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CYLINDRIC ALGEBRAS AND ALGEBRAS OF SUBSTITUTIONS(1)

BY

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ABSTRACT. Several new formulations of the notion of cylindric algebra are presented. The class CA_α of all cylindric algebras of degree α is shown to be definitionally equivalent to a class of algebras in which only substitutions (together with the Boolean $+$, $\cdot$, and $-$) are taken to be primitive operations. Then CA_α is shown to be definitionally equivalent to an equational class of algebras in which only substitutions and their conjugates (together with $+$, $\cdot$, and $-$) are taken to be primitive operations.

1. **Introduction.** In the theory of cylindric algebras,(2) certain operations called *substitutions* are defined in terms of cylindrifications and diagonal elements. Specifically, if $\mathfrak{A}$ is a cylindric algebra of degree α and $\kappa, \lambda < \alpha$, then the operation of λ-*for*-κ *substitution*, denoted by S^κ_λ, is defined by the formula

$$\text{(A)} \qquad S^\kappa_\lambda x = x \quad \text{if } \kappa = \lambda; \qquad S^\kappa_\lambda x = c_\kappa(x \cdot d_{\kappa\lambda}) \quad \text{if } \kappa \neq \lambda.$$

Just as (A) defines substitutions in terms of cylindrifications and diagonal elements, it is known that both cylindrifications and diagonal elements may be defined in terms of substitutions. Thus, it is clearly possible to develop a theory of algebraic logic in which substitutions are taken to be the only primitive operations.

It is proved in [6] that (a) in every CA_α, $d_{\kappa\lambda}$ is the least element x such that $S^\kappa_\lambda x = 1$, and (b) in every dimension-complemented CA_α, $c_\kappa x = \sum_{\lambda<\alpha} S^\kappa_\lambda x$ for each $\kappa < \alpha$. It is noted in [6] that (a) and (b) suggest a possible new treatment of dimension-complemented cylindric algebras as algebraic structures $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ where the S^κ_λ are unary operations subject to certain conditions. A simple and elegant characterization of dimension-complemented cylindric algebras in this sense has been obtained by Anne Preller in [10].

There is more than one way of defining cylindrifications in terms of substitutions. The definition based upon (b), above, has the disadvantage of yielding cylindrifications only in the case of dimension-complemented algebras; alternative

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(2) For notation and terminology, see Henkin, Monk and Tarski [6].

definitions are available, where the assumption of dimension complementedness is not needed. Using any one of these, it is possible to characterize *all* cylindric algebras in terms of substitutions.

In this paper, we describe two methods for defining cylindrifications in terms of substitutions without the assumption of dimension complementedness. Using these, we get two simple characterizations of cylindric algebras in terms of substitutions, which take an especially simple form in the presence of a condition somewhat weaker than dimension complementedness.

In a recent paper, William Craig [2] has shown that in a cylindric algebra every substitution S^κ_λ has a conjugate,(3) and that cylindrifications and diagonal elements may be defined entirely in terms of substitutions and their conjugates. Using these ideas, we derive a set of axioms for an equational class of algebras which is definitionally equivalent to CA_α, and in which substitutions and their conjugates (together with the Boolean $+$, $\cdot$, and $-$) are the only primitive operations.

All the cylindric and related algebras considered in this paper are assumed to be of degree $\alpha > 2$. For the case $\alpha = 2$, simpler results hold which will be presented elsewhere, in the form of an abstract. For the cases $\alpha = 0, 1$, the questions we shall examine here are trivial.

2. **Substitution algebras and cylindric algebras.** By a "substitution algebra" we shall mean a Boolean algebra with "substitution" operations. Our precise formulation is as follows:

2.1. **Definition.** By a substitution algebra of degree α, briefly an SA_α, we mean a system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda \rangle_{\kappa,\lambda<\alpha}$ where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra and the S^κ_λ are unary operations on A which satisfy the following conditions for all $x, y \in A$ and all $\kappa, \lambda, \mu, \nu < \alpha$:

(S_1) $S^\kappa_\lambda(x + y) = S^\kappa_\lambda x + S^\kappa_\lambda y$,

(S_2) $S^\kappa_\lambda(-x) = -S^\kappa_\lambda x$,

(S_3) $S^\kappa_\kappa x = x$,

(S_4) $S^\mu_\lambda S^\kappa_\mu x = S^\mu_\lambda S^\kappa_\lambda x$,

(S_5) $S^\kappa_\mu S^\kappa_\lambda x = S^\kappa_\lambda x$, if $\kappa \neq \lambda$,

(S_6) $S^\kappa_\lambda S^\mu_\nu x = S^\mu_\nu S^\kappa_\lambda x$, if $\kappa \neq \mu, \nu$ and $\mu \neq \lambda$.

Our notion of substitution algebras is similar to that of Preller in [10], but, in the latter, some additional conditions are assumed. In addition, substitution algebras, as we have just defined them, are closely related to the *transformation algebras* invented by P. R. Halmos [3], with the difference that we restrict transformations to *replacements* in the sense of Halmos [5, p. 289]. Thus, our operations S^κ_λ may be compared with the operations $S(\tau)$ of [3] where τ is restricted to

(3) For the notion of conjugate, due to Jónsson and Tarski [7], see our Definition 2.5.

replacements. Then our axioms (S_1) and (S_2) correspond to the condition in [3] that the operations $S(\tau)$ be Boolean endomorphisms, and our axiom (S_3) is essentially the same as Halmos' Axiom S1. Instead of Halmos' Axiom S2, which transcends the conditions assumed in our notion of substitution algebra, we adopt three immediate consequences of this axiom restricted to replacements—namely (S_4)–(S_6). This is all we need for our purposes.

In every substitution algebra (and in every cylindric algebra if operations S^κ_λ are defined by (A)), the range of S^κ_λ, for $\kappa \neq \lambda$, is independent of λ. Indeed, it follows from (S_5) that range S^κ_λ = range S^κ_μ for all λ, $\mu \neq \kappa$. In view of this observation, we shall henceforth write rg κ for the range of S^κ_λ, $\kappa \neq \lambda$.

