define a set which we will call H'_i, which will essentially be a thickening of the set Z'_i in $\partial N(n-3) \times I$.

Let $C_i = X_i \cap T_i$; C_i corresponds to ∂X_i and also ∂T_i. Then we define T'_i by

$$T'_i = C_i \times [0, 1/3] \cup \left(\overline{T_i - \sum_j W_{ij}}\right) \times (1/3)$$

$$\cup \left(\sum_j \partial W_{ij}\right) \times [1/3, 2/3] \cup \left(\sum_j W_{ij}\right) \times (2/3).$$

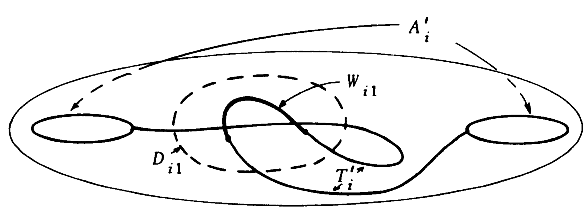
Let $E_i = Y_i \cap Z_i$. Let U_{ij} be the subset of Z_i corresponding to $W_{ij} \times D^2$; let U'_{ij} be the subset of Z_i corresponding to $\partial W_{ij} \times D^2$. Now we will define Z'_i by

$$Z'_i = E_i \times [0, 1/3] \cup \left(\overline{Z_i - \sum_j U_{ij}}\right) \times (1/3)$$

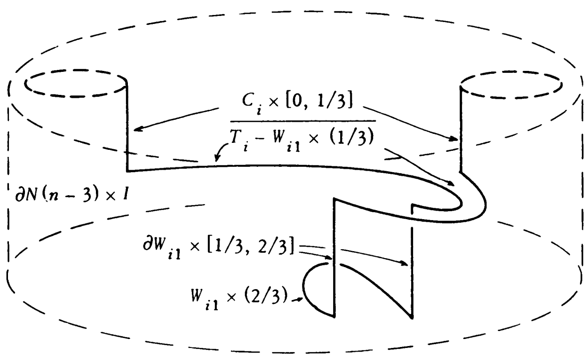
$$\cup \left(\sum_j U'_{ij}\right) \times [1/3, 2/3] \cup \left(\sum_j U_{ij}\right) \times (2/3).$$

Note that if $p\colon \partial N \times I \to \partial N$ is the projection on the first factor, then $p(T'_i) = T_i$ and $p(Z'_i) = Z_i$.

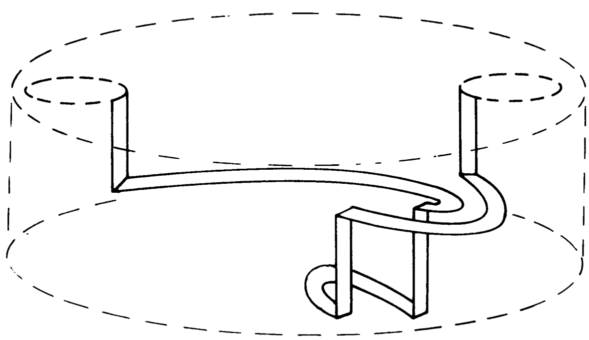
We will illustrate these definitions by the following example:



Here A_i' is the a-set of the ith $(n-3)$-handle, b_i. This example shows a situation with a 1-handle and a 2-handle. These are not correct dimensions for our hypothesis, but we will be able to depict our sets with these examples. T_i' will then look like this:



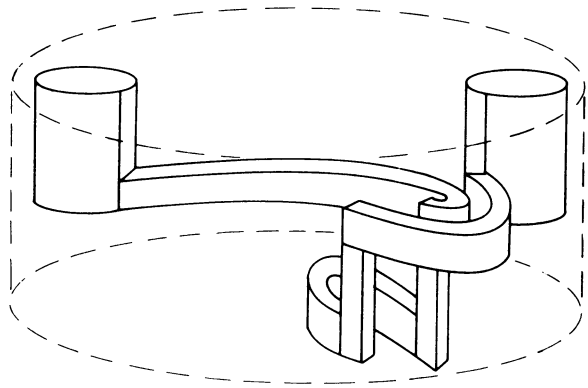
And then Z_i' will look like this:



