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INSTABILITY IN $\mathrm{Diff}^r(T^3)$ AND THE NONGENERICITY OF RATIONAL ZETA FUNCTIONS

BY

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ABSTRACT. In the search for an easily-classified Baire set of diffeomorphisms, all the studied classes have had the property that all maps close enough to any diffeomorphism in the class have the same number of periodic points of each period. The author constructs an open subset U of $\mathrm{Diff}^r(T^3)$ with the property that if f is in U there is a g arbitrarily close to f and an integer n such that f^n and g^n have a different number of fixed points. Then, using the open set U, he illustrates that having a rational zeta function is not a generic property for diffeomorphisms and that Ω-conjugacy is an ineffective means for classifying any Baire set of diffeomorphisms.

A. **Introduction and statement of theorems.** Let $\mathrm{Diff}^r(M^n)$ be the space of C^r diffeomorphisms of a compact C^∞ n-manifold M with the C^r topology, $1 \le r \le \infty$. Central problems in the study of differentiable dynamical systems, as formulated by Smale ([24], [26]) are

(a) Find a Baire subset B of $\mathrm{Diff}^r(M^n)$ with strong stability properties.

(b) Find a practical means of classifying the elements of B.

Let $f \in \mathrm{Diff}^r(M)$. The *nonwandering set* of f, $\Omega(f)$, is the invariant set $\{x \in M$: for any neighborhood U of x there is a positive integer n with $f^nU \cap U \neq \emptyset\}$. f satisfies *Axiom A* if the periodic points of f are dense in $\Omega(f)$ and if $\Omega(f)$ has a *hyperbolic structure*, i.e., there is an invariant splitting of the tangent bundle of M restricted to $\Omega(f)$

$$TM\,|\,\Omega(f) = E^u \oplus E^s$$

with $Tf: E^u \longrightarrow E^u$ an expansion and $Tf: E^s \longrightarrow E^s$ a contraction. Hirsch and Pugh [9] have shown that if f satisfies Axiom A, then for each $x \in \Omega(f)$ the stable manifold of x, $W^s(x, f) \equiv \{y \in M: d(f^mx, f^my) \longrightarrow 0$ as $m \longrightarrow \infty\}$, is a smooth, injectively immersed open cell through x and depends smoothly on x and f. The unstable manifolds of f, $W^u(x, f)$, are the stable manifolds of f^{-1}. f is *structurally stable* (*Ω-stable*) if for each g in some neighborhood of f in $\mathrm{Diff}^r(M)$ there is a homeomorphism $h: M \longrightarrow M$ ($h: \Omega(f) \longrightarrow \Omega(g)$) with $gh = hf$ on M (on $\Omega(f)$). A *generic property* is a property that holds for a Baire subset of $\mathrm{Diff}^r(M)$. For a general

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reference, see Smale's survey article [24] or Nitecki's book [14].

Finally, the reader is referred to [8] and [31] for the definition and properties of a k-foliation $\mathfrak{F}$ on M. $f\colon M \longrightarrow M$ *respects the foliation* $\mathfrak{F}$ if the image of a leaf of $\mathfrak{F}$ by f is another leaf of $\mathfrak{F}$. f *preserves the foliation* $\mathfrak{F}$ if f maps each leaf onto itself.

To put the results of this paper into perspective, we discuss briefly the recent history of problems (a) and (b). There have been a number of unsuccessful candidates for B, beginning with *Morse-Smale maps*, [20], i.e., diffeomorphisms whose nonwandering set is hyperbolic and consists of a finite number of points, whose stable and unstable manifolds intersect only transversally (*strong transversality condition*). Such maps were later shown to be structurally stable [15] but by no means dense in $\mathrm{Diff}^r(M)$ ([22], [24]). Smale showed that structurally stable maps are not dense in [23], where he conjectured that diffeomorphisms that satisfy Axiom A and the strong transversality condition might form a Baire subset of $\mathrm{Diff}^r(M)$. Later, he demonstrated [25] that maps satisfying Axiom A and the "no-cycle property" were Ω-stable. However, in 1968 Abraham and Smale [2] showed that neither Ω-stable maps nor ones satisfying Axiom A form a Baire subset of $\mathrm{Diff}^r(M^n)$ for $r \geq 1$, $n \geq 4$. Newhouse [13] has the corresponding result for $r \geq 2$, $n = 2$. However, both Abraham and Smale [26] have emphasized that many more such counterexamples must be constructed and analyzed for the theory to advance, especially since each new conjecture for B has arisen from careful analysis of past counterexamples. The examples we construct in this paper are the first C^1 counterexamples to the genericity of Axiom A and Ω-stability on 3-manifolds. More significantly, all the above classes of diffeomorphisms conjectured to solve problem (a) have had the following property: all maps close enough to any diffeomorphism in the class have the same number of periodic points of each period as the original map. Theorem 1 below illustrates that this is not a generic property, i.e. there is an open set in $\mathrm{Diff}^r(T^3)$ with the property that as close as you wish to any map in the set there is another map with a different number of periodic points of some period.

Theorem 1. *Let* $1 \leq r \leq \infty$. *For* $f \in \mathrm{Diff}^r(T^3)$ *and positive integer* n, *let* $N_n(f) =$ *number of fixed points of* $f^n = f \circ f \circ \cdots (n \text{ times}) \cdots f\colon T^3 \longrightarrow T^3$. *Then, there exists an open set* U *in* $\mathrm{Diff}^r(T^3)$ *such that if* $f_0 \in U$ *and* U_0 *is any neighborhood of* f_0 *in* U, *there are* $f_1 \in U_0$ *and integer* n *such that* $N_n(f_0) \neq N_n(f_1)$ *and all periodic points of* f_1 *of period* $\leq n$ *are hyperbolic.*

The proof of Theorem 1 is contained in §§B–K. First, let us see what effect it has on problem (b), the classification problem. In [24], Smale conjectured that an effective means of classifying the maps in B might be the zeta function. The *zeta function* of a diffeomorphism f is

$$\zeta(f) = \zeta_f(t) = \exp\left(\sum_{i=1}^{\infty} \frac{N_i t^i}{i}\right) \quad \text{where } N_i = N_i(f)$$

as in Theorem 1. Artin and Mazur [3] demonstrated that a dense (not Baire) set of diffeomorphisms have zeta functions with a positive radius of convergence. Meyer [12] and Shub [19] showed that if f satisfies Axiom A, $\zeta_f(t)$ has a positive radius of convergence. Williams [28] demonstrated that if Λ is a hyperbolic attractor of f, $\zeta(f|\Lambda)$ is rational. Bowen and Lanford ([4], [5]) showed the same for Λ zero-dimensional and hyperbolic. Recently, Guckenheimer [7] has shown that if f satisfies Axiom A and the no-cycle property, $\zeta(f)$ is rational. However, in order to be at all effective and practical as a means of classification, $\zeta(f)$ must be rational for a Baire set of diffeomorphisms. Whether or not $\zeta(f)$ is generally rational was asked in [24, Problem 4.5], [29], [27], and [28]. Theorem 2 uses Theorem 1 to answer this question.

Theorem 2. *Diffeomorphisms with rational zeta functions do not form a Baire subset of* $\mathrm{Diff}^r(T^3)$, $1 \le r \le \infty$.

Proof of Theorem 2. Since there are only a countable number of rational zeta functions [5], enumerate them as $Z_1, Z_2, \cdots, Z_j, \cdots$. Say $Z_j(t) = \exp(\Sigma_{i=1}^{\infty} N_i^j t^i/i)$. Let U be the open set in $\mathrm{Diff}^r(T^3)$ from Theorem 1. Let $V_j = \{f \in U|$ for some k in $\mathbf{N}$, (1) $N_k(f) \neq N_k^j$ and (2) f^k has only hyperbolic fixed points$\}$. So, if $f \in V_j$, $\zeta(f) \neq Z_j$. By the hyperbolicity in the definition of V_j, each V_j is open. We claim each V_j is also dense. Then, we will have $V = \bigcap V_j$, a Baire subset of U; and no diffeomorphism in V can have a rational zeta function.

Suppose the above claim is false, i.e. that there is an open set W in U with $W \cap V_j = \emptyset$. By the Kupka-Smale Theorem [21], there is $g_1 \in W$ with all periodic points hyperbolic. Since $g_1 \notin V_j$, $N_k(g_1) = N_k^j$ for all k. By Theorem 1, there are $g_2 \in W$ and integer i with $N_i(g_2) \neq N_i(g_1) = N_i^j$ and $\mathrm{Fix}(g_2^i)$ hyperbolic. Thus, $g_2 \in V_j$, contradicting $W \cap V_j = \emptyset$.

Finally, Theorem 3 below deals with another aspect of the classification problem. It states that Ω-conjugacy is not a reasonable equivalence relation to use in classifying diffeomorphisms. The same result holds for any equivalence relation which has all $N_n(f)$ constant in each equivalence class. The proof of Theorem 3 is the same as that of Theorem 2 with V_j replaced by $\{f \in U|$ for some k in $\mathbf{N}$, (1) $N_k(f) \neq N_k(h_j)$ and (2) f^k has only hyperbolic fixed points$\}$.

Theorem 3. *There do not exist a countable set* $\{h_j\}$ *and a Baire subset* B *in* $\mathrm{Diff}^r(T^3)$ *such that each* f *in* B *is* Ω*-conjugate to some* h_j.

Let us outline the construction used to prove Theorem 1. In §B, we construct a hyperbolic "D-A" diffeomorphism g of T^2. $\Omega(g)$ consists of a fixed point source θ and a one-dimensional expanding attractor Σ. The one-dimensional $\{W^s(x, g): x \in \Sigma\}$ fill up $T^2 \backslash \{\theta\}$ and extend to a g-invariant foliation $\mathcal{S}$ of T^2. If $h: S^1 \rightleftarrows S^1$ has $\{+1\}$ as a fixed point source, $g \times h$ is a diffeo of T^3 respecting the foliation $\mathcal{F}$ whose leaves are a product of S^1 and the leaves of $\mathcal{S}$. In §D, we construct b: T^3

$\to T^3$ which is the identity on $\Sigma_1 \equiv \Sigma \times \{+1\}$, which preserves $\mathcal{F}$, and which forces the two-dimensional local unstable manifolds of points of Σ_1 to intersect the one-dimensional stable manifolds from Σ_1 transversally. $\mathcal{F}$ is a normally-hyperbolic foliation (§F) for $f = h \circ (g \times h)$. So, maps near f will respect foliations $\mathcal{F}'$ near $\mathcal{F}$.

In §D, we single out an open subset B_1 of Σ_1 and for each $x \in B_1$ a 2-disk $F(x)$ in the leaf of $\mathcal{F}$ through x, so that $\bigcup \{F(x): x \in B_1\}$ is a 3-disk. Each $f|F(x)$ contains a Smale "horseshoe" as drawn in Figures 6, 9, and 10, yielding a one-parameter family of horseshoe maps. In §§H, I, and J, we show how an arbitrarily small change in f can radically change the topological type of one of these horseshoes so that, for some x, $f|F(x)$ will have a different number of periodic points than the corresponding $f'|F'(x')$. In §J, we achieve the hyperbolicity of Theorem 1 by using the Kupka-Smale Theorem. See also [32].

Theorems 1, 2, 3 hold at least for all manifolds which are the product of T^2 with any manifold. The author has benefited from many valuable and encouraging discussions with R. Clark Robinson, Sheldon Newhouse, and especially from the inspiration and counsel of R. F. Williams.

B. Anosov diffeomorphisms and derived-from-Anosov diffeomorphisms. Let A_0 be a 2×2 matrix with all integer entries, determinant 1, and no eigenvalues of norm one. A_0 induces a hyperbolic automorphism A of the 2-torus via the canonical quotient map $\pi: R^2 \to T^2$. A_0 has eigenvalues λ, μ with $0 < |\lambda| < 1 < |\mu|$ and eigenspaces L_0, M_0 respectively. Let $\mathcal{L}$ and $\mathcal{M}$ be the families of all lines in T^2 parallel to $\pi(L_0)$ and $\pi(M_0)$ respectively. $\mathcal{L}$ and $\mathcal{M}$ become the stable and unstable manifolds for A giving us two transversal foliations of the torus. For example, $W^s(\theta, A) = \pi(L_0)$ where $\theta = \pi(0, 0)$.