An analogous observation relates the range of S^κ_λ to the range of c_κ in cylindric algebras. Indeed, in every cylindric algebra, $S^\kappa_\lambda c_\kappa = c_\kappa$ and $c_\kappa S^\kappa_\lambda = S^\kappa_\lambda$ for $\kappa, \lambda < \alpha$, $\kappa \neq \lambda$ (see [6, 1.5.8 and 1.5.9]). Consequently,

(*) $$\text{range } c_\kappa = \text{rg } \kappa \quad \text{for every } \kappa < \alpha.$$

Now it follows from the elementary properties of cylindrification that

(**) $$c_\kappa x \text{ is the least } y \in \text{range } c_\kappa \text{ such that } y \geq x.$$

Combining (*) and (**), we get the following important property of cylindric algebras:

(2.2) $$c_\kappa x \text{ is the least } y \in \text{rg } \kappa \text{ such that } y \geq x.$$

(2.2) clearly shows that cylindrifications can be expressed in terms of substitutions in an algebra which does not need to be dimension complemented. The proposition

(2.3) $$d_{\kappa\lambda} \text{ is the least } y \text{ such that } S^\kappa_\lambda y = 1$$

shows that diagonal elements likewise can be so expressed. This suggests that we consider substitution algebras having the properties

(π1) for each $x \in A$ and $\kappa < \alpha$, $\{y \in \text{rg } \kappa : y \geq x\}$ has a least element, and

(π2) for every $\kappa, \lambda < \alpha$, $\{x : S^\kappa_\lambda x = 1\}$ has a least element,

and introduce for them operations c_κ and $d_{\kappa\lambda}$ by means of the definitions

(B) $c_\kappa x$ is the least element of $\{y \in \text{rg } \kappa : y \geq x\}$, and

(C) $d_{\kappa\lambda}$ is the least element of $\{x : S^\kappa_\lambda x = 1\}$.

The notion that (B) may be used to introduce cylindrifications in substitution algebras satisfying (π1) appears clearly in the work of P.-F. Jurie. In [8], Jurie has shown that a condition similar to (B) holds in every polyadic algebra, and that it may be used to introduce quantifiers in transformation algebras satisfying a condition similar to (π1).

Before going on, we consider some consequences of (π1) and (π2).

2.4. **Lemma.** *Let* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda \rangle_{\kappa,\lambda<\alpha}$ *be a substitution algebra subject to condition* (π1). *If* c_κ *is defined by* (B), *then the following statements* *hold for all* $x \in A$ *and all* $\kappa, \lambda, \mu < \alpha$:

(i) $c_\kappa S^\kappa_\lambda = S^\kappa_\lambda$ *if* $\lambda \neq \kappa$,

(ii) $S^\kappa_\lambda c_\kappa = c_\kappa$,

(iii) $c_\kappa 0 = 0$,

(iv) $x \leq c_\kappa x$,

(v) $c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y$,

(vi) $x \leq y$ *implies* $c_\kappa x \leq c_\kappa y$,

(vii) $S^\kappa_\lambda x \leq c_\kappa x$,

(viii) $c_\kappa S^\lambda_\mu x \leq S^\lambda_\mu c_\kappa x$ *if* $\kappa \neq \lambda, \mu$.

Proof. It is clear from (π1) and (B) that if $y = c_\kappa x$, then $y \in \mathrm{rg}\,\kappa$, and also that if $y \in \mathrm{rg}\,\kappa$, then $y = c_\kappa y$. Thus,

(a) range $c_\kappa = \mathrm{rg}\,\kappa$.

Furthermore, it follows from (S_5) and (B), respectively, that

(b) $S^\kappa_\lambda S^\kappa_\lambda = S^\kappa_\lambda$ and $c_\kappa c_\kappa = c_\kappa$.

Now (i) and (ii) follow immediately from (a) and (b).

By (S_1) and (S_2), S^κ_λ is an endomorphism of $\mathfrak{A}$, hence $\mathrm{rg}\,\kappa$ is a Boolean subalgebra of $\mathfrak{A}$. In fact, by (π1), $\mathrm{rg}\,\kappa$ is a *relatively complete* Boolean subalgebra of $\mathfrak{A}$ (see Halmos [4, §4]). Thus, by Halmos [4, Proof of Theorem 5], c_κ is a quantifier of $\mathfrak{A}$, and therefore (iii)–(v) hold.

Now (vi) follows from (iii)–(v) by [6, 1.2.7], and (vii) is an immediate consequence of (iv), (S_1) and (ii). It remains to prove (viii): by (π1), $c_\kappa S^\lambda_\mu x$ is the least element $y \in \mathrm{rg}\,\kappa$ such that $y \geq S^\lambda_\mu x$. Now $S^\lambda_\mu c_\kappa x \geq S^\lambda_\mu x$ by (iv) and (S_1); furthermore, by (ii) and (S_6), if $\nu \neq \lambda$, then

$$S^\lambda_\mu c_\kappa x = S^\lambda_\mu S^\kappa_\nu c_\kappa x = S^\kappa_\nu S^\lambda_\mu c_\kappa x \in \mathrm{rg}\,\kappa;$$

thus, $c_\kappa S^\lambda_\mu x \leq S^\lambda_\mu c_\kappa x$. □

The next definition introduces a concept which plays an important role in the next section.

2.5. **Definition** (**Jónsson and Tarski** [7, **Definition 1.11**]). If A is a Boolean algebra and $f, g \in {}^A A$, we say that g is the *conjugate* of f if, for any $x, y \in A$, $f(x) \cdot y = 0 \Leftrightarrow x \cdot g(y) = 0$.

2.6. **Lemma.** *Let* $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ *be a substitution algebra subject to conditions* (π1) *and* (π2). *If* c_κ *and* $d_{\kappa\lambda}$ *are defined by* (B) *and* (C), *respectively, then the following hold for all* $x \in A$ *and all* $\kappa, \lambda, \mu < \alpha$:

(i) $S^\kappa_\lambda x \cdot d_{\kappa\lambda} = x \cdot d_{\kappa\lambda}$,

(ii) $S^\kappa_\lambda x = c_\kappa(x \cdot d_{\kappa\lambda})$ *if* $\kappa \neq \lambda$,

(iii) $S^\mu_\lambda d_{\kappa\mu} = S^\mu_\lambda d_{\kappa\lambda}$ *if* $\kappa \neq \mu$,

(iv) S^κ_λ *admits a conjugate*.(4)

Proof. (i) Clearly $S^\kappa_\lambda(S^\kappa_\lambda x + -x) = 1$, hence by ($\pi$2) and (C), $d_{\kappa\lambda} \leq S^\kappa_\lambda x + -x$, which is the same as $x \cdot d_{\kappa\lambda} \leq S^\kappa_\lambda x$. Analogously, $S^\kappa_\lambda x \cdot d_{\kappa\lambda} \leq x$, so we get (i).