Let R_{ij} be a small collar neighborhood of U_{ij} in Z_i. We will define a subset H_i' of $\partial N \times I$ by

$$H_i' = A_i' \times [0, 1/2] \cup \left(\overline{Z_i - \sum_j U_{ij}}\right) \times [1/3, 1/2]$$
$$\cup \left(\sum_j R_{ij}\right) \times [1/2, 1] \cup \left(\sum_j U_{ij}\right) \times [2/3, 1].$$

Now H_i' is an n-ball in $\partial N \times I$ which intersects $\partial N \times (0)$ in A_i'. Furthermore, Z_i' lies on the boundary of this ball. Thus H_i' is just like the handle b_i, but it is inside N rather than outside:



Now let $J = [-1, 0]$. We are given an embedding $N \subseteq Q$. Let $N' = N + (\partial N \times J)$, where we identify ∂N with $\partial N \times (0)$; then we can extend the embedding of N to an embedding of N'.

We now consider $\partial N \times D^2$. Using polar coordinates for D^2 we will consider $D^1 = \{(r, \theta) \in D^2$ with $\theta = 0$ or $\theta = \pi\}$, $I = \{(r, \theta) \in D^2$ with $\theta = 0\}$ and $J = \{(r, \theta) \in D^2$ with $\theta = \pi\}$. We now define an isotopy F_t of $\partial N \times D^2$ by rotating half a revolution by means of the disk $F_t(x, (r, \theta)) = (x, (r, \theta + t\pi))$. Let $H_i = F_1(H_i')$; these sets will correspond to our handles h_i.

To see that the a-map of H_i is the same up to isotopy as the a-map of h_i, we may argue as follows. Write $D^n = D^{n-3} \times D^3$, then h_i is a smooth embedding of $\partial D^{n-3} \times D^3$. Suppose we write $D^{n-3} \times D^3 = D^{n-3} \times D^2 \times I$. Then since our handle decomposition is a nicely handled decomposition, if we may think of C_i as the 2-disk $C_i = \{0\} \times D^2 \times \{0\}$ then Y_i will correspond to $D^{n-3} \times C_i \times \{0\} \subseteq \partial h_i$. If we let $h_i' = h_i | \partial D^{n-3} \times C_i \times \{0\}$, then h_i' determines h_i (up to isotopy) as we may see by applying Lemma 2.4 with $P = \partial D^{n-3} \times C_i$, $Q = h_i(\partial D^{n-3} \times C_i \times \{0\})$ and thus the a-set of h_i corresponds to $Q \times I$. If we consider H_i' to be a handle attached from the inside to ∂N, then the a-map of this handle, call it also H_i', will similarly be determined on the set corresponding to C_i and thus we may have $H_i' = h_i \circ \psi$ where ψ is the orientation reversing diffeomorphism on $\partial D^{n-3} \times D^3 = \partial D^{n-3} \times D^2 \times I$ defined as the cartesian product of the identity on ∂D^{n-3}, the identity on D^2 and the linear orientation reversing map on I. Similarly, if H_i denotes the a-map of the handle H_i, we will have $H_i = H_i' \circ \psi$ and so $H_i = h_i \circ \psi \circ \psi = h_i$.

Thus we may define an embedding $\phi\colon N + h_1^{(n-2)} + \cdots + h_{s_{n-2}}^{(n-2)} \to N'$ so that $\phi(N) = N$, $\phi(h_i) = H_i$ and $\phi(Y_i) = Z_i''$.

Consider the isotopy F_t as giving a map $F\colon (\partial N \times I) \times I \to \partial N \times D^2$; then define $P_i = F(Z_i \times I)$. Viewing $\partial N \times D^2 \approx \partial N \times D' \times I$, P_i will be an n-ball which will hit $\partial N \times D' \times \{0\}$ in $Z_i' \cup Z_i''$ where $Z_i'' = F_1(Z_i')$.