We now construct a C^0 perturbation of A, using a surgery described by Smale [24] and Williams [30].

Theorem (Smale-Williams). *Let $A: T^2 \to T^2$ be a hyperbolic toral automorphism. Then there exists $g: T^2 \to T^2$ such that*

(a) *g is smoothly isotopic to A,*

(b) *nonwandering set $\Omega(g) = \{\theta\} \cup \Sigma$, where $\theta = \pi(0, 0)$ is a point source and Σ is a one-dimensional attractor with hyperbolic structure,*

(c) *the stable manifolds of $g|\Sigma$ are the lines of $\mathcal{L}$ except for L_0 which divided by θ now forms two stable manifolds,*

(d) *g respects the foliation $\{W^s(x, A): x \in T^2\}$.*

g is usually called a D-A map, since it is derived from the Anosov diffeomorphism A. In the construction of g, one chooses a small rectangle Q (in the canonical coordinates of [24]) about θ. Then, $g = \phi \circ A$ where ϕ is a C^∞ diffeomorphism of T^2 that is the identity outside $Q \cap A(Q)$ and on D_0, the path component of M_0

$\cap\, Q$ containing θ. One requires that $\phi(C) = C$ for each path component C of members of $\mathfrak{L}$ in Q and that, on each C, ϕ is expanding away from $D_0 \cap C$. The expansive constant of ϕ on the path component of $L_0 \cap Q$ containing θ need be greater than μ. In effect, one changes A on Q so that g has 2 saddle-like fixed points $\{x_0, \bar{x}_0\}$ in Q and one point source θ, as in Figure 1; while A had only one fixed point in Q, the saddle point θ. Williams ([27], [30]) has shown that Σ, a "generalized solenoid," is locally the product of a Cantor set and an interval, periodic points of g are dense in Σ, $W^u(\Sigma, g) = \Sigma$, and $\Sigma = \overline{W^u(x_0, g)}$.

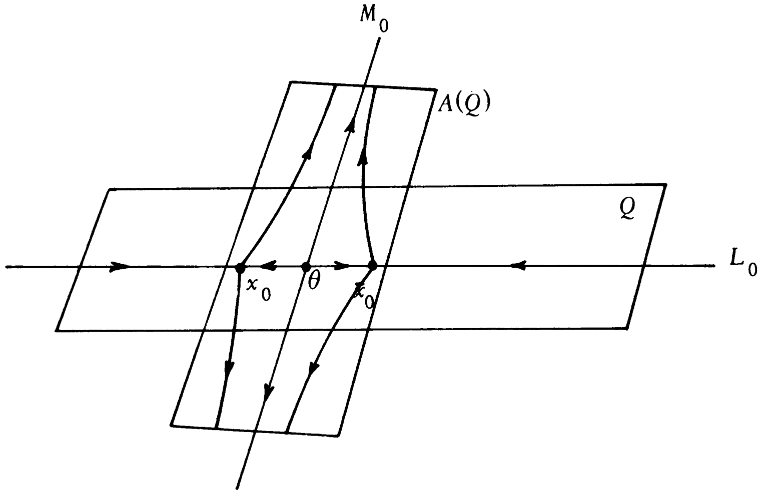


Figure 1

The leaves of our foliation are now the generalized stable manifolds of points of Σ with the exception that $W^s(x_0, g) \cup W^s(\bar{x}_0, g) \cup \{\theta\}$ forms one leaf. Now Σ is a *basic set* for g, i.e. a closed invariant subset of $\Omega(g)$ with a hyperbolic structure, a dense orbit, and a dense subset of periodic points. So, $T_\Sigma M$ has an invariant splitting $E^+ \oplus E^-$ and there are constants $0 < \lambda_1 < 1 < \mu_1$ such that $|TgX| \le \lambda_1 |X|$ for $X \in E^-$ and $|TgX| \ge \mu_1 |X|$ for $X \in E^+$. By choosing ϕ so that the rate of expansion of $g = \phi \circ A$ on all the above-mentioned intervals C is less than μ_2 where $1 < \mu_2 << \mu_1$, one makes the rate of expansion normal to the foliation larger than any rate of expansion on any leaf.

Consider now g^k for any integer $k > 0$. $\Omega(g^k) = \Omega(g)$. x_0 is a fixed point of g^k and $W^s(x, g) = W^s(x, g^k)$ for all $x \in \Omega(g)$. g^k respects the above foliation. In addition, $|T(g^k)X| < \lambda_1^k |X|$ for $X \in E^-$ and $|T(g^k)X| > \mu_1^k |X|$ for $X \in E^+$. If r in Theorem 1 is finite, choose k so that $\mu_1^k > 4^r$ and $\lambda_1^k < 1/4$. If $r = \infty$, make $\mu_1^k > 16$. g^k will be denoted as g in the remainder of this paper.

C. $g \times h$: $T^3 \to T^3$. Let h: $S^1 \to S^1$ be a C^∞ diffeomorphism of the circle with exactly two (hyperbolic) fixed points: $\{+1\}$ a source and $\{-1\}$ a sink. Choose h so that $T_{+1}h(s) = as$ where $3 < a < 4$ and h increases no arc of S^1 by a factor greater than 4.

$g \times h$ is a hyperbolic C^∞ diffeomorphism of the 3-torus, $T^3 \cong T^2 \times S^1$. Since $\Omega(g \times h) = \Omega(g) \times \Omega(h)$ [24, §10], $\Omega(g \times h) = \Sigma \times \{+1\} \cup (\theta, +1) \cup \Sigma \times \{-1\} \cup (\theta, -1)$.

For convenience, we introduce the following notation: $T^2_+ \equiv T^2 \times \{+1\}$, $\Sigma \equiv \Sigma \times \{+1\}$, $\theta \equiv (\theta, +1)$, $x_0 \equiv (x_0, +1)$, $g_+ \equiv (g \times h)|T^2_+$.

Since g respected the foliation $\{W^s(x, A)\}$ on T^2, $g \times h$ respects the foliation $\{W^s(x, A) \times S^1\}$ on $T^2 \times S^1$. We will denote this C^∞ foliation with cylindrical leaves by $\mathfrak{F}$ and the leaf of $\mathfrak{F}$ containing $x \in T^3$ by $F(x)$.

Note also that, around the fixed point x_0, $W^s(x_0, g \times h)$ is a 1-disk lying in T^2_+ and equal to $W^s(x_0, g_+)$. $W^u(x_0, g \times h)$ is a 2-disk transversal to $W^s(x_0, g \times h)$, and equal to $W^u(x_0, g_+) \times [S^1 - \{-1\}]$.

D. **The bump function h with support near x_0.** Choose 2-disk B_1 in T^2_+ such that

(1) $x_0 \in$ interior (as 2-disk) of B_1.

(2) $\theta \notin B_1$.

(3) $B_1 \subset Q$, where Q is as in §B.

(4) If $x \in J^s$, the path component of $W^s(x_0, g \times h) \cap B_1$ containing x_0, $d(gx, x_0) < 1/3d(x, x_0)$. If $y \in J^u$, the path component of $W^u(x_0, g_+) \cap B_1$ containing x_0, $d(gy, x_0) > 3d(y, x_0)$. This is possible because eigenvalues λ and μ of $T_{x_0}g$ are such that $|\lambda| < 1/4$ and $|\mu| > 4$.

(5) $B_1 = J^s \times J^u$ in T^2_+.

(6) Let v_0 be the point of ∂J^s closest to θ as in Figure 2. $\{v_0\} \times J^u \subset W^u_{\mathrm{loc}}(\theta, g_+)$, a fixed local unstable manifold of θ for g_+; while $g^n_+(\{v_0\} \times J^u) \cap W^u_{\mathrm{loc}}(\theta, g_+) = \varnothing$ for all $n \geq 1$.

(7) For each $x \in B_1 \cap \Sigma$, let $W^s_L(x, g)$ be the path component of $W^s(x, g_+) \cap B_1$ containing x. Choose B_1 so that, for $x \in B_1 \cap \Sigma$, $g(W^s_L(x, g)) \subset W^s_L(gx, g)$ or misses B_1.

Choose interval B_2 in S^1 so that $+1 \in \operatorname{int} B_2 \subset S^1$ and $z \in B_2 \Rightarrow d(hz, +1) > 3d(z, 1)$. Then, $B = B_1 \times B_2$ is a 3-disk about x_0 in T^3.

Notation. The following notation will be helpful:

$\tilde{\Sigma} \equiv$ path component of $\Sigma \cap B$ containing x_0, i.e. J^u;

$\tilde{F}(x) \equiv$ path component of $F(x) \cap B$ containing x for $x \in B$;

$W^s_L(x, g \times h) \equiv$ the local stable manifold of x, i.e. path component of $W^s(x, g \times h) \cap B$ containing x, for $x \in \Sigma \cap B$;

$W^u_L(x, g \times h) \equiv$ the local unstable manifold of x, i.e. path component of $W^u(x, g \times h) \cap B$ containing x, for $x \in \Sigma \cap B$;

$W^s_L(\tilde{\Sigma}) \equiv B_1 = \bigcup W^s_L(x, g)$ for $x \in \tilde{\Sigma}$.

Note that $W^s_L(x, g \times h)$ is an interval and equals $\tilde{F}(x) \cap T^2_+$, while $W^u_L(x, g \times h)$ is a 2-disk. Now, choose 2-disk N_1 in B_1 so that

(a) $N_1 \cap \Sigma = \varnothing$,
(b) $g_+^{-n}(N_1) \cap B_1 = \varnothing$ and $g_+^n(N_1) \cap N_1 = \varnothing$ for all $n > 0$,
(c) $W_L^s(x_0, g \times h)$ divides N_1 into two 2-disks (as in Figure 2),
(d) if $W_L^s(x, g \times h) \cap N_1 = \varnothing$, then $g_+ W_L^s(x, g \times h) \cap B_1 = \varnothing$.