(4) Theorem 2.6(iv) is a special case of a more general result proved in Craig [2, Lemma 27].

(ii) By 2.4(i) and 2.6(i), $c_\kappa S^\kappa_\lambda x \cdot d_{\kappa\lambda} = S^\kappa_\lambda x \cdot d_{\kappa\lambda} = x \cdot d_{\kappa\lambda}$. Thus, $c_\kappa(x \cdot d_{\kappa\lambda}) = c_\kappa(c_\kappa S^\kappa_\lambda x \cdot d_{\kappa\lambda})$, so by 2.4(i) and (v), $c_\kappa(x \cdot d_{\kappa\lambda}) = c_\kappa S^\kappa_\lambda x \cdot c_\kappa d_{\kappa\lambda} = S^\kappa_\lambda x \cdot c_\kappa d_{\kappa\lambda}$. But by (C) and 2.4(vii), $c_\kappa d_{\kappa\lambda} = 1$, so $c_\kappa(x \cdot d_{\kappa\lambda}) = S^\kappa_\lambda x \cdot 1 = S^\kappa_\lambda x$.

(iii) If $\lambda = \mu$, there is nothing to prove. Now by (S_3) and (S_4), $S^\mu_\kappa S^\kappa_\mu x = S^\mu_\kappa S^\kappa_\kappa x = S^\mu_\kappa x$, hence $S^\kappa_\mu x = 1 \Rightarrow S^\mu_\kappa x = 1$; symmetrically, $S^\mu_\kappa x = 1 \Rightarrow S^\kappa_\mu x = 1$; thus, by (C), $d_{\kappa\mu} = d_{\mu\kappa}$. Therefore if $\lambda \neq \mu$, then by 2.6(ii), $S^\mu_\lambda d_{\kappa\mu} = c_\mu(d_{\kappa\mu} \cdot d_{\mu\lambda}) = c_\mu(d_{\mu\lambda} \cdot d_{\mu\kappa}) = S^\mu_\kappa d_{\mu\lambda}$. Thus,

$$
\begin{aligned}
S^\kappa_\lambda(S^\mu_\lambda d_{\kappa\mu}) = S^\kappa_\lambda(S^\mu_\kappa d_{\mu\lambda}) &= S^\kappa_\lambda S^\mu_\lambda d_{\mu\lambda} && \text{by } (S_4),\\
&= 1 && \text{by (C), } (S_1) \text{ and } (S_2).
\end{aligned}
$$

It follows by (C) that $d_{\kappa\lambda} \leq S^\mu_\lambda d_{\kappa\mu}$, and analogously, $d_{\kappa\mu} \leq S^\lambda_\mu d_{\kappa\lambda}$. Thus, by ($S_1$)–($S_4$), $S^\mu_\lambda d_{\kappa\lambda} \leq S^\mu_\lambda S^\mu_\lambda d_{\kappa\mu} = S^\mu_\lambda d_{\kappa\mu} \leq S^\mu_\lambda S^\lambda_\mu d_{\kappa\lambda} = S^\mu_\lambda d_{\kappa\lambda}$.

(iv) If $\kappa = \lambda$, we define T^κ_λ by $T^\kappa_\lambda x = x$; in this case, it is obvious that T^κ_λ is the conjugate of S^κ_λ. *If* $\kappa \neq \lambda$, we define T^κ_λ by

(a) $T^\kappa_\lambda x = c_\kappa x \cdot d_{\kappa\lambda}$,

and we proceed to show that T^κ_λ is the conjugate of S^κ_λ:

$$
\begin{aligned}
& T^\kappa_\lambda x \cdot y = 0 \Rightarrow c_\kappa x \cdot d_{\kappa\lambda} \cdot y = 0 && \text{by (a)},\\
\text{(applying } S^\kappa_\lambda) \quad & \qquad\qquad \Rightarrow c_\kappa x \cdot 1 \cdot S^\kappa_\lambda y = 0 && \text{by } (S_1)\text{–}(S_2)\text{, 2.4(ii) and (C)},\\
& \qquad\qquad \Rightarrow x \cdot S^\kappa_\lambda y = 0 && \text{by 2.4(iv)}.
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
x \cdot S^\kappa_\lambda y = 0 &\Rightarrow c_\kappa(x \cdot S^\kappa_\lambda y) = 0 && \text{by 2.4(iii)},\\
&\Rightarrow c_\kappa x \cdot S^\kappa_\lambda y = 0 && \text{by 2.4(i) and (v)},\\
&\Rightarrow c_\kappa x \cdot d_{\kappa\lambda} \cdot S^\kappa_\lambda y = 0 &&\\
&\Rightarrow c_\kappa x \cdot d_{\kappa\lambda} \cdot y = 0 && \text{by 2.6(i)},\\
&\Rightarrow T^\kappa_\lambda x \cdot y = 0 && \text{by (a)}. \quad \square
\end{aligned}
$$

In our first theorem, we will show that the theory of cylindric algebras is definitionally equivalent to the theory of substitution algebras which satisfy (π1), (π2) and the following additional condition:

(π3) $S^\kappa_\lambda c_\mu = c_\mu S^\kappa_\lambda$ provided $\mu \neq \kappa, \lambda$, where c_μ is defined by (B).

Note that only half of this condition is really needed, as the other half is derived in 2.4(viii).

2.7. **Theorem.** (i) *If* $\langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ *is a cylindric algebra and* S^κ_λ *is defined by* (A), *then* $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ *is a substitution algebra satisfying* (π1)–(π3), *together with* (B) *and* (C).