We will next use Lemma 3.9 below to obtain corresponding n-balls, V_i in $\partial N \times D' \times \{0\}$ with $Z_i' \cup Z_i'' \subseteq \partial V_i$; we can arrange it so that the collection $\{V_i\}$ is a disjoint collection of balls.

Lemma 3.9. *Suppose $A \approx S^k \times D^r$, $A \subseteq W^w$ with $w - k \geq 3$, $w > k + r$; then A lies on the boundary of a $(k+r+1)$-disk in W iff $A \times \{0\}$ lies on the boundary of a $(k+r+1)$-disk in $W \times I$. (In the case $r = 0$, this is Lemma 1.5 of Haefliger [2].)*

Next, we define a homotopy, ψ_t, of $\partial N \times D'$ such that $\psi_t | Z_i' \cup Z_i''$ is an isotopy and $\psi_1(Z_i' \cup Z_i'') = \phi(A_i)$. This homotopy will be essentially given by $K_t(x, t) = G_{1-t}(x, 1/t)$ on $\partial N \times I$; $K_t(x, t) = [\text{identity on } \partial N(n-3) \times J]$. However, $K_t | Z_i' \cup Z_i''$ will not be an isotopy since K_1 will collapse subsets of Z_i' corresponding to $E_i \times [0, 1/3]$ and $(\Sigma U_{ij}) \times [1/3, 2/3]$. We may avoid this problem by first applying an isotopy that tilts these sets slightly in the I direction so that the restriction of the projection $\partial N \times I \to \partial N$ to Z_i' will be a one-to-one map of Z_i' onto Z_i. If we now apply K_t, we will obtain our ψ_t.

Next we may extend the isotopies $\psi_t | Z_i' \cup Z_i''$ to isotopies $\overline{\psi_t}$ of the n-balls V_i; let $V_i^* = \overline{\psi_1}(V_i)$. Then $\{V_i^*\}$ is a disjoint collection of n-balls in N', and the restriction of the normal bundle of N' in Q to ΣV_i^* is trivial; thus we may extend the embedding of N' to an embedding of $X = N' + \Sigma V_i^* \times I$.

Let K_i be the n-ball of X corresponding to $A_i \times I \cup V_i^* \times \{1\}$. These K_i will be our $(n-2)$-handles. We may check from our construction that the a-map of K_i, call it K_i, is the same as k_i. However, it is easier to argue that these maps are isotopic as follows. The a-spheres of these handles are clearly the same, so we may view K_i and k_i as giving two framings by 2-frames of this $(n-3)$-sphere. These framings determine elements of $\pi_{n-3}(SO_2)$ and the framings will be equivalent (and thus K_i and k_i isotopic) iff they determine the same element; which they must since, in fact, $\pi_{n-3}(SO_2) = 0$ if $n \geq 5$. Thus the desired embedding of M is given by $N + H_1 + \cdots + H_s + K_1 + K_2 + \cdots + K_s$. □

Proof of Theorem A. We begin by considering some handle decomposition of M on N. Since M has nonempty, connected boundary, we may assume that this decomposition has no n-handles.

We will eliminate the problem of having handles of type one less than the dimension of the manifold by considering $M' = M \times I$, and $N' = N \times I$. Let $m = \operatorname{Dim} M' = n + 1$. By Lemma 2.1 we can get a handle decomposition of M' on N' where the largest type handle is of type $n - 1 = m - 2$; so M' has a handle decomposition on N' with no m-handles and no $(m-1)$-handles.

Since we can embed N in Q, we certainly can embed $N \times I$ in $Q \times I$. Now we consider $(N')^*$. By Lemma 3.5 we can embed $(N')^*$ in $(Q \times I) \times I$.