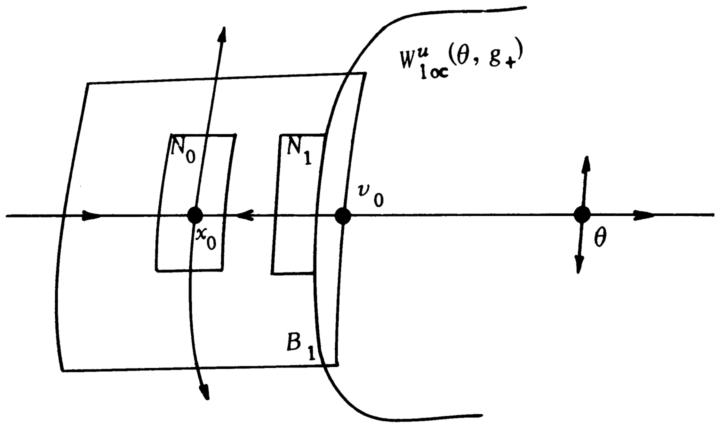


Figure 2

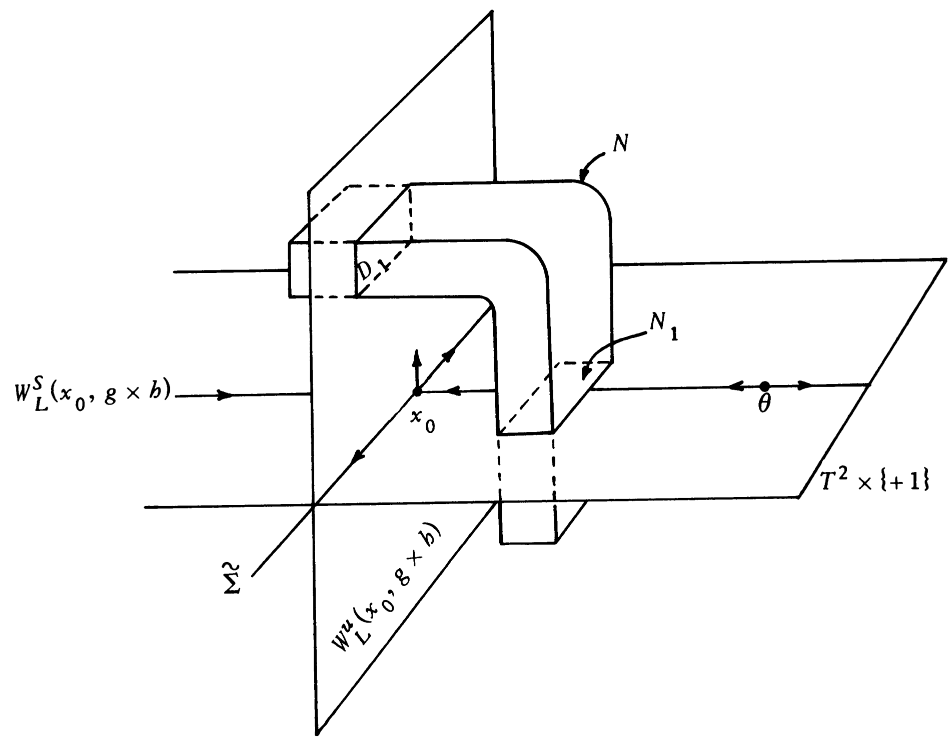


Figure 2A

In addition, as in Figure 2, about x_0 choose 2-disk N_0 in the interior (as 2-disk) of B_1 so that $N_0 \cap N_1 = \emptyset$ but $W^s_L(x, g \times h)$ meets N_0 iff it meets N_1 for $x \in \Sigma$.

At this point, it will be helpful to name a collection of intervals in S^1. First, write S^1 as the union of two intervals, S_+ and S_-, where $S_+ \cap S_- = \{-1, +1\}$. Then, choose open intervals N_2 and N_3 in $B_2 \subset S^1$ such that

(i) $+1 \in N_2$,

(ii) $\overline{N}_3 \subset [B_2 - N_2] \cap S_+$,

(iii) $h^{-1}N_3 \cap N_3 = \emptyset$,

(iv) $hN_2 \supset N_3$.

Also, let N_5 be an interval in B_2 such that

(v) $N_2 \subset$ interior (as 1-disk) of N_5,

(vi) $\overline{N}_3 \cap \overline{N}_5 = \emptyset$.

Let c be the point $\partial N_5 \cap S_+$. Finally, let N_4 be a subinterval of N_2 about $+1$, contained in $h^{-1}N_5$ with length at most $1/3$ the length of N_2.

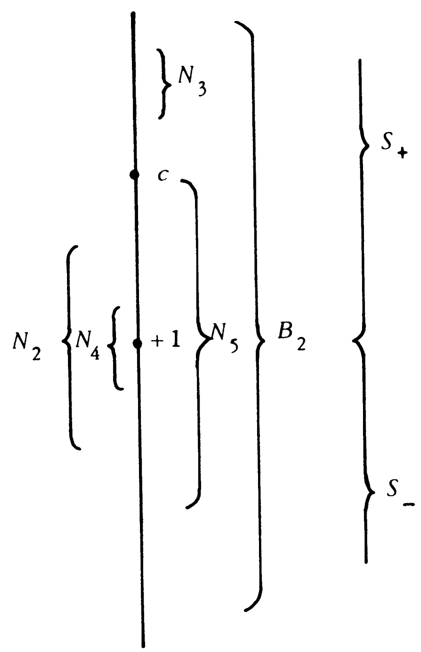


Figure 3

Let $D_1 = (N_0 \times N_3) \cap W^u_L(x_0, g \times h)$, a 2-disk in T^3. Finally, choose open set N in T^3 such that

(1) $N \cap (T^2 \times N_2) = N_1 \times N_2$,

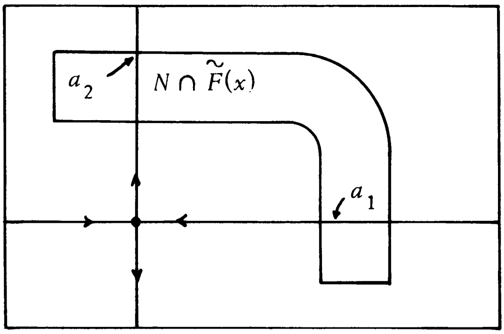
(2) $N \cap (N_0 \times S^1) = N_0 \times N_3$,

(3) $\bar{N} \subset$ interior B,

(4) $B \cap (g \times h)^{-1}N \subset B_1 \times h^{-1}(B_2 - N_5)$,

(5) $[g_+ B_1 \times \{c\}] \cap N = \emptyset$.

So, $N \cap W^u_L(x_0, g \times h) = D_1$ and $N \cap T^2_+ = N_1$. Pictorially, we want $N \cap \tilde{F}(x)$ to be empty or as in Figure 4 for $x \in \tilde{\Sigma}$, where $a_1 = N \cap \tilde{F}(x) \cap N_1 \subset T^+_2$ and $a_2 = N \cap \tilde{F}(x) \cap D_1 \subset W^u_L(x, g \times h)$.



$\tilde{F}(x)$, **Figure 4**

One now can construct a C^∞ diffeomorphism h of T^3, a "bump function" whose main purpose is to force $W^u_L(x_0, g \times h)$ to intersect T^2_+ transversally. h need have the following properties:

(a) h = identity outside N,

(b) $h(D_1)$ intersects T^2_+ transversally (in N_1, of course),

(c) $h(\tilde{F}(x)) \subset \tilde{F}(x)$ for all $x \in B$, i.e. h preserves the foliation $\mathfrak{F}$,

(d) $h[\{x_0\} \times N_3]$ intersects $W^s_L(x_0, g \times h)$ in two points,

(e) the largest increase of arc length under h occurs at $\{x_0\} \times N_3$ where length $h[\{x_0\} \times N_3]$/length of $\{x_0\} \times N_3 = P$,

(f) for all $x \in \tilde{\Sigma}$, $\tilde{F}(x)$ intersects $h(D_1)$ transversally in $N_1 \times N_2$.

Pictorially, h sends points from left to right in $N \cap \tilde{F}(x)$ in Figure 4; and for $x = x_0$, $h(a_2)$ intersects a_1 in two points. Finally, choose k at the end of §B so that $\mu_1^k > [4(1 + P)]^r$ and again denote g^k by g. N_0, N_1, and B_1 will still have the desired properties for our new D-A g.

E. Stable and unstable manifolds for $h \circ (g \times h)$. Let $f = h \circ (g \times h)$. f is a C^∞ diffeomorphism of T^3, and f respects the foliation $\mathfrak{F} = \{W^s(x, A) \times S^1\}$ since h preserves $\mathfrak{F}$. To obtain U, the open set of diffeomorphisms in the statement of our theorem, we will construct a ball about f in $\mathrm{Diff}^r(T^3)$.

Since a study of the orbit structure of maps near f is parallel to such a study of f, we will try to understand the stable and unstable manifolds for f in this section. First, note that since we did not alter $g \times h$ near $\Omega(g \times h)$ and periodic points are dense in $\Omega(g \times h)$, $\Omega(g \times h) \subset \Omega(f)$ with the same hyperbolicity constants there for f as for $g \times h$.

We will make frequent use of the following simple lemma:

Lemma 1. *Let f, f_1 be diffeomorphisms of compact manifold M. Let Σ be a hyperbolic compact invariant subset in $\Omega(f)$ with periodic points of Σ dense in Σ. Let N be a subset of $M \ni f_1 = f$ outside N and $N \cap \Sigma = \emptyset$.*

(a) *For $x \in \Sigma$, let $W^s_{\mathrm{loc}}(x, f)$ be a subset of $W^s(x, f)$. If $f^n W^s_{\mathrm{loc}}(x, f) \cap N = \emptyset$ for all $n \geq 0$, then $x \in \Omega(f_1)$ and $W^s_{\mathrm{loc}}(x, f) \subset W^s(x, f_1)$.*

(b) *If $f_1 = b \circ f$ where* supp *$b \subset N$ and $f^n W^s_{\mathrm{loc}}(x, f) \cap N = \emptyset$ for all $n \geq 1$, then $x \in \Omega(f_1)$ and $W^s_{\mathrm{loc}}(x, f) \subset W^s(x, f_1)$.*

(c) *Let $W^u_{\mathrm{loc}}(x, f)$ be a subset of $W^u(x, f)$. If $f^{-n} W^u_{\mathrm{loc}}(x, f) \cap N = \emptyset$ for all $n \geq 0$, then $x \in \Omega(f_1)$ and $W^u_{\mathrm{loc}}(x, f) \subset W^u(x, f_1)$.*

(d) *If $f_1 = b \circ f$ where* supp *$b \subset N$ and $f^{-n} W^u_{\mathrm{loc}}(x, f) \cap N = \emptyset$ for all $n \geq 1$, then $x \in \Omega(f_1)$ and $b[W^u_{\mathrm{loc}}(x, f)] \subset W^u(x, f_1)$.*

Proof of Lemma 1. Let $x \in \Sigma$. As in [9] and [21], $W^s(x, f) = \{y \in M: d(f^n x, f^n y) \longrightarrow 0$ as $n \longrightarrow +\infty\}$.

Let $y \in W^s_{\mathrm{loc}}(x, f)$. $x, y \notin N \Longrightarrow fx = f_1 x$ and $fy = f_1 y$. In fact, $f^n x, f^n y \notin N$ for all $n \geq 0 \Longrightarrow f_1^n x = f^n x$ and $f_1^n y = f^n y$ for all $n \geq 0$. So,

$$d(f_1^n y, f_1^n x) = d(f^n y, f^n x) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

x is nonwandering for f_1 since $\Sigma \cap N = \emptyset$ and periodic points are dense in Σ. $y \in W^s_{\mathrm{loc}}(x, f_1)$, proving (a).

If $f_1 = b \circ f$ and $y \in W^s_{\mathrm{loc}}(x, f)$(possibly in N), $fy \notin N$ by hypothesis and therefore $f_1 y = b \circ fy = fy$. Then, argue as in the proof of part (a) to obtain (b). (c) follows, since $W^u(x, f) = W^s(x, f^{-1})$.

(d) $f^{-1} W^u_{\mathrm{loc}}(x, f)$ is a subset of $W^u(f^{-1}x, f)$ and $f^{-n}[f^{-1} W^u_{\mathrm{loc}}(x, f)] \cap N = f^{-(n+1)} W^u_{\mathrm{loc}}(x, f) \cap N = \emptyset$ for $n \geq 0$ by hypothesis. By (c), $f^{-1} W^u_{\mathrm{loc}}(x, f) \subset W^u(f^{-1}x, f_1) = W^u(f_1^{-1}x, f_1)$; therefore,

$$f_1 f^{-1} W^u_{\mathrm{loc}}(x, f) \subset f_1 W^u(f_1^{-1} x, f_1) = W^u(x, f_1).$$

But, $f_1 \circ f^{-1} = b \circ f \circ f^{-1} = b$. This proves Lemma 1.

Let $f = b \circ (g \times h)$ be as defined above. As above, for $x \in \tilde{\Sigma}$, let $W^s_L(x, f)$, the local stable manifold for x, be the path component of $W^s(x, f) \cap B$ that contains x; and let $W^u_L(x, f)$, the local unstable manifold for x, be the path component of $W^u(x, f) \cap B$ that contains x.

Lemma 2. *For $x \in \tilde{\Sigma}$,*

(a) $W^s_L(x, f) = W^s_L(x, g \times h)$,

(b) $W^s_L(\tilde{\Sigma}, f) = W^s_L(\tilde{\Sigma}, g \times h) = B_1$,

(c) $W^u_L(x, f) = W^u_L(x_0, f) = b[W^u_L(x_0, g \times h)]$.