(ii) *Conversely, if* $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ *is a substitution algebra in which conditions* (π1)–(π3) *hold, and if* c_κ *and* $d_{\kappa\lambda}$ *are defined by* (B) *and* (C) *re-*

spectively, then $\langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ *is a cylindric algebra where* (A) *holds.*

Proof. (i) Let $\langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ be a cylindric algebra, and define S^κ_λ by (A). Then (S_1)–(S_6) hold by [6, 1.5.3 and 1.5.10], and $(\pi 2)$ holds by [6, 1.5.7]. Next, range $c_\kappa = \mathrm{rg}\,\kappa$ by [6, 1.5.8(i) and 1.5.9(ii)], hence by [6, 1.2.4 and 1.2.9], we have $(\pi 1)$ and (B). Finally, $(\pi 3)$ holds by [6, 1.5.8(ii)], and (C) holds by [6, 1.5.7].

(ii) Let $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ be a substitution algebra in which $(\pi 1)$–$(\pi 3)$ hold, and let c_κ and $d_{\kappa\lambda}$ be defined by (B) and (C) respectively. We will show that conditions (C_0)–(C_7) of [6, Definition 1.1.1], are satisfied. Well, (C_0) holds by 2.1, and (C_1)–(C_3) are given by 2.4(iii)–(v). We prove (C_4) as follows: If $\kappa = \lambda$ there is nothing to prove; hence, assume $\kappa \neq \lambda$. By 2.4(iv), $x \leq c_\kappa x$, so $c_\kappa c_\lambda x \leq c_\kappa c_\lambda c_\kappa x$. Now if $\mu \neq \kappa, \lambda$, then

$$\begin{aligned}
c_\kappa c_\lambda c_\kappa x &= c_\kappa c_\lambda S^\kappa_\mu c_\kappa x && \text{by 2.4(ii)},\\
&= c_\kappa S^\kappa_\mu c_\lambda c_\kappa x && \text{by } (\pi 3),\\
&= S^\kappa_\mu c_\lambda c_\kappa x && \text{by 2.4(i)},\\
&= c_\lambda S^\kappa_\mu c_\kappa x && \text{by } (\pi 3),\\
&= c_\lambda c_\kappa x && \text{by 2.4(ii)}.
\end{aligned}$$

This shows that $c_\kappa c_\lambda x \leq c_\lambda c_\kappa x$; symmetrically, $c_\lambda c_\kappa x \leq c_\kappa c_\lambda x$, proving (C_4). (C_5) follows immediately from $(\pi 2)$ and (C).

To prove (C_6), let $c^\partial_\mu x = -(c_\mu - x)$. It easily follows from $(\pi 3)$ and (S_2) that $S^\kappa_\lambda c^\partial_\mu = c^\partial_\mu S^\kappa_\lambda$ for $\mu \neq \kappa, \lambda$. Thus, $S^\kappa_\lambda c^\partial_\mu d_{\kappa\lambda} = c^\partial_\mu S^\kappa_\lambda d_{\kappa\lambda} = 1$, so by $(\pi 2)$, $d_{\kappa\lambda} \leq c^\partial_\mu d_{\kappa\lambda}$. On the other hand, it follows from 2.4(iv) that $c^\partial_\mu d_{\kappa\lambda} \leq d_{\kappa\lambda}$, so $d_{\kappa\lambda} = c^\partial_\mu d_{\kappa\lambda} \in \mathrm{rg}\,\mu$, hence, in particular, $d_{\kappa\lambda} = S^\mu_\lambda d_{\kappa\lambda}$. Thus, by 2.6(iii), $S^\mu_\lambda d_{\kappa\mu} = d_{\kappa\lambda}$ for $\kappa \neq \mu$. Consequently, if $\mu \neq \kappa, \lambda$, then

$$\begin{aligned}
c_\mu(d_{\kappa\mu}\cdot d_{\mu\lambda}) &= S^\mu_\lambda d_{\kappa\mu} && \text{by 2.6(ii)},\\
&= d_{\kappa\lambda},
\end{aligned}$$

which proves (C_6). Lastly, (C_7) follows immediately from 2.6(ii) and (S_2), while (A) follows from 2.6(ii) and (S_1). □

Remark. It follows from 2.7(i) that if $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ is a substitution algebra, then there is *at most* one family of functions $\langle c_\kappa\rangle_{\kappa<\alpha}$ and one family of elements $\langle d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ such that $\langle A, +, \cdot, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a cylindric algebra; indeed, the c_κ and $d_{\kappa\lambda}$ are completely determined by (B) and (C). A similar remark was made for polyadic algebras by P.-F. Jurie [8], and, earlier, for locally finite polyadic algebras by L. LeBlanc [9].

We will show next that Theorem 2.7 can be improved significantly in the presence of either of two conditions somewhat weaker than dimension complementedness. We begin with some definitions:

2.8. **Definition.** Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S^{\kappa}_{\lambda}\rangle_{\kappa,\lambda<\alpha}$ be a substitution algebra. For each $x \in A$, the *dimension set* of x, in symbols Δx, is the set of all $\kappa < \alpha$ such that $c_{\kappa} x \neq x$. $\mathfrak{A}$ is said to be *dimension complemented* if, for every $x \in A$, $\Delta x \neq \alpha$.

2.9. **Definition.** Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S^{\kappa}_{\lambda}\rangle_{\kappa,\lambda<\alpha}$ be a substitution algebra. We write $Z(\kappa, \lambda) = \{x \in A : S^{\kappa}_{\lambda} x = 1\}$. Then $\mathfrak{A}$ is said to be *normal* if $\bigcap_{\kappa\neq\lambda;\,\lambda<\alpha} Z(\kappa, \lambda) = \{1\}$ for each $\kappa < \alpha$.

A cylindric algebra will be called *normal* if it has the same property.