Now let $M'' = (M')^*((n-4))$; this will be all of the decomposition of M' on N' except the last h-decomposition of type $(n-3, n-2)$. Now we may use Lemma 3.2 on M'', since Lemma 2.10 assures us of the condition on the fundamental groups; therefore we can embed M'' in $Q \times I^2$. Finally we use Theorem 3.8 to get an embedding of all of $(M')^*$ in $Q \times I^2$. But then we have $M' = M \times I \subseteq Q \times I^2$. □

Proof of Theorem B. We will define N' and M' as in the previous theorem. We are given $N' \subseteq Q$; this time we will use Lemma 3.4 to embed (N') in Q.

We define M'' as in the previous theorem, and the same argument will show us that we can embed M'' in Q, and thus we can embed $M' \subseteq M \times I$ in Q. □

Proofs of Theorems A′ and B′. If we now proceed with the proofs of the previous theorems, using M and N instead of M' and N', and assuming $n > 5$, we could conclude in Theorem A that we could embed M in $Q \times I$; and we could conclude in Theorem B that if $N \subseteq Q$, then $M \subseteq Q$. □

4. **Embedding homology connected manifolds.** A homology cobordism is a triple $(W; M_0, M_1)$ where ∂W is the disjoint union of M_0 and M_1 and $H_*(W, M_0) = H_*(W, M_1) = 0$. If M_0 is a manifold with boundary, a homology cobordism of M_0 is a 4-tuple $(W; H, M_0, M_1)$ where $\partial W = H \cup M_0 \cup M_1$, $H \cap M_0 = \partial M_0$, $H \cap M_1 = \partial M_1$, $H_*(W, M_0) = H_*(W, M_1) = 0$ and $(H, \partial M_0, \partial M_1)$ is a homology cobordism as above.

In the theorem below, we will assume for convenience only that M_0 has no boundary. If M_0 has boundary then the homology cobordism we would obtain in the proof of the theorem would be a 4-tuple $(W; H, M_0, M_1)$ with $H \approx \partial M_0 \times I$.

Theorem 4.1. *Let M_0^n be a smooth compact, orientable manifold $n \geq 5$, with $H_i(M_0^n) = 0$, $1 \leq i \leq k$, then there is a homology cobordism $(W; M_0, M_1)$ such that M_1 is k-connected.*

Furthermore, if M_0 is a π-manifold, so is M_1; if M_0 is almost parallelizable, so is M_1.

Proof. As in the proof of Theorem 2.9, we may find a handle decomposition of M^0 of the following form:

$$M_0 = h_1^0 + h_1^1 + \cdots + h_r^1 + k_1^2 + \cdots + k_r^2 + \mathcal{H}$$

where $\mathcal{H}$ is the sum of handles of type two or greater and $h_1^0 + \cdots + k_r^2$ is an h-decomposition of type $(1, 2)$. Let $N_0 = h_1^0 + \cdots + k_r^2$, then $\widetilde{H}_*(N_0) = 0$. Also, the map $\pi_1(N_0) \to \pi_1(M_0)$ induced by inclusion is onto since every loop in M_0 can be represented by a loop in $h_1^0 + \cdots + h_r^1$.

Thus N_0 is a smooth homology disk and since all obstructions to trivializing the tangent bundle plus a trivial line bundle vanish, N_0 is a π-manifold. Now $\pi_1(N_0)$ has k generators; each may be represented by an embedded circle with a product normal bundle. We may attach 2-handles to $M_0 \times I$ along these circles considered as subsets of $N_0 \times \{1\} \subseteq M_0 \times \{1\}$; call the resulting manifold W', then W' will be a cobordism between M_0 (corresponding to $M_0 \times \{0\}$) and another manifold which we will denote by M'. We will let V' denote the subset of W' corresponding to $N_0 \times I$ plus those 2-handles; V' will be a cobordism between N_0 and another manifold, denoted N'. By a proper choice of trivializations of the product bundles, we may add our handles in such a way as to have N' a π-manifold [5, Theorem 5.5].

Also, from the same theorem, we may conclude that N' is simply connected. Since $\pi_1(N_0) \to \pi_1(M_0)$ was onto, we will then have M' simply connected.