Proof of Lemma 2. We will use Lemma 1, with Σ = solenoid in T^2_+ and N as constructed in §D. $N \cap \Sigma = \emptyset$ and $f = g \times h$ outside N. By (b) in definition of N_1 in §D, $(g \times h)^{-n} N \cap B_1 = \emptyset$ for all $n > 0$. Since $B_1 = W^s_L(\tilde{\Sigma}, g \times h)$, $N \cap (g \times h)^n W^s_L(x, g \times h) = \emptyset$ for $n \geq 1$. (a) and (b) follow now from Lemma 1 and the definition of W^s_L. For (c), recall that $\tilde{\Sigma} \subset W^u(x_0, g \times h)$ and so $W^u_L(x, g \times h) = W^u_L(x_0, g \times h)$ for all $x \in \tilde{\Sigma}$. $W^u_L(x_0, g \times h)$ meets N only in $T^2 \times N_3$. Since $h^{-n} N_3 \cap N_3 = \emptyset$ for all $n \geq 1$ and $W^u_L(x_0, g \times h)$ is invariant under $(g \times h)^{-1}$, $(g \times h)^{-n} W^u_L(x_0, g \times h) \cap N = \emptyset$ for $n \geq 1$. By (d) of Lemma 1 and the definition of W^u_L, $h[W^u_L(x_0, g \times h)] = W^u_L(x_0, f)$.

The local stable and unstable manifolds for f around x_0 are pictured in Figure 5.

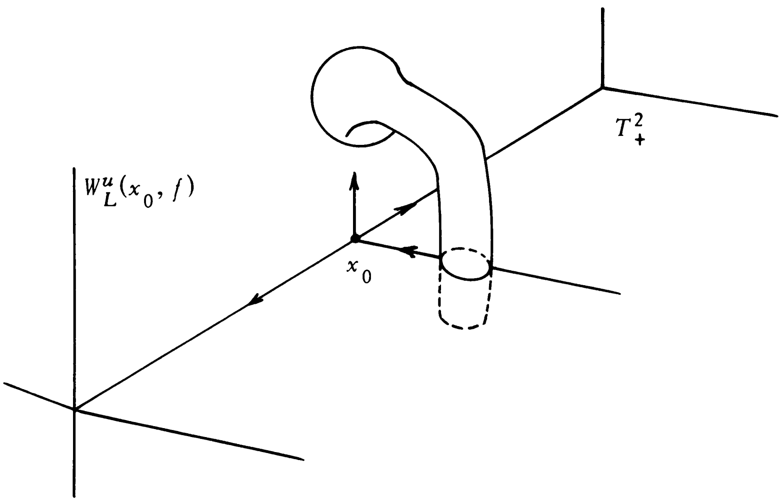


Figure 5

It will be helpful to have some notation for the three-dimensional local unstable manifold of θ. Considering θ first as a source for $g_+: T^2 \to T^2$, let $W^u_{\mathrm{loc}}(\theta, g_+)$ be a 2-disk in its unstable manifold, with $\{v_0\} \times J^u$ in its interior, as in Figure 2. $W^u_{\mathrm{loc}}(\theta, g_+)$ can be constructed so that

(1) interior $N_1 \cap$ interior $W^u_{\mathrm{loc}}(\theta, g_+) = \emptyset$,

(2) boundary $N_1 \cap$ boundary $W^u_{\mathrm{loc}}(\theta, g_+) \neq \emptyset$,

(3) $g_+^{-n} W^u_{\mathrm{loc}}(\theta, g_+) \subset W^u_{\mathrm{loc}}(\theta, g_+)$ and is disjoint from B_1 for all $n > 0$,

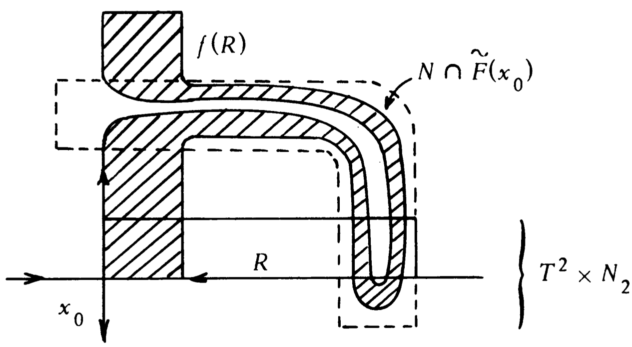
(4) g reduces lengths on stable manifolds outside $W^u_{\mathrm{loc}}(\theta, g_+)$ by at least one-third. (g does so near Σ and away from Q.)

Define $W^u_{\mathrm{loc}}(\theta, g \times h) = W^u_{\mathrm{loc}}(\theta, g_+) \times N_2$ in $T^2 \times S^1$. Since $N \cap (T^2 \times N_2) = N_1 \times N_2$, we can define $W^u_{\mathrm{loc}}(\theta, f) = W^u_{\mathrm{loc}}(\theta, g \times h)$ by Lemma 1 and property (3) above.

Since f respects foliation $\mathcal{F}$, $x \in T^3$ is periodic under f only if leaf $F(x)$ is periodic under f. Since the leaves for F are products of the stable manifolds of Σ and S^1, x must lie on $F(y)$ where y is a periodic point on Σ. Consequently, a good way to study $\Omega(f)$ is by examining f^n restricted to a leaf of period n.

Lemma 3. $f|\Omega(f|\tilde{F}(x_0))$ *is conjugate to the shift automorphism on the bisequence space of* 3 *symbols, i.e.* 3^Z.

Since Lemma 3 is superfluous to the proof of Theorem 1, we merely sketch its proof. One constructs by the methods of §I, a closed rectangle R in $\tilde{F}(x_0)$ such that $f\colon R \to F(x_0)$ looks like the standard geometric realization of the shift on 3^Z, as in [22]. See Figure 6.



$\tilde{F}(x_0)$, **Figure 6**

To show the conjugacy to the shift, one easily applies the methods of [18]. Finally, by using the properties of the subsets constructed in §D, one shows that $\Omega(f|\tilde{F}(x_0)) \subset R$.

Let z be a periodic point in $\Sigma \cap N_0$ of least period k. Suppose $f^i z \in \Sigma \cap N_0$ only for $i = 0, i_1, \cdots, i_s$ if $i < k$. Then, an analysis like that of Lemma 3 will show that $f^k|\Omega(f^k|\tilde{F}(z))$ is conjugate to the shift map acting on a quotient space of $(3^s)^Z$.

F. **Normally-hyperbolic foliations.**

Definition. Let f be a diffeomorphism of compact C^∞-manifold M^n that respects a foliation $\mathfrak{F}$ on M. We call f *r-normally-hyperbolic* (with respect to $\mathfrak{F}$) if $\exists$ a continuous splitting $TM = E_+ \oplus E_- \oplus T\mathfrak{F}$ invariant under Tf such that the following conditions hold: for some Riemannian metric on M $\exists$ constants λ, μ with $0 < \lambda < 1 < \mu$ such that if $0 \neq X \in TM$,

$$|TfX| \leq \lambda|X| \quad \text{if } X \in E_-,$$

$$|TfX| \geq \mu|X| \quad \text{if } X \in E_+,$$

$$\lambda|X| < |Tf^iX| < \mu|X| \quad \text{for } i = 0, 1, \cdots, r, \quad \text{if } X \in T\mathfrak{F}.$$

Intuitively, this condition means that the contracting (expanding) effect of f normal to the leaves of the foliation is at least r times greater than the contracting (expanding) effect of f on the leaves.

Definition. For foliation $\mathcal{F}$ on M, let $Q(\mathcal{F})$ be the quotient space obtained by identifying leaves of $\mathcal{F}$ to points. If f respects $\mathcal{F}$, $f: Q(\mathcal{F}) \to Q(\mathcal{F})$ is well defined. If g respects foliation $\mathcal{G}$ on M, $(\mathcal{F}, f)$ is *conjugate* to $(\mathcal{G}, g)$ if there is homeomorphism $h: Q(\mathcal{F}) \to Q(\mathcal{G})$ such that the following diagram commutes:

$$\begin{array}{ccc} Q(\mathcal{F}) & \xrightarrow{f} & Q(\mathcal{F}) \\ h \downarrow & & \downarrow h \\ Q(\mathcal{G}) & \xrightarrow{g} & Q(\mathcal{G}) \end{array}$$

Theorem (Hirsch-Pugh-Shub [10]). *Let $1 \leq r < \infty$ and M be a compact C^∞-manifold. Let f be a C^r diffeomorphism of M that is r normally-hyperbolic with respect to some foliation $\mathcal{F}$ where the leaves of $\mathcal{F}$ are C^r-manifolds. Then, there exists an open set U in* $\mathrm{Diff}^r(M)$ *about f such that if $g \in U$, then g respects a foliation $\mathcal{G}$ whose leaves are C^r-manifolds. $(\mathcal{F}, f)$ is conjugate to $(\mathcal{G}, g)$.*

Remark 1. As constructed in §B, the D-A map g is r normally-hyperbolic with respect to the foliation $\mathcal{F} = \{W^s(x, A): x \in T^2\}$. In fact, one can construct an invariant foliation $\mathcal{G}$, everywhere transverse to $\mathcal{F}$ and containing the path components of Σ as leaves. g is expanding on leaves of $\mathcal{G}$, but contracting on leaves of $\mathcal{F}$ except near θ where by proper choice of ϕ the expansion can be made arbitrarily slow compared to the expansion along leaves of $\mathcal{G}$. Take $E^+(x)$ to be the tangent space to the leaf of $\mathcal{G}$ through x and $E^-(x)$ to be empty. [$\mathcal{G}$ is tangent to a "Denjoy vector field" on T^2.]

Remark 2. $g \times h: T^3 \to T^3$ is r normally-hyperbolic with respect to $\mathcal{F} = \{W^s(x, A) \times S^1\}$. To see this, one constructs a one-dimensional invariant foliation $\mathcal{G}^+$ on T^3, expanding under f and everywhere normal to $\mathcal{F}$, by putting the foliation $\mathcal{G}$ of Remark 1 on each $T^2 \times \{s\}$ for all $s \in S^1$.

Remark 3. $f = b \circ (g \times h)$ is r normally-hyperbolic with respect to $\mathcal{F} = \{W^s(x, A) \times S^1\}$. It is not as simple a task to construct the invariant subbundle E^+ for $b \circ (g \times h)$ as it was for g and $g \times h$. However, b takes each leaf of $\mathcal{F}$ into itself and expands lengths by a factor $\leq P$ (as defined in §D) while expansion normal to leaves under $b \circ (g \times h)$ remains greater than $[4(P + 1)]^r$. The stability of foliation $\mathcal{F}$ follows then from the methods of §2 of an expanded version of [10] where Hirsch, Pugh, and Shub characterize normal hyperbolicity by comparing the spectrum of $f_\#$ restricted to $T\mathcal{F}$ (where $f_\#(v) = Tf \circ v \circ f^{-1}$ for sections v of TM) to the spectrum of $f_\#$ restricted to the formal normal bundle of $T\mathcal{F}$. Furthermore, in [6, esp. §VI], Fenichel proves a similar perturbation theorem using only the asymptotic behavior of such a map f without assuming any invariant splitting of TM.

G. **The open set U in Theorem 1.**

1. If the r in Theorem 1 is finite, the last section indicated how g can be chosen so that $f = b \circ (g \times h)$ is r normally-hyperbolic with respect to $\mathfrak{F}$. If $r = \infty$, choose g so that f is at least 2 normally-hyperbolic. Then, let U in each case be as in the conclusion of the Hirsch-Pugh-Shub Theorem.