If $\mathfrak{A}$ satisfies (π2) and $d_{\kappa\lambda}$ satisfies (C) for all κ, $\lambda < \alpha$ (for example, if $\mathfrak{A}$ is a cylindric algebra), then clearly $\mathfrak{A}$ is normal if and only if $\sum_{\lambda\neq\kappa;\,\lambda<\alpha} d_{\kappa\lambda} = 1$ for each $\kappa < \alpha$.(5)

2.10. **Definition.** Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S^{\kappa}_{\lambda}\rangle_{\kappa,\lambda<\alpha}$ be a substitution algebra satisfying (π1), and for each $\kappa < \alpha$, let c_{κ} be defined by (B). We say that $\mathfrak{A}$ has the RS property if

$$c_{\kappa} x = \sum_{\lambda\neq\kappa;\,\lambda<\alpha} S^{\kappa}_{\lambda} x \tag{2.11}$$

for all $\kappa < \alpha$ and $x \in A$. A cylindric algebra is likewise said to have the RS property if it satisfies (2.11).

Equation (2.11) is due to Rasiowa and Sikorski [11].

2.12. **Theorem.** (a) *Every dimension complemented substitution algebra is normal.*

(b) *Every normal substitution algebra satisfying* (π1) *and* (π2) *has the RS property.*

Proof. (a) If $\mathfrak{A}$ is not normal, then there is some $x \neq 1$ in A and some $\kappa < \alpha$ such that $S^{\kappa}_{\lambda} x = 1$ for every $\lambda \neq \kappa$. But $\mathfrak{A}$ is dimension complemented; thus, $\mu \notin \Delta x$ for some $\mu < \alpha$. If $\mu = \kappa$, then for any $\lambda \neq \mu$, $x = S^{\mu}_{\lambda} x = S^{\kappa}_{\lambda} x = 1$; if $\mu \neq \kappa$, then $x = S^{\mu}_{\kappa} x = S^{\mu}_{\kappa} S^{\kappa}_{\mu} x = S^{\mu}_{\kappa} 1 = 1$; in either case, we have a contradiction.

(b) Let $\mathfrak{A}$ be a normal substitution algebra satisfying (π1) and (π2). Then,

$$\begin{aligned}
\sum_{\lambda\neq\kappa;\,\lambda<\alpha} S^{\kappa}_{\lambda} x &= \sum_{\lambda\neq\kappa;\,\lambda<\alpha} c_{\kappa}(x \cdot d_{\kappa\lambda}) && \text{by 2.6(ii),}\\
&= c_{\kappa} \sum_{\lambda\neq\kappa;\,\lambda<\alpha} (x \cdot d_{\kappa\lambda}) && \text{by [6, 1. 2. 6],}\\
&= c_{\kappa}\Big(x \cdot \sum_{\lambda\neq\kappa;\,\lambda<\alpha} d_{\kappa\lambda}\Big) &&\\
&= c_{\kappa} x && \text{by the remark following 2.9.} \quad \square
\end{aligned}$$

2.13. **Lemma.** *If* $\langle A, +, \cdot, -, 0, 1, S^{\kappa}_{\lambda}\rangle_{\kappa,\lambda<\alpha}$ *is a substitution algebra with the RS property satisfying* (π1) *and* (π2), *then* (π3) *holds.*

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(5) It follows easily that every normal CA_{α}, for $\alpha < \omega$, is discrete. Thus, for $\alpha < \omega$, a CA_{α} is normal iff it is dimension-complemented.

Proof. We have proved, Lemma 2.6(iv), that each S^κ_λ admits a conjugate. It follows by Jónsson and Tarski [7, Theorem 1.14] that each S^κ_λ is completely additive. Thus, if $\kappa \neq \lambda, \mu$, we have

$$\begin{aligned} S^\lambda_\mu c_\kappa x = S^\lambda_\mu \sum_{\nu \neq \kappa;\, \nu<\alpha} S^\kappa_\nu x &= \sum_{\nu \neq \kappa;\, \nu<\alpha} S^\lambda_\mu S^\kappa_\nu x = S^\lambda_\mu S^\kappa_\lambda x + \sum_{\nu \neq \kappa,\lambda;\, \nu<\alpha} S^\lambda_\mu S^\kappa_\nu x \\ &= S^\kappa_\mu S^\lambda_\mu x + \sum_{\nu \neq \kappa,\lambda;\, \nu<\alpha} S^\kappa_\nu S^\lambda_\mu x \quad \text{by } (S_4) \text{ and } (S_6), \\ &= \sum_{\nu \neq \kappa,\lambda;\, \nu<\alpha} S^\kappa_\nu S^\lambda_\mu x. \end{aligned}$$

Consequently, $S^\lambda_\mu c_\kappa x + S^\kappa_\lambda S^\lambda_\mu x = \Sigma_{\nu\neq\kappa;\,\nu<\alpha} S^\kappa_\nu S^\lambda_\mu x = c_\kappa S^\lambda_\mu x$, hence $S^\lambda_\mu c_\kappa x \leq c_\kappa S^\lambda_\mu x$. Combining this with 2.4(viii) gives us our result. □

We conclude this section with the following result:

2.14. **Theorem.** (i) *If* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ *is a cylindric algebra with the RS property (resp. a normal cylindric algebra, resp. a dimension complemented cylindric algebra) and* S^κ_λ *is defined by* (A), *then* $\mathfrak{A}' = \langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ *is a substitution algebra with the RS property (resp. a normal substitution algebra, resp. a dimension complemented substitution algebra) satisfying* (π1) *and* (π2), *as well as* (B) *and* (C).

(ii) *Conversely, if* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ *is a substitution algebra with the RS property (resp. a normal SA, resp. a dimension complemented SA) satisfying* (π1) *and* (π2), *and if* c_κ *and* $d_{\kappa\lambda}$ *are defined by* (B) *and* (C), *respectively, then* $\mathfrak{A}' = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ *is a cylindric algebra with the RS property (resp. a normal cylindric algebra, resp. a dimension complemented cylindric algebra) in which* (A) *holds.*

Thus, in the special case of algebras which are normal, dimension complemented, or have the RS property, Theorem 2.7 may be re-stated without condition (π3). Theorem 2.14, stated for dimension complemented algebras, is germane to the theorem in Preller [10]. In fact, Theorem 2.14 provides a partial solution to a problem posed by Preller; in the theorem of Preller [10], it is required that the substitution algebra possess a property stronger than dimension complementedness, namely the property

(T) For every x, y there exists κ such that $S^\kappa_\lambda x = x$ and $S^\kappa_\lambda y = y$ for all $\lambda < \alpha$. Preller raised the question whether the theorem in [10] holds in case condition (T) is replaced by dimension complementedness (our Definition 2.8). We have given an affirmative answer to this question, with the proviso that c_κ be defined by (B), and, correspondingly, that Preller's (S_7) be replaced by (π1).