Now $H_2(W', M_0) = H_2(V', N_0)$ is free on k generators; we will now show that each of these can be represented by an embedded 2-sphere in N'. We consider the exact sequence

$$H_2(V') \xrightarrow{j_*} H_2(V', N_0) \longrightarrow H_1(N_0).$$

Since $H_1(N_0) = 0$, $j_*\colon H_2(V') \to H_2(V', N_0)$ is onto. Viewing V' as a cobordism obtained by adding handles to $N' \times I$ we see these must be $(n-2)$-handles. Since $n \geq 5$, these are handles of type 3 or greater, thus adding them to $N' \times I$ does not introduce any (nontrivial) relative 2-cycles and so $H_2(V', N') = 0$; thus we obtain an onto map $i_*\colon H_2(N') \to H_2(V')$ via the exact sequence

$$H_2(N') \xrightarrow{i_*} H_2(V') \longrightarrow H_2(V', N').$$

Thus, via $i_* \circ j_*$, any element of $H_2(V', N_0')$ can be represented by an element of $H_2(N')$. But since N' is simply connected, we have $\pi_2(N') \approx H_2(N')$; and since $n \geq 5$ any element of $\pi_2(N')$ may be represented by an embedded 2-sphere. Since N' is a π-manifold of dimension larger than 5, the 2-spheres will have trivial normal bundle (Lemma 5.3 of [5]). We will now add to N' k 3-handles, one for each generator of $H_2(V', N_0)$ so that if we let V denote V' plus these 3-handles, then V will be a cobordism between N and, say, N_1. Also, since $N' \subseteq M'$, we may consider these handles as being added to W'; if we do, the resulting manifold will be called W and will be a cobordism between M_0 and a manifold M_1. M_1 will be simply connected: since it is obtained from M' by adding 3-handles; M' was simply connected and the process of adding a 3-handle does not affect the fundamental group.

We next claim that $(W; M_0, M_1)$ is a homology cobordism. By duality, it is sufficient to show $H_*(W, M_0) = 0$. The skeletal chain complex $C_*(W, M_0)$ is zero except in dimensions 2 and 3 where in each case it is free on k generators, one for each handle. By our construction, the boundary map will give an isomorphism between the 3-chains and the 2-chains since it gives a 1-1 correspondence on the generators; thus $C_*(W, M_0)$ is acyclic and $H_*(W, M_0) = 0$.

Now by duality [11], we will also have $H_*(W, M_1) = 0$. From the exact sequences of the pairs, we see that if M_0 is homology k-connected, so is M_1. But since M_1 is simply connected, it must be (homotopy) k-connected.

To prove the second assertion of the theorem, we note that our surgery was performed in such a way that if M_0 were a π-manifold then so would be M_1; if M_0 were almost parallelizable, so would be M_1. □

Many embedding theorems assume that one has a k-connected manifold. The theorem of this paper in conjunction with theorems of [5] allow us to use some of these theorems to obtain results on embedding homology k-connected manifolds, such as the following

Corollary 4.2. *If M_0^n is a smooth compact homology k-connected manifold, $k \geq 1$, $n \geq 5$, then M_0^n embeds smoothly in R^{2n-k+1}.*

Proof. By Theorem 4.1 there is a smooth homology cobordism $(W; M_0, M_1)$ with M_1 a k-connected manifold. Let C denote a collar neighborhood of M_1 in W; $C \approx M_1 \times I$. In the proof of the theorem we have seen that W will have a handle decomposition relative to M_1 consisting of handles of type three or greater, thus W will be simply connected and thus the map $\pi_1(M_0) \to \pi_1(\overline{W - C})$ induced by inclusion will be added onto the map and we may use Theorem B$'$. By [3], since M_1 is k-connected, M_1 embeds in R^{2n-k}, thus C embeds in R^{2n-k+1}.

Remarks. The hypotheses and conclusion of the above corollary are weaker than those of Theorem E of [4]. It is also interesting to note in view of the particular handle decomposition of W relative to M_1, we are essentially making direct use of Theorem 3.8.

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