2. Part of the Ω-stability theorem [25] states that if Λ is a hyperbolic basic set (as defined in §B) for f, then each g close enough to f has an invariant basic set Λ' that is near to and conjugate to Λ. So, we can choose U so that for $f' \in U$, there is a one-dimensional set Σ' with $f|\Sigma$ conjugate to $f'|\Sigma'$. For all $f' \in U$, let x_0 denote the fixed point corresponding to the fixed point x_0 for f. Let $\tilde{F}'(x) = F'(x) \cap B$ for $x \in \Sigma' \cap B$ where $F' \in \mathfrak{F}'$, the foliation of f'. For $f'C^r$ close to f, $\tilde{F}'(x)$ is C^r close to $\tilde{F}(x)$, where again for notation's sake, we are assuming the conjugacy between $f|\Sigma$ and $f'|\Sigma'$ is the identity.

Let $W_L^{s,u}(x, f')$ for $x \in \Sigma'$ be the path component of $W^{s,u}(x, f') \cap B$ containing x. By the Hirsch-Pugh Stable Manifold Theorem [9], for $f'C^r$ near f, $W_L^s(x, f')$ is C^r near $W_L^s(x, f)$ and $W_L^u(x, f')$ is C^r near $W_L^u(x, f)$.

3. $W_L^u(x_0, f)$ intersects $W_L^s(x_0, f)$ transversally in two points in $N_1 \times N_2$. Choose U so that this is true for all $f' \in U$. In particular, we can demand that, for $f' \in U$, $f'|\Omega(f'|\tilde{F}'(x_0))$ is conjugate to the standard 3-shift since this open condition ([18], [22]) is true for f.

4. If f' is C^r near $f = b \circ (g \times h)$, $f' = b' \circ (g \times h)$ where b' is C^r near b. Choose U so that, for $f' \in U$, $T^2 \times \{+1\}$ intersects $W_L^u(x_0, f')$ transversally in $N_1 \times N_2$.

5. Let $N_4 \subset N_2 \subset S^1$ be as in §D. Using [9], choose U so that, for all $f' \in U$, $W_L^s(x, f') \subset B_1 \times N_4$ for all x in interior $\tilde{\Sigma}'$.

6. By (b) of §D, $f(N_1) \cap N_1 = \emptyset$. Choose U so that this holds for all f' in U.

7. $W_L^u(x_0, f)$ is transverse to the boundary of B. Choose U so that this is true for all $f' \in U$. In particular, $\tilde{\Sigma}'$ will be an interval for all f'.

8. Choose U so that $W_L^u(x_0, f') \cap (N_1 \times N_2) \subset [\text{interior of } N_1] \times N_2$ for all $f' \in U$.

9. Using stable manifold theory again, choose U so that $f'^{-1}W_L^u(x_0, f') \cap N = \emptyset$ for all $f' \in U$.

10. Using (4) in construction of N, choose U so that $B \cap f'^{-1}N \subset B_1 \times h^{-1}(B_2 - N_5)$.

11. Since b = identity on $gB_1 \times c$ (cf. (5) in construction of N) and $c \notin N_2$, $b[gB_1 \times \{c\}] \cap T^2 \times N_2 = \emptyset$. Choose U so that for b' as in (4) above, $b'[gB_1 \times \{c\}] \cap T^2 \times N_2 = \emptyset$, i.e. $f'[B_1 \times h^{-1}c] \cap T^2 \times N_2 = \emptyset$.

12. B is the union of 2-disks $\tilde{F}(x)$ for $x \in \tilde{\Sigma}$. For $f' \in U_1$ demand that either $\tilde{F}(x)$ is a 2-disk whose "interior" lies in B or $f'(\tilde{F}(x)) \cap B = \emptyset$.

13. Consider $W^u_{\mathrm{loc}}(\theta, f)$ described in §D. Choose U so that for $f' \in U$, $W^u_{\mathrm{loc}}(\theta, f) \subset W^u(\theta, f')$, and $f^{-1}\overline{W^u_{\mathrm{loc}}(\theta, f)}$ lies in the interior of $W^u_{\mathrm{loc}}(\theta, f)$ and in the complement of B.

14. Since $h(N_2) \supset B_2$, one can choose U so that $f'(T^2 \times N_2) \supset T^2 \times B_2$ and $f'[\overline{\mathcal{C}(T^2 \times B_2)}] \subset \overline{\mathcal{C}(T^2 \times B_2)}$.

15. Let $\partial N_2 = \{a_1, a_2\} \subset S^1$ with $a_1 \in S_+$. In $[T^2 \times N_2] - f^{-1}N$, f increases distances normal to $T^2 \times \{a_1, a_2\}$ by a factor greater than 3 by construction of h. Choose U so that this holds for all $f' \in U$.

16. For $x \in T^2 \times \{a_1, a_2\}$, let $K(x)$ be distance measured along $F(x)$ from x to $T^2 \times \{a_1\}$ if $x \in T^2 \times \{a_2\}$ or to $T^2 \times \{a_2\}$ if $x \in T^2 \times \{a_1\}$. For $f = b \circ (g \times h)$, $K(x) =$ length of N_2, for all $x \in T^2 \times \partial N_2$. Choose U so that for all $f' \in U$ and all x as above, $K(x) < 3 \times$ length of $N_2 \equiv K$.

17. If f' is C^r near $b \circ (g \times h)$, $f' = b \circ k$, where k is C^r near $(g \times h)$. $g \times h$ satisfies Axiom A and strong transversality condition. Therefore, by [17], it is structurally stable. Choose $U \ni$ if $f' \in U$, $f' = b \circ k$ where k is topologically conjugate to $g \times h$.

H. Perturbing maps in U.

Notation. If f_t is in U, let $\mathcal{F}_t$ be the foliation on T^3 as in (1) in §G; let Σ_t or $\Sigma(f_t)$ denote the important solenoid as in (2) in §G; let $\tilde{\Sigma}_t$ be the path component of $\Sigma_t \cap B$ containing x_0; $\tilde{\Sigma}_t$ is an interval by (7) in §G. Let $W^s_L(x, f_t)$, $W^u_L(x, f_t)$ and $\tilde{F}_t(x)$ be as defined in (2) of §G. In this section, we want to prove

Lemma 4. *Given $f_0 \in U$, there is a point $z \in \tilde{\Sigma}(f_0)$ and a one-parameter family of maps in U, $\{f_t\}$, $0 \le t \le 1$, such that the following hold*:

(1) $\Sigma(f_t) = \Sigma(f_0)$ *for all* $t \in [0, 1]$.

(2) $W^s_L(z, f_0)$ *and* $W^u_L(z, f_0)$ *have linking number* 0 *in* $N_1 \times N_2$; *in fact, they intersect but* W^s_L *lies on one side of* W^u_L.

(3) $W^s_L(z, f_1)$ *and* $W^u_L(z, f_1)$ *have linking number at least* 2 *in* $N_1 \times N_2$.

Figures 7 and 8 describe the difference between (2) and (3).

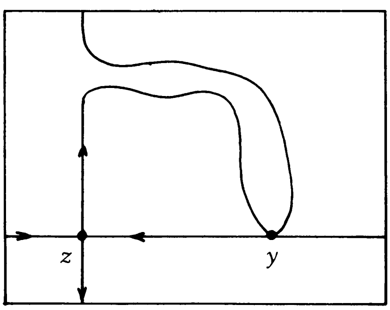


Figure 7. $\tilde{F}_0(z)$

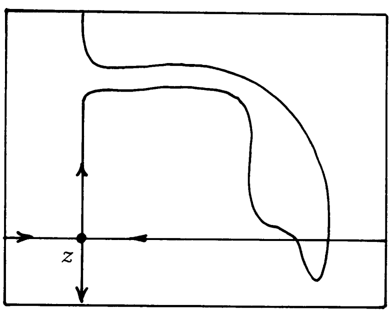


Figure 8. $\tilde{F}_1(z)$

Let f_0 be an arbitrary map in $\mathrm{Diff}^r(T^3)$. Let $\Lambda_0 = \{x \in \tilde{\Sigma}_0 : W_L^s(x, f_0) \cap W_L^u(x_0, f_0) \neq \emptyset$ in $N_1 \times N_2\}$. Λ_0 is a nonempty proper closed subset of $\tilde{\Sigma}_0$ by (3) and (8) of §G, since $W_L^s(\tilde{\Sigma}_0)$ is a two-dimensional topological disk, a result of the stable manifold theorem. Ordering the points in the interval $\tilde{\Sigma}_0$ naturally, there is a unique z in Λ_0 such that if $z' > z$ in $\tilde{\Sigma}_0$, then $z' \notin \Lambda_0$. So,

$$W_L^u(z, f_0) \cap W_L^s(z, f_0) \neq \emptyset \quad \text{in } N_1 \times N_2$$

$$W_L^u(z', f_0) \cap W_L^s(z', f_0) = \emptyset \quad \text{in } N_1 \times N_2 \text{ for } z' > z.$$

Recall that since $z, z' \in \tilde{\Sigma}_0 \subset W_L^u(x_0, f_0)$, $W_L^u(x_0, f_0) = W_L^u(z, f_0) = W_L^u(z', f_0)$. Since the zero linking number is a closed condition, $W_L^u(x_0, f_0)$ and $W_L^s(z, f_0)$ have linking number zero in $N_1 \times N_2$. However, they do intersect there, as in Figure 7.

We now construct our one-parameter family of maps. Let $y \in W_L^s(z, f_0) \cap W_L^u(x_0, f_0) \cap N_1 \times N_2$, as in Figure 7. Choose y to be the furthest such point on $W_L^s(z, f_0)$ from z. $T_y W_L^s(z, f_0) \subset T_y W_L^u(x_0, f)$. Choose nonzero vector $X(y)$ normal to $T_y W_L^u(x_0, f_0)$, tangent to $\tilde{F}_0(z)$, and pointing in the S_--direction, i.e. away from $W_L^u(x_0, f_0)$. Extend $X(y)$ to a C^∞ constant vector field on T^3. Now select an open set V in $N_1 \times N_2$ around y with $f_0(V) \cap V = \emptyset$ and with $W_L^u(x_0, f)$ dividing V into two parts. Let $k\colon T^3 \to \mathbf{R}$ be a C^∞ Urysohn function that is 1 near y but 0 outside V; and consider vector field $Y(x) = k(x)X(x)$ for $x \in T^3$ with flow α_t. Defining $f_t = \alpha_t \circ f_0$, let $t_1 > 0$ be such that, for all $t \in [0, t_1]$, f_t is in the open set U in $\mathrm{Diff}^r(T^3)$. By Lemma 1, $\tilde{\Sigma}_t = \tilde{\Sigma}_0$, $W_L^s(x, f_t) = W_L^s(x, f_0)$ for all $x \in \tilde{\Sigma}_0$, and $W_L^u(x_0, f_t) = f_t f_0^{-1} W_L^u(x_0, f_0) = \alpha_t W_L^u(x_0, f_0)$.

All one need show now is that, for $t > 0$, $W_L^s(z, f_t)$ and $W_L^u(x_0, f_t)$ have linking number greater than zero in $N_1 \times N_2$. $W_L^u(x_0, f_0)$ divides V into two parts, with $W_L^s(z, f_0) \cap V$ lying in the lower (S_-) part. Since $W_L^s(z, f_0)$ is tangent to $W_L^u(x_0, f_0)$ at y and α_t pushes $W_L^u(x_0, f_0)$ in the normal direction, there is t_2 with $0 < t_2 \leq t_1$ so that, for $t \in (0, t_2]$, some of $W_L^s(z, f_0)$ lies above $\alpha_t W_L^u(x_0, f_0) \cap V$ and some lies below. Thus, the linking number of $W_L^s(z, f_0)$ and $\alpha_t W_L^u(x_0, f_0)$ is greater than 0 in V for $t \in (0, t_2]$. Now reparameterize $[0, t_2]$ to $[0, 1]$. Since $W_L^s(z, f_0) = W_L^s(z, f_t)$ and $\alpha_t W_L^u(x_0, f_0) = W_L^u(x_0, f_t)$, the proof of Lemma 4 is complete.

If $r < \infty$ in the statement of Theorem 1, the $F_0(x)$ are C^r manifolds by (1) in §G and X can be chosen everywhere tangent to F_0 (e.g., using foliation charts of [8]). In this case, $F_t = F_0$ and one merely pulls $W_L^u(x_0, f_0) \cap \tilde{F}_0(z)$ down along $\tilde{F}_0(z)$ to proceed from the situation of Figure 7 to that of Figure 8.