3. **Some equivalences in substitution algebras.** In a recent paper, William Craig [2] showed that in a polyadic algebra with generalized diagonal elements every substitution admits a conjugate, and that cylindrifications and generalized diagonal elements may be defined in terms of substitutions and their conjugates.

In this section we shall use the ideas introduced by Craig to formulate several conditions which are shown to be equivalent to ($\pi 1$)–($\pi 3$), and thus obtain another characterization of cylindric algebras in terms of substitutions and Boolean notions alone.

In this and the succeeding pages, if B is an ordered set and B has a least element, then the least element of B will be designated by min B. In our next lemma, we record a few elementary properties of conjugates.

3.1. **Lemma.** *Let A be a Boolean algebra and let f be an endomorphism of A.*

(i) *If f has a conjugate g, then*

(a) $x \leq f(y) \Leftrightarrow g(x) \leq y$,

(b) $g(x) = \min\{y \in A : f(y) \geq x\}$, *for each* $x \in A$,

(c) $fg(x) = \min\{y \in \text{range } f : y \geq x\}$, *for each* $x \in A$,

(d) $g(1) = \min\{y \in A : f(y) = 1\}$.

(ii) *Suppose that for each* $x \in A$, $\{y \in A : f(y) \geq x\}$ *has a least element. If* $g(x) = \min\{y \in A : f(y) \geq x\}$ *for each* $x \in A$, *then g is the conjugate of f.*

Proof. (i) (a) is an immediate consequence of Definition 2.5 and the fact that f is an endomorphism. For (b), we have $g(x) \leq g(x)$, so by (a), $x \leq fg(x)$, hence $g(x) \in \{y \in A : f(y) \geq x\}$; now if $f(y) \geq x$, then by (a), $g(x) \leq y$, which proves (b). For (c), we have just seen that $x \leq fg(x)$, so $fg(x) \in \{y \in \text{ran } f : y > x\}$; now if $y \in \text{ran } f$, say $y = f(z)$, and $y = f(z) \geq x$, then by (a), $g(x) \leq z$, so $fg(x) \leq f(z) = y$, which proves (c). Finally, (d) is a special case of (b).

(ii) Clearly $f(y) \geq x \Rightarrow g(x) \leq y$. By hypothesis, $fg(x) \geq x$, so $g(x) \leq y \Rightarrow fg(x) \leq f(y) \Rightarrow x \leq f(y)$. Thus, $x \leq f(y) \Leftrightarrow g(x) \leq y$; from this and the fact that f is a Boolean endomorphism, we immediately deduce that g is the conjugate of f. □

We introduce two new conditions for substitution algebras:

($\pi 4$) for all $\kappa, \lambda < \alpha$, S^κ_λ admits a conjugate;

($\pi 5$) for all $\kappa, \lambda < \alpha$ and every $x \in A$, $\{y : S^\kappa_\lambda y \geq x\}$ has a least element.

From Lemma 3.1, with S^κ_λ replacing f, together with Lemma 2.6(iv), we immediately get the following result:

3.2. **Theorem.** *In every substitution algebra, the following conditions are equivalent:*

(i) ($\pi 1$) *and* ($\pi 2$) *hold*;

(ii) ($\pi 4$) *holds*;

(iii) ($\pi 5$) *holds*.

A substitution algebra satisfying these conditions is called a *substitution algebra admitting conjugates*.

From 3.1(i)b, with S^κ_λ replacing f and T^κ_λ replacing g, we deduce that the following two conditions are equivalent for every SA_α admitting conjugates:

(D) T^κ_λ is the conjugate of S^κ_λ, for all $\kappa, \lambda < \alpha$,

(E) $T^\kappa_\lambda x = \min\{y: S^\kappa_\lambda y \geq x\}$, for all $\kappa, \lambda < \alpha$ and every $x \in A$.

Similarly, by 3.1(i)c, with S^κ_λ replacing f and T^κ_λ replacing g, it follows that in every SA_α admitting conjugates, (B) is equivalent to

(F) $c_\kappa x = S^\kappa_\lambda T^\kappa_\lambda x$ if $\lambda \neq \kappa$, where T^κ_λ is defined by (D) (resp. (E)),

and by 3.1(i)d, (C) is equivalent to

(G) $d_{\kappa\lambda} = T^\kappa_\lambda 1$, where T^κ_λ is defined by (D) (resp. (E)).

By 3.2 and our last remarks, Theorems 2.7 and 2.14 can be re-stated with (π4) (resp. (π5)) replacing (π1) and (π2), (F) replacing (B), and (G) replacing (C). The purpose of our next result is to show that (π3) may likewise be replaced by equivalent conditions.

We introduce three new conditions for SA's admitting conjugates:

(π6) If $x \in \mathrm{rg}\ \kappa$ and $y \in \mathrm{rg}\ \lambda$ and $x \leq y$, then there is some $z \in \mathrm{rg}\ \kappa \cap \mathrm{rg}\ \lambda$ such that $x \leq z \leq y$.

(π7) $\Delta d_{\kappa\lambda} \neq \alpha$ for all $\kappa, \lambda < \alpha$, where $d_{\kappa\lambda}$ is given by (C).

(π8) $S^\kappa_\lambda T^\mu_\nu = T^\mu_\nu S^\kappa_\lambda$ if $\kappa \neq \mu, \nu$ and $\lambda \neq \mu$, where T^μ_ν is defined by (D) or (E).

Condition (π6) is a special case of condition (Ind) in Jurie [8].

3.3. **Theorem.** *If* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ *is a substitution algebra admitting conjugates, and* c_κ, $d_{\kappa\lambda}$ *and* T^κ_λ *are defined by* (B), (C) *and* (D), *respectively, then the following three conditions are equivalent*:

(i) (π3) *holds*;

(ii) (π6) *and* (π7) *hold*;

(iii) (π8) *holds*.

Proof. (π3) $\Rightarrow$ (π6) $\wedge$ (π7). If (π3) holds, then $\mathfrak{A}$ is a cylindric algebra by 2.7. Thus, if $x \in \mathrm{rg}\ \kappa$ and $y \in \mathrm{rg}\ \lambda$ and $x \leq y$, then $x \leq c_\lambda x \leq c_\lambda y = y$; but $c_\lambda x \in \mathrm{rg}\ \lambda$ and $c_\lambda x = c_\lambda c_\kappa x = c_\kappa c_\lambda x \in \mathrm{rg}\ \kappa$, so $c_\lambda x \in \mathrm{rg}\,\kappa \cap \mathrm{rg}\ \lambda$. This proves ($\pi$6); ($\pi$7) holds by [6, 1.3.3].

(π6) $\wedge$ (π7) $\Rightarrow$ (π3). Assume that (π6) and (π7) hold. We have

$$d_{\kappa\lambda} \in \mathrm{rg}\ \nu \quad \text{for all } \nu \neq \kappa, \lambda. \tag{3.4}$$

If $\alpha = 3$, this is the same statement as (π7); thus, we assume that $\alpha > 3$. Let $\nu \neq \kappa, \lambda$; for any $\pi \neq \kappa, \nu$, $S^\kappa_\lambda S^\nu_\pi d_{\kappa\lambda} = S^\nu_\pi S^\kappa_\lambda d_{\kappa\lambda} = 1$, so by ($\pi$2), $d_{\kappa\lambda} \leq S^\nu_\pi d_{\kappa\lambda}$. It follows by 2.4(i) and (vi) that $c_\nu d_{\kappa\lambda} \leq c_\nu S^\nu_\pi d_{\kappa\lambda} = S^\nu_\pi d_{\kappa\lambda}$. But $S^\nu_\pi d_{\kappa\lambda} \leq c_\nu d_{\kappa\lambda}$ by 2.4(vii), hence

$$S^\nu_\pi d_{\kappa\lambda} = c_\nu d_{\kappa\lambda} \quad \text{for any } \pi \neq \kappa, \nu. \tag{*}$$

Now by (π7), there is some $\mu < \alpha$ such that $d_{\kappa\lambda} \in \mathrm{rg}\ \mu$; we may assume that $\mu \neq \kappa, \lambda$, for otherwise $d_{\kappa\lambda} = 1$, and therefore (3.4) certainly holds; similarly, we may assume that $\mu \neq \nu$, for otherwise (3.4) holds immediately. Now, let $\pi \neq \kappa, \mu, \nu$; by ($S_6$), $S^\mu_\pi S^\nu_\pi d_{\kappa\lambda} = S^\nu_\pi S^\mu_\pi d_{\kappa\lambda} = S^\nu_\pi d_{\kappa\lambda}$, so $S^\nu_\pi d_{\kappa\lambda} \in \mathrm{rg}\ \mu$. But by (*), $S^\nu_\pi d_{\kappa\lambda} = S^\nu_\mu d_{\kappa\lambda}$, hence $S^\nu_\mu d_{\kappa\lambda} \in \mathrm{rg}\ \mu$. Thus, by ($S_3$) and ($S_4$), $S^\nu_\mu d_{\kappa\lambda} = S^\mu_\nu S^\nu_\mu d_{\kappa\lambda} = S^\mu_\nu d_{\kappa\lambda} = d_{\kappa\lambda}$, so $d_{\kappa\lambda} \in \mathrm{rg}\ \nu$. Next, we have

$$c_\kappa c_\lambda = c_\lambda c_\kappa \quad \text{for all } \kappa, \lambda < \alpha. \tag{3.5}$$

Note first that $c_\kappa c_\lambda x \in \mathrm{rg}\ \lambda$. Indeed, by 2.4(iv), $c_\lambda x \leq c_\kappa c_\lambda x$; but $c_\lambda x \in \mathrm{rg}\ \lambda$ and $c_\kappa c_\lambda x \in \mathrm{rg}\ \kappa$, so by ($\pi$6), there is some $z \in \mathrm{rg}\ \kappa \cap \mathrm{rg}\ \lambda$ such that $c_\lambda x \leq z \leq c_\kappa c_\lambda x$. By (B), $z = c_\kappa c_\lambda x$, hence $c_\kappa c_\lambda x \in \mathrm{rg}\ \lambda$. Now $x \leq c_\lambda x$, so $c_\lambda c_\kappa x \leq c_\lambda c_\kappa c_\lambda x$; but $c_\kappa c_\lambda x \in \mathrm{rg}\ \lambda$, so $c_\lambda c_\kappa c_\lambda x = c_\kappa c_\lambda x$. This shows that $c_\lambda c_\kappa x \leq c_\kappa c_\lambda x$; symmetrically, $c_\kappa c_\lambda x \leq c_\lambda c_\kappa x$, proving (3.5). Finally, if $\mu \neq \kappa, \lambda$, then

$$\begin{aligned} c_\mu S^\kappa_\lambda x &= c_\mu[c_\kappa(x \cdot d_{\kappa\lambda})] && \text{by 2.6(ii),} \\ &= c_\kappa c_\mu(x \cdot d_{\kappa\lambda}) && \text{by 3.5,} \\ &= c_\kappa(c_\mu x \cdot d_{\kappa\lambda}) && \text{by 2.4(v) and 3.4,} \\ &= S^\kappa_\lambda c_\mu x && \text{by 2.6(ii).} \end{aligned}$$

(π8) $\Longrightarrow$ (π3). By (π8), (S_6) and (F), if $\mu \neq \kappa, \lambda$, then $S^\kappa_\lambda c_\mu x = S^\kappa_\lambda S^\mu_\nu T^\mu_\nu x = S^\mu_\nu S^\kappa_\lambda T^\mu_\nu x = S^\mu_\nu T^\mu_\nu S^\kappa_\lambda x = c_\mu S^\kappa_\lambda x$.