Lemma 5. *If $f_0 \in U$, f_0 does not satisfy Smale's Axiom* A, *i.e. f_0 has a nonhyperbolic nonwandering point.*

Proof. The point y of Lemma 4 is nonhyperbolic yet nonwandering.

$y \in \Omega(f_0)$: $y \in W^s(z, f_0) \cap W^u(x_0, f_0)$. But $W^u(z, f_0)$ and $W^s(x_0, f_0)$ intersect transversally since $W^u(z, f_0) = W^u(x_0, f_0)$ and $x_0 \in W^u(x_0, f_0) \pitchfork W^s(x_0, f_0)$. By "Cloud Lemma," [24, (7.2)] or [2], $y \in \Omega(f_0)$.

y not hyperbolic: $y \in W^s(z, f_0)$ *and* $y \in W^u(x_0, f_0)$. If y were hyperbolic, $W^s(y, f_0) = W^s(z, f_0)$, $W^u(y, f_0) = W^u(x_0, f_0)$ and $y \in W^u(y, f_0) \pitchfork W^s(y, f_0)$. But $W^s(z, f_0)$ and $W^u(x_0, f_0)$ do not meet transversally at y. Q.E.D.

1. **Construction of special 2-disks in the** $F(x)$**'s.** In this section, f will denote an arbitrary element of U, not necessarily $b \circ (g \times h)$ as in previous sections. For each $f \in U$ and periodic point x in $\Sigma(f) \cap [N_0 \times N_2]$, we construct a "rectangular" 2-disk $R(x) \subset \tilde{F}(x)$, which will have roughly the same purpose as the R in Figure 6. If $x \in \Sigma(f) \cap [N_0 \times N_2]$ and x' is the corresponding point in $\Sigma(f')$, $R(x, f)$ will be C^0 close to $R(x', f')$.

Lemma 6. *Let $f \in U$ and let s be a path in $\tilde{F}(x, f) \cap T^2 \times N_2$ for $x \in \Sigma \cap [N_0 \times N_2]$. Suppose $s \cap \Sigma = \emptyset$. If $f^j s \subset T^2 \times N_2$ for $0 \le j \le k$, then $f^j s \cap N = \emptyset$ for $1 \le j \le k$. If also $f^m s \cap W^u_L(x_0, f) = \emptyset$, $f^n s \cap W^u_L(x_0, f) = \emptyset$ for all $n \ge m$.*

Proof. Last sentence follows from $f^{-1}W^u_L(x_0, f) \subset W^u_L(x_0, f)$. The geometric reason for $fs \cap N = \emptyset$ is that f sends points in $T^2 \times N_2$ closer to $W^u_L(x_0, f)$ and away from N. To send s back to N, f would have to map some of s out of $T^2 \times N_2$. Suppose $fs \cap N \ne \emptyset$. Since $s \subset B$, $s \cap [f^{-1}N \cap B] \ne \emptyset$. By (10) of §G, s contains a path from Σ to $B_1 \times b^{-1}(B_2 - N_5)$ and so must intersect $B_1 \times \{b^{-1}c\}$ where c is in Figure 3. So, $fs \cap f[B_1 \times b^{-1}c] \ne \emptyset$. By (11) of §G, fs has a point outside $T^2 \times N_2$. This contradicts the hypothesis and shows $fs \cap N = \emptyset$.

For $j = 2$, argue as for $j = 1$ if $fs \subset B$. Otherwise, $f^2 s \cap B = \emptyset$ by (12) in §G; and $N \subset B$. Let i be the first integer > 2 with $f^i s \cap B \ne \emptyset$ but $f^{i-1} s \cap B = \emptyset$. If $f^i s$ met N, it would have to do so in $N_1 \times N_2$ since $N \cap T^2 \times N_2 = N_1 \times N_2$. Then, $f^i s$ would join $N_1 \times N_2$ to Σ by a curve in $F(f^i s) \cap T^2 \times N_2$. In $\tilde{F}(f^i s) \cap T^2 \times N_2$, $W^u_L(x_0, f)$ separates Σ from $N_1 \times N_2$. But $f^i s \cap W^u_L(x_0, f) = \emptyset$ since $W^u_L(x_0, f)$ lies in B and is invariant under f^{-1}. So, $f^i s$ must leave $\tilde{F}(f^i s)$ and intersect $W^u_{\mathrm{loc}}(\theta, f)$. But then, $s \cap f^{-i} W^u_{\mathrm{loc}}(\theta, f) \ne \emptyset$. Since s lies in B, this contradicts (13) of §G. So, $f^i s \cap N = \emptyset$. An inductive argument then finishes the proof of this lemma.

Now let q be a periodic point for f, say of (least) period m, in $\Sigma(f) \cap N_0 \times N_2$ with $W^s_L(q, f) \cap N_1 \times N_2 \ne \emptyset$. We are going to construct rectangle $R(q)$ in $\tilde{F}(q)$. Let s_1 be a closed interval in $W^s_L(q, f)$ with endpoints q and w_1 that is maximal in that $s_1 \subset s_1' \subset W^s_L(q, f)$ and $s_1 \ne s_1'$ implies $\partial s_1' \cap N = \emptyset$. So w_1 is the point on $\partial[N \cap \tilde{F}(q)] \cap W^s_L(q)$ furthest from q. Let s_2 be the path along $\partial[N \cap \tilde{F}(q)]$ from w_1 to w_2 where $w_2 \in T^2 \times \{a_1\}$ (cf. (15) of §G). Let s_3 be the path in $T^2 \times \{a_1\} \cap \tilde{F}(q)$ from w_2 to w_3, a point on $f^{-1}W^u_L(q, f) \cap \tilde{F}(q)$. Let s_4 be the path in $f^{-1}W^u_L(q, f)$

$\cap \tilde{F}(q)$ from w_3 to q. $s_1 \cup s_2 \cup s_3 \cup s_4$ encloses a rectangle $R_0(q) \subset \tilde{F}(q)$. [$R$ in Figure 6 is $R_0(x_0)$.]

Let $R_1(q)$ be the component of $fR_0 \cap T^2 \times N_2$ containing fs_1. [In Figure 6, $R_1(x_0)$ is $R \cap fR$.] Define inductively $R_j(q) \equiv$ component of $fR_{j-1}(q) \cap T^2 \times N_2$ containing $f^j s_1$. By Lemma 6, $R_j(q) \cap N = \varnothing$ for $1 \leq j < m$; and so $R_j(q) =$ component of $f^j R_0(q) \cap T^2 \times N_2$ containing $f^j s_1$. Finally, recalling $f^m q = q$, define $R_m(q) = f(R_{m-1}(q)) \subset \tilde{F}(q)$ and denote $f^{-m}R_m(q) = f^{-(m-1)}R_{m-1}(q)$ as $R(q)$ or $R(q, f)$. See Figure 9.

Note that for $0 \leq i \leq m$ each $R_i(q)$ is a "rectangle" with one side, viz. $f^i s_1$, lying in $f^i W^s_L(q, f)$; and each $f^{-i}R_i(q)$ is a rectangle in $R_0(q)$ with s_1 as one of its sides. For notation's sake, label the sides of $R_i(q)$ as $s_{i1}, s_{i2}, s_{i3}, s_{i4}$ and the sides of $R(q)$ as s_1', s_2', s_3', s_4' where s_{ij} and s_j' correspond to s_j in $R_0(q)$, $j = 1, 2, 3, 4$. For each $i < m$ and each $R_i(q)$, call the maximum distance measured along $F(f^i q)$ from x in $s_{i1} = f^i s_1$ to $s_{i3} \subset T^2 \times \{a_1\}$ the *height* of $R_i(q)$. For each $i \leq m$ and each $f^{-i}R_i(q)$, call the maximum distance measured along $F(q)$ from x in s_1 to the opposite side of $f^{-i}R_i(q)$, viz. $f^{-i}s_{i3}$, the *height* of $f^{-i}R_i(q)$. By (16) of §G, the height of each $R_i(q) < K$. By (15) of §G, height of $f^{-i}R_i(q) < K/3^i$ for $0 \leq i < m$ and so height of $R(q) < K/3^{m-1}$.

We now describe the sides of $R_m(q)$, $s_{m,i} = fs_{m-1,i}$. $s_{m-1,3} \subset T^2 \times \{a_1\}$ and (14) of §G imply that $s_{m,3}$ lies above $T^2 \times B_2$, i.e. above $F(q)$, as in Figure 9. Since $s_{m-1,4}$ is the path component of

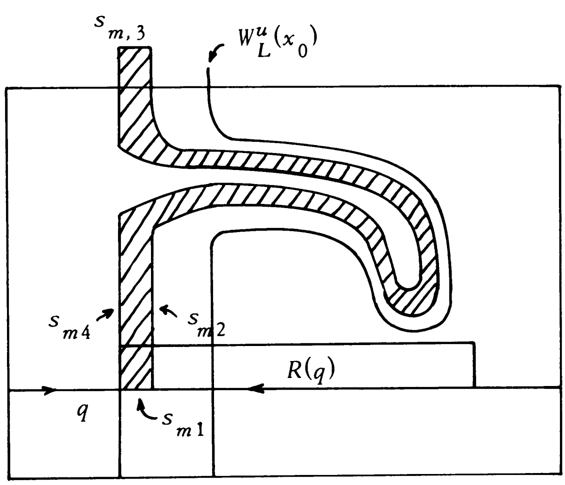


Figure 9. $R_m(q)$(shaded area) in $\tilde{F}(q)$.

$W^u(f^{m-1}q) \cap F(f^{m-1}q) \cap T^2 \times N_2$ containing $f^{m-1}q$ (by induction), s_{m4} is that part of $W^u(q, f) \cap F(q)$ between q and $s_{m,3}$. $s_{m,1} \subset s_1 \subset W^s_L(q, f)$ by construction and since f preserves orientation. By the second part of Lemma 6, $s_{m,1}$ and $s_{m,2}$

are disjoint from $W^u_L(x_0, f)$ as in Figure 9. Putting all this together, one obtains

Lemma 7. *Let q be a periodic point for $f \in U$, of (least) period m, in $\Sigma(f) \cap [N_0 \times N_2]$ such that $W^s_L(q, f)$ meets P, the boundary of $N_1 \times N_2$ nearest θ. Then, there is a "rectangle" $R(q)$ in $\tilde{F}(q) \cap T^2 \times N_2$ with boundary $s_1' \cup s_2' \cup s_3' \cup s_4'$ where*

s_1' is the arc of $W^s_L(q, f)$ from q to P,

$s_2' \subset P \cap \tilde{F}(q)$,

$s_4' \subset W^u_L(q, f) \cap \tilde{F}(q)$, and

s_3' joins s_2' to s_4' and is opposite s_1'.

Height of $R(q) < K/3^{m-1}$. Let $R_m(q) = f^m R(q)$ with sides $s_{mi} = f^m s_i'$, $i = 1, \cdots, 4$.

$s_{m1} \subset s_1'$; s_{m3} lies above $\tilde{F}(q)$ in $F(q)$;

$s_{m,4} \subset W^u(q, f) \cap F(q)$ and joins s_{m1} to s_{m3};

$s_{m,2}$ lies strictly between $W^u_L(q, f) \cap \tilde{F}(q)$ and $W^u_L(x_0, f) \cap \tilde{F}(q)$, as in Figure 9.

$R(q)$ varies continuously with $f \in U$. If f_t is a one-parameter family of maps in U which agree outside N and respect the same foliation, then one $R(q)$ works for all the f_t's.

Now $R(q)$ contains at least one point period $\leq m$, viz. q. In Lemma 8, one constructs another 2-disk $R^\#(q)$ about $R(q)$ in $F(q)$ such that f has no points of period $\leq m$ in $R^\#(q) - R(q)$. For Lemma 9, one thickens $R^\#(q)$ to a 3-disk $V(q)$ such that f has no points of period $\leq m$ in $V(q)\backslash R(q)$.