(π3) $\Longrightarrow$ (π8). First, we note that in every SA_α admitting conjugates,

$$T^\kappa_\lambda x = c_\kappa x \cdot d_{\kappa\lambda}. \tag{3.6}$$

Indeed, it is easily verified that $c_\kappa x \cdot d_{\kappa\lambda} = \min\{y\colon S^\kappa_\lambda y \geq x\}$, so by (E) we get (3.6). We have already proved that (π3) $\Longrightarrow$ (π6) $\wedge$ (π7), hence we may assume that (3.4) holds. Thus, by (π3), (3.4) and (3.6), $S^\kappa_\lambda T^\mu_\nu x = S^\kappa_\lambda(c_\mu x \cdot d_{\mu\nu}) = S^\kappa_\lambda c_\mu x \cdot S^\kappa_\lambda d_{\mu\nu} = c_\mu S^\kappa_\lambda x \cdot d_{\mu\nu} = T^\mu_\nu S^\kappa_\lambda x$. □

We shall now consider algebras whose primitive operations are the Boolean operations together with substitutions *and their conjugates*. By a *substitution algebra* (of degree α) *with conjugates*, briefly a SAC_α, we mean an algebra $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda, T^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ where $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ is a substitution algebra and T^κ_λ is the conjugate of S^κ_λ for all $\kappa, \lambda < \alpha$. It is obvious that $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ will satisfy the conditions of Theorem 3.2.

The interest of SAC_α is that it is an equational class(6) of algebras. For example, SAC_α may be axiomatized by (S_1)–(S_6) together with

(S_7) $T^\kappa_\lambda(x + y) = T^\kappa_\lambda x + T^\kappa_\lambda y$,

(S_8) $x \leq S^\kappa_\lambda T^\kappa_\lambda x$, and

(S_9) $x \geq T^\kappa_\lambda S^\kappa_\lambda x$.

Indeed, suppose $\mathfrak{A}$ is an SAC_α; then (S_7) holds by Jónsson and Tarski [7, Theorem 1.14]. Next, $-T^\kappa_\lambda x \cdot T^\kappa_\lambda x = 0$, hence $-S^\kappa_\lambda T^\kappa_\lambda x \cdot x = 0$, giving ($S_8$). Finally, $S^\kappa_\lambda x \cdot S^\kappa_\lambda - x = 0$, so $T^\kappa_\lambda S^\kappa_\lambda x \cdot - x = 0$, giving ($S_9$). Conversely, one verifies immediately that if (S_7)–(S_9) hold, then T^κ_λ is the conjugate of S^κ_λ.

From our last remarks, together with Theorem 2.7, it is clear that the theory of substitution algebras with conjugates satisfying (π8) is definitionally equivalent

(6) For the notion of equational class, see, for example, [6, p. 44].

to the theory of cylindric algebras. In fact, they are definitionally equivalent in the strong sense that all added definitions are equational. This raises the question whether it is possible to define a class K of algebras whose only primitive operations are the Boolean operations and substitutions (the conjugates of substitutions are not to be taken as primitive) such that K is an *equational* class and is definitionally equivalent to CA_α.

We will now answer this question in the negative: we will show that any set of axioms for a class of algebras which is definitionally equivalent to CA_α, where each algebra is a structure $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ subject to (S_1)–(S_6) and additional conditions, must include axioms which are not equations.

Let CA^*_α designate the class of all $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha} \in SA_\alpha$ which admit operations c_κ and $d_{\kappa\lambda}$ such that $\langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a cylindric algebra and $S^\kappa_\lambda x = c_\kappa(x \cdot d_{\kappa\lambda})$ for all $x \in A$ and $\kappa \neq \lambda$. We will prove that CA^*_α is not an equational subclass of SA_α.

To prove this, we use the criterion given in Birkhoff [1]: a class K of similar algebras is equational if and only if K is closed under the formation of subalgebras, homomorphic images and arbitrary direct products. For our purposes, it will suffice to give an example of a CA^*_α with a subalgebra which is not a CA^*_α.

3.7. **Example.** Let $\mathfrak{B} = \langle B, \cup, \cap, \sim, 0, {}^\omega\omega, C_\kappa, D_{\kappa\lambda}\rangle_{\kappa,\lambda<\omega}$ be a full cylindric set algebra of dimension ω with base ω (see [6, Definition 1.1.5]). If we define S^κ_λ by

$$S^\kappa_\lambda X = C_\kappa(X \cap D_{\kappa\lambda}) \quad \text{for each } X \in B,$$

then $\mathfrak{B}' = \langle B, \cup, \cap, \sim, 0, {}^\omega\omega, S^\kappa_\lambda\rangle_{\kappa,\lambda<\omega}$ is a CA^*_ω.

Let $P \in B$ be the set of all strictly increasing sequences of natural numbers, that is, $P = \{\mu \in {}^\omega\omega\colon \kappa < \lambda \Rightarrow \mu_\kappa < \mu_\lambda\}$. Let $C = \{0, {}^\omega\omega, P, \sim P\}$; it is easy to see that for all $\kappa, \lambda < \omega$, $S^\kappa_\lambda 0 = 0$, $S^\kappa_\lambda({}^\omega\omega) = {}^\omega\omega$, $S^\kappa_\lambda P = 0$, and $S^\kappa_\lambda \sim P = {}^\omega\omega$. Thus, $\mathfrak{C} = \langle C, \cup, \cap, \sim, 0, {}^\omega\omega, S^\kappa_\lambda\rangle_{\kappa,\lambda<\omega}$ is a subalgebra of $\mathfrak{B}'$. If $\mathfrak{C}$ were a CA^*_ω, it would not be discrete, for $P \neq 0$ and $S^\kappa_\lambda P = 0$; but this contradicts [6, 2.1.9(ii)]; thus $\mathfrak{C}$ is not a CA^*_ω.

3.8. **Remark.** In Example 3.7, one easily verifies that $\mathfrak{C}$ satisfies (π5). But we have shown that $\mathfrak{C}$ is not a CA^*_ω; this observation makes it clear that in Theorem 2.7, condition (π3) is indispensable.

It might be of some interest to consider algebras $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda, c_\kappa\rangle_{\kappa,\lambda<\alpha}$ to be called *substitution algebras with quantifiers*, where $\langle A, +, \cdot, -, 0, 1, S^\kappa_\lambda\rangle_{\kappa,\lambda<\alpha}$ is a substitution algebra and (B) holds. The class of these algebras is easily shown to be equational; for example, it may be axiomatized by (S_1)–(S_6) together with 2.4 (i)–(v); (we omit the simple details). The relationship between