Lemma 8. *Let q, m, f be as in Lemma 7 with $R(q) \subset \tilde{F}(q)$ as constructed in Lemma 7. Then, there is another 2-disk $R^\#(q) \subset \tilde{F}(q)$ such that*

(i) *$R^\#(q)$ contains $R(q)$ in its interior as a 2-disk, and*

(ii) *f has no points of period $\leq m$ in $R^\#(q)\backslash R(q)$.*

$R^\#(q)$ varies continuously for f in U. If f' near f respects the same foliation and equals f off N, then $R^\#(q, f) = R^\#(q, f')$.

Proof. The proof is simple but a little tedious. So, we will sketch it geometrically, using Figure 9. Let s_1', s_2', s_3', s_4' be the edges of $R(q)$ as in Lemma 7. There are no points of period $\leq m$ in $\tilde{F}(q)$ below $W^s_L(q, f)$. To see this, write f as $b \circ k$ where k is topologically conjugate to $g \times b$ as in (17) of §G. k has two invariant tori, $T^2_+(k)$ and $T^2_-(k)$, with $\Sigma(f)$ and the $W^s_L(x, f)$ contained in $T^2_+(k)$ by Lemma 1. k and b send all points "below" $T^2_+(k)$ toward $T^2_-(k)$. Thus, there are no nonwandering points "below" $T^2_+(k)$, and hence below $W^s_L(q, f)$ for $f = b \circ k$.

There are no nonwandering points to the right of s_2' in $\tilde{F}(q)$ since by its construction in §E and by (13) of §G such points are in the three-dimensional $W^u_{\mathrm{loc}}(\theta, f)$. There are no points of period $\leq m$ to the left of s_4' in $\tilde{F}(q)$. One way to see this is to extend $R_0(q)$ to a rectangle $\underline{R}_0(q)$ with boundary $\underline{s}_1, \underline{s}_2, \underline{s}_3, \underline{s}_4$ where

$$s_1 \subset \underline{s}_1 \subset W_L^s(q, f), \quad s_2 = \underline{s}_2, \quad s_3 \subset \underline{s}_3 \subset T^2 \times \{a_1\}, \quad \underline{s}_4 \subset \text{left boundary of } \tilde{F}(q).$$

Define $\underline{R}_j(q)$ inductively as above and let $\underline{R}(q) = f^{-m}\underline{R}_m(q)$, an extension of $R(q)$ to the left. If $x \in \underline{R}(q)\setminus R(q)$, $f^i(x) \in (T^2 \times N_2)\setminus N$ for $i = 0, \cdots, m-1$, and so $f^i(x) = k^i(x)$ where k is conjugate to $g \times h$. x cannot have period $\leq m$ for f since q is the only point of period $\leq m$ for k in $\tilde{F}(q)$.

Finally, we need to see that we can extend $R(q)$ beyond s_3'. $f^m s_3' \cap T^2 \times B_2 = \emptyset$ by Lemma 7 and, by (14) and (17) of §G, $\Omega(f) \cap \mathcal{C}(T^2 \times B_2) \subset T^2_-(k)$. So $f^m s_3' \cap \Omega(f) = \emptyset$ and there is a 2-disk V_3 about $f^m s_3'$ but missing closed set $\Omega(f)$. $f^{-m}V_3$ is disjoint from $\Omega(f)$ and extends $R(q)$ above s_3'. This finishes our sketch of the construction of $R^\#(q)$.

We want to thicken $R^\#(q)$ to a 3-disk $V(q)$ such that all points of period $\leq m$ in $V(q)$ actually lie in $R(q)$.

Lemma 9. *Let q, m, f be as in Lemma 7. Let $R(q)$ and $R^\#(q)$ be as constructed in Lemmas 7 and 8. Then, there is a 3-disk $V(q)$ in T^3 such that $R(q) \subset R^\#(q) \subset V(q)$. If $x \in V(q)$ with $f^j x = x$ and $0 < j \leq m$, then $x \in R(q)$. $V(q) \cap \tilde{F}(q) = R^\#(q)$ and $V(q)$ varies continuously with $f \in U$.*

Proof. We first show that points of period j not on $R^\#(q)$ do not accumulate on $R^\#(q)$. Suppose the contrary, i.e. suppose there exists a sequence of points $\{x_n\}$ such that

(i) $x_n \notin R^\#(q)$ for all n,

(ii) $f^j x_n = x_n$ for all n, with $0 < j \leq m$, and

(iii) the sequence $\{x_n\}$ accumulates on $R^\#(q)$.

By compactness and since $\mathrm{Fix}(f^j)$ is a closed set, there is a point $\bar{x} \in R^\#(q) \ni x_n \to \bar{x}$, where $\{x_n\}$ is now a subsequence of the original sequence and $f^j(\bar{x}) = \bar{x}$. Therefore, $j = m$. Otherwise, $f^j F(q) = F(q)$ and $f^j W^s(q, f)$. But $W^s(q, f) \cap f^j(W^s(q, f)) = \emptyset$ for $0 < j < m$.

Choose chart $\mathbf{R}^3$ about $F(q)$ where $\mathbf{R}^2 \times \{0\}$ contains $\tilde{F}(q)$ and $\mathbf{R}^2 \times \{t\} \subset$ leaf of foliation. Let $\pi_3\colon \mathbf{R}^3 \to \underline{0} \times \mathbf{R}^1$ be the projection on the third factor. Using [10], we can choose our chart so that for f' near f:

(i) new chart $\mathbf{R}^2 \times \mathbf{R}^1$ is close to the original one,

(ii) $\mathbf{R}^2 \times \{t\} \subset$ leaf of foliation for f',

(iii) $\mathbf{R}^2 \times \{0\} \supset \tilde{F}(q, f')$,

(iv) π_3 for f' is C^2-close to π_3 for f.

Now, $\pi_3 \circ (f^m - \mathrm{id})\colon \mathbf{R}^3 \to \mathbf{R}^1$ with $\pi_3 \circ (f^m - \mathrm{id})\underline{x} = 0$. $\partial \pi_3 \circ (f^m - \mathrm{id})/\partial x_3(\underline{x}) \neq 0$ since f is expanding in the x_3 direction, i.e. normal to the foliation. By the implicit function theorem, $[\pi_3 \circ (f^m - \mathrm{id})]^{-1}(\underline{0})$ forms a two-dimensional submanifold through $\bar{x}$ in our chart. $f^m R^{\#}(q) \subset F(q)$. So, $y \in R^{\#}(q)$ implies $y \in \mathbf{R}^2 \times \{0\}$ and $f^m y \in \mathbf{R}^2 \times \{0\}$. $f^m y - y \in \mathbf{R}^2 \times \{0\}$ or $\pi_3 \circ (f^m - \mathrm{id})y = 0$. Therefore, $R^{\#}(q) \subset [\pi_3 \circ (f^m - \mathrm{id})]^{-1}(0)$. Since $f^m x_n = x_n$ for all n, all $x_n \in [\pi_3 \circ (f^m - \mathrm{id})]^{-1}(0)$. But by the submanifold property, the x_n cannot accumulate to $R^{\#}(q)$ without being on $R^{\#}(q)$. So, points of period j not on $R^{\#}(q)$ do not accumulate on $R^{\#}(q)$ and, consequently, there is an open neighborhood $V(q)$ about $R^{\#}(q)$ as in the conclusion of this lemma. As f varies, π_3 and f^m vary smoothly; so $[\pi_3 \circ (f^m - \mathrm{id})]^{-1}(0)$ and $V(q)$ vary continuously with $f \in U$.

J. **Comparison of f_0 and f_1.** As in the statement of Theorem 1, let f_0 be an arbitrary map in U and U_0 an arbitrary neighborhood of f_0 in U. For convenience, we can without loss of generality consider U as our U_0 since every open subset of U_0 has the properties in §G. Let us now use the 2-disk $R(q)$ constructed in §I to study the one-parameter family of maps $\{f_t\}$ discussed in §H. Recall that for all $f \in U$, $\Sigma(f)$ is locally the product of a Cantor set and an interval ([27], [30]). For $z \in \tilde{\Sigma}(f)$, $W^s_L(z, f) \cap \Sigma(f)$ is a Cantor set and so points of $W^s_L(z, f) \cap \Sigma(f)$ accumulate on z.

Let f_t and z be as in Lemma 4. Choose $z' > z$ in $\tilde{\Sigma}(f_1)$ $[\equiv \tilde{\Sigma}(f_0)]$, using the order in §H, such that $W^s_L(z', f_1)$ and $W^u_L(z', f_1)$ $[\equiv W^u_L(x_0, f_1)]$ have nonzero linking number in $N_1 \times N_2$. By choice of z in $\tilde{\Sigma}(f_0)$, $W^s_L(z', f_0)$ and $W^u_L(z', f_0)$ do not intersect in $N_1 \times N_2$. Using the stable manifold theorem [9], the openness of nonzero linking number and of nonempty intersection, and $\Sigma(f_0) = \Sigma(f_1)$, one can choose a neighborhood H of z' in $\Sigma(f_0)$ such that, for all $y \in H$,

(a) $W^s_L(y, f_1) \cap W^u_L(y, f_1)$ intersect in $N_1 \times N_2$ with nonzero linking number,

(b) $W^s_L(y, f_0) \cap W^u_L(x_0, f_0)$ [and consequently $W^s_L(y, f_0) \cap W^u_L(y, f_0)$] is empty in $N_1 \times N_2$.

Let H_1 be a compact nbd of z' satisfying (a) and (b) and homeomorphic to the product of a Cantor set and an interval. Since H_1 is closed, there is an $\epsilon > 0$ such that for $y \in H_1$ the distance (measured along $\tilde{F}(y)$) between $W^u_L(y, f_0)$ and $W^s_L(y, f_0)$ in $N_1 \times N_2$ is at least ϵ, using (b).

Since periodic points are dense in $\Sigma(f)$ and there are finitely many points of each period [27], there are periodic points in H_1 of arbitrarily high period. Choose $q \in H_1$ of (least) period m where $K/3^m < \epsilon$. Construct $R(q, f_0)$ and $R(q, f_1)$ as in Lemma 7. So $f_t^m R(q, f_t) = R_m(q, f_t)$ lies in $\tilde{F}(q)$ and is bounded by $W^s_L(q, f_t)$, $W^u_L(q, f_t) \cap \tilde{F}(q)$, and $W^u_L(x_0, f_t) \cap \tilde{F}(q)$ as in Figure 9. Also, in $N_1 \times N_2$, $f_t^m R(q)$ lies below $W^u_L(q, f_t) \cap \tilde{F}(q)$ and above $W^u_L(x_0, f_t) \cap \tilde{F}(q)$. In the C^r case, $r < \infty$, $R(q, f_0) = R(q, f_1)$ by Lemma 4 and the last sentence of Lemma 7.

However, the height of $R(q) < K/3^m < \epsilon$, while the distance between $W^u_L(x_0, f_0)$ and $W^s_L(q, f_0)$ is at least ϵ in $\tilde{F}(q) \cap N_1 \times N_2$. So we have exactly the situation of Figure 9 with $R(q)$ and $f_0^m(R(q))$ not intersecting in $N_1 \times N_2$. On the other hand, since $W^u_L(q, f_1)$ has nonzero linking number with $W^s_L(q, f_1)$ in $N_1 \times N_2$, Figure 10 would more accurately describe the situation for $R(q, f_1)$.

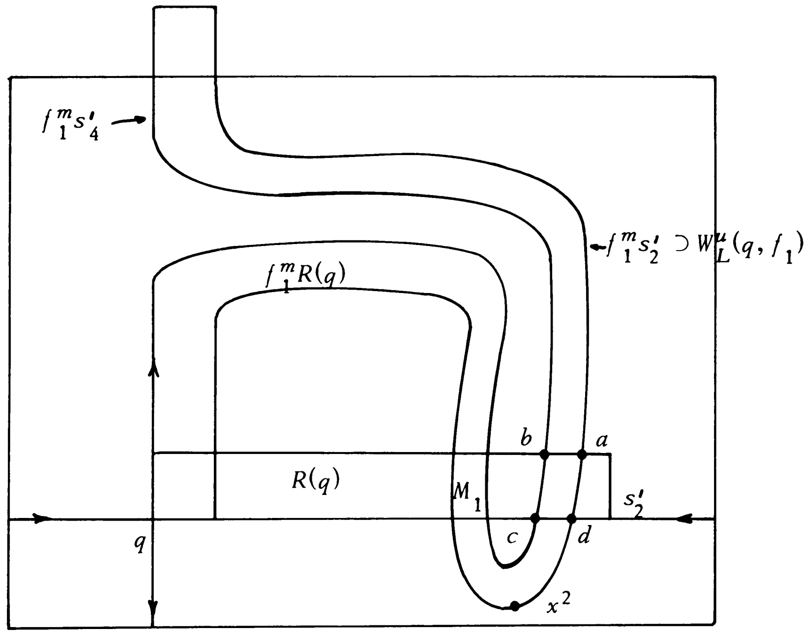


Figure 10. $\tilde{F}(q, f_1)$

Lemma 10. *q is the only point of period $\leq m$ for f_0 in $R(q, f_0)$. However, $f_1^m R(q, f_1) \cap R(q, f_1)$ has at least three components each of which contains a fixed point of f_1^m.*

Proof. Since q has least period m, $f_t^i F(q, f_t) \cap F(q, f_t) = \emptyset$ for $0 < i < m$ and so there are no points of period $< m$ in $R(q, f_t)$. Let $x \in R(q, f_0)$ with $f_0^m x = x$. Since $R \cap f_0^m R \cap N_1 \times N_2 = \emptyset$, $x \in N$. By construction of R, $f_0^j x \notin N$ for $j < m$. Using (17) of §G, $f_0 = b \circ k_0$ where k_0 is conjugate $g \times h$ and support $b \subset N$. So $f_0^j x = k_0^j x$ for $j = 0, 1, \cdots, m$ and $x \in \operatorname{Fix}(k_0^m) \cap \tilde{F}(q, f_0)$. Therefore, $x = q$ and q is the only point of period m in $R(q, f_0)$.

The situation is different for f_1. Let s_1', s_2', s_3', s_4' be the sides of $R(q, f_1)$ as in Lemma 7. As in Figure 10, $f_1^m s_4'$ and $f_1^m s_2'$ cut across $R(q)$ in $N_1 \times N_2$, dip below $R(q)$, and then cross it again. More precisely, there exist closed subintervals I_1^4, I_2^4 of $f_1^m s_4'$ and closed subintervals I_1^2, I_2^2 of $f_1^m s_2'$ such that

(a) there is x^4 between I_1^4 and I_2^4 on $f_1^m s_4'$ lying below $R(q)$,

(b) there is x^2 between I_1^2 and I_2^2 on $f_1^m s_2'$ lying below $R(q)$,

(c) each I_i^j has one endpoint on s_1' and the other on s_3', e.g., the points $\{a, b, c, d\}$ in Figure 10 where $\overline{ad}$ is I_2^2 and $\overline{bc}$ is I_2^4.

Choose I_2^2 and I_2^4 so that the subinterval $\overline{ab}$ of s_3' and the subinterval $\overline{cd}$ of s_1' have minimal length. Similarly, choose I_1^2 and I_1^4. Let $M_1 \subset R \cap f_1^m R$ be the 2-disk bounded by I_1^4 and I_1^2 and let M_2 ($= abcd$ in Figure 10) $\subset R \cap f_1^m R$ be the 2-disk bounded by I_2^4 and I_2^2.

Claim. f_1^m has a fixed point in M_1 and another one in M_2. We will work on M_2; the proof for M_1 is isomorphic, modulo a change in orientation.

To facilitate the analysis of M_2, one introduces a coordinate system on $\tilde{F}(q) \cap T^2 \times N_2$ with $W_L^s(q, f_1)$ the x-axis, q the origin and the positive direction toward $N_1 \times N_2$, i.e. to the right in Figure 10. Let s_4' be the y-axis with positive direction toward s_+, i.e. "up" in Figure 10. Now $f_1^{-m}\overline{ad} \subset s_2'$ and thus lies to the right of M_2 and $f_1^{-m}\,\overline{bc} \subset s_1'$ and lies to the left of M_2. $f_1^{-m}\overline{ab}$ lies in $R(q)$ below s_3' and $f_1^{-m}\overline{cd}$ lies in $R(q)$ above s_1'.

Williams has shown me the following simple technique for exhibiting a fixed point for $f_1^{-m}|M_2$ given the above situation. The set $E_y = \{z \in M_2 : f_1^{-m}z$ and z have the same y-coordinate$\}$ separates M_2 into two disjoint open sets, $\{z : f_1^{-m}$ increases y-coordinate of $z\}$ containing $\overline{cd}$ and $\{z : f_1^{-m}$ decreases y-coordinate of $z\}$ containing $\overline{ab}$. Similarly, $E_x = \{z \in M_2 : f_1^{-m}z$ and z have the same x-coordinate$\}$ separates M_2 into two disjoint open sets, one containing $\overline{bc}$ and the other containing $\overline{ad}$. Since M_2 is closed, $E_x \cap E_y \neq \emptyset$ by point-set topology arguments. But $E_x \cap E_y = \{z \in M_2 : f_1^m z = z\}$, proving this lemma.

Summarizing, we have a one-parameter family of diffeomorphisms in U: $f_t^m|R(q, f_t)$: $R(q, f_t) \longrightarrow R_m(q, f_t)$. $R(q, f_t)$ varies continuously with t and in the C^r case, $r < \infty$, do not vary at all. f_0^m has exactly one fixed point in $R(q, f_0)$, while f_1^m has at least three fixed points in $R(q, f_1)$. The set of f in U that have q as the only fixed point of f^m in $R(q, f)$ is open. So, there is a T with $0 < T < 1$ such that f_T^m has more than one fixed point in $R(q, f_T)$ but f_t^m has q as its only fixed point in $R(q, f_t)$ for all t with $0 \leq t < T$.

K. Three perturbations of f_t in U. In this section $\bigcup$ will mean $\bigcup_{j=0}^{m-1}$. First, one makes hyperbolic all periodic points of f_T of period $\leq m$ not in the orbit of $R(q, f_T)$. From Lemma 9, there is a 3-disk V such that $R \subset$ interior V and all points of V of period $\leq m$ are actually in R. Choose V small enough so that V, $f_T V, \cdots, f_T^{m-1}V$ are mutually disjoint. By Peixoto's proof of the Kupka-Smale Theorem [16], one can choose $\bar{f}_T$ so that

(1) $\bar{f}_T = f_T$ in $Y \equiv \bigcup f_T^j V$,

(2) if $\bar{f}_T^n z = z$, $0 < n \leq m$, and $z \notin Y$, then z is a *hyperbolic* fixed point of $\bar{f}_T^n$,

(3) $\bar{f}_T \in U$.

Since $\bar{f}_T = f_T$ in Y, $\bar{f}_T$ has at least $3m$ points of period m in $\bigcup \bar{f}_T^j R$. Now, perturbing $\bar{f}_T$ in Y to make all points in Y of period m hyperbolic, one obtains, via [16] again, g_T where

(1) $g_T = \bar{f}_T$ outside Y,

(2) $g_T \in U$,

(3) g_T has at least $3m$ points of period m in $\bigcup \bar{f}_T^j R$ and all points of

period $\leq m$ are hyperbolic.

We now want to perturb $\bar{f}_T$ in another way to $\bar{g}_T \in U$ where

(i) $\bar{g}_T = \bar{f}_T$ outside Y,

(ii) $\bar{g}_T = f_{T_1}$ on R for some $T_1 < T_0$,

(iii) $\bar{g}_T$ has exactly m points of period $\leq m$ in Y, all of which are hyperbolic.

Then, we will have g_T and $\bar{g}_T$ in U such that

(a) g_T and $\bar{g}_T$ have all points of period $\leq m$ hyperbolic,

(b) $N_m(g_T) \geq N_m(\bar{g}_T)$.

One of $\{g_T, \bar{g}_T\}$ must satisfy the conclusion of Theorem 1, i.e. $N_m(f_0) \neq N_m(g_T)$ or $N_m(f_0) \neq N_m(\bar{g}_T)$.

So, we need only construct $\bar{g}_T$ as above. Let $Y_i = \bigcup f^j_T V_i$, $i = 0, 1, 2, 3$ where $V_0 \supset V \supset V_1 \supset V_2 \supset V_3$ are all closed 3-disks with the properties that all points of V_0 of period $\leq m$ for f_T lie in R, int $V_3 \supset \bar{R}$, and $\bar{V}_i \supset$ int V_{i+1} for each i. $Y = \bigcup f^j_T V = \bigcup \bar{f}^j_T V$.

Let $\phi\colon T^3 \to \mathbf{R}$ be C^r with the property that $\phi = 0$ outside Y but $\phi = 1$ inside Y_1 and consider the one-parameter family of maps of T^3, $k_t = (1-\phi)\bar{f}_T + \phi f_t$. k_t is C^r for all t. $k_T = \bar{f}_T$ since $\bar{f}_T = f_T$ where $\phi \neq 0$, i.e. in Y. $k_t = \bar{f}_T$ outside Y for all t since $\phi = 0$ there.

Let R_t be the 2-disk $R(q, f_t)$ in $\tilde{F}(q, f_t)$. R_t varies continuously with t. So, there is an open interval (t_1, t_2) about T such that $R_t \subset V_2$ for $t \in (t_1, t_2)$. Choose V_1 and (t_1, t_2) so that all points of period $\leq m$ for f_t in V_1 lie on R_t when $t \in (t_1, t_2)$. Since $\bigcup f^j_T \bar{V}_2 \subset \bigcup \text{int } f^j_T V_1$, one can choose an open interval (t_3, t_4) about T so that $\bigcup f^j_t \bar{V}_2 \subset \bigcup \text{int } f^j_T V_1 = Y_1$ for $t \in (t_3, t_4)$. Choose an open interval (t_5, t_6) about T so that $k_t \in U$ for $t \in (t_5, t_6)$. Choose an open interval (t_7, t_8) about T so that for such t, k_t has no points of period $\leq m$ in $\bigcup f^j_T (V \setminus V_3)$. This is possible since $k_T = f_T$ has no such periodic points. Finally, choose an open interval (t_9, t_{10}) about T so that $\bigcup f^j_T V_3 \subset \bigcup k^j_t V_2$ for $t \in (t_9, t_{10})$.

Now choose $t < T$ with $t \in \bigcap_{i=1}^5 (t_{2i-1}, t_{2i})$. Claim k_t is our desired $\bar{g}_T$. $k_t \in U$ and $k_t = \bar{f}_T$ outside Y.

So it suffices to show that k_t has only m points of period $\leq m$ in Y. Let $x \in Y$ with $k^i_t x = x$ for some $i \leq m$. $Y = [\bigcup f^j_T V_3] \cup [\bigcup f^j_T (V \setminus V_3)]$. $x \in \bigcup f^j_T V_3$ since $t \in (t_7, t_8)$. $x \in \bigcup k^j_t V_2$ since $t \in (t_9, t_{10})$. Also, $k_t = f_t$ in $\bigcup f^j_t V_2 = \bigcup k^j_t V_2 \subset Y_1$ since $t \in (t_3, t_4)$. So $x \in \bigcup f^j_t V_2$ and $f^i_t x = x$ for some i, $1 \leq i \leq m$. Because $t \in (t_1, t_2)$, $x \in \bigcup f^j_t R_t$. Since $t < T$, q is the only point of period $\leq m$ for f_t in R_t. Therefore, $x = f^{-j} q$ for some $j \leq m$. So, k_t can be the $\bar{g}_T$ needed to finish the proof of this theorem.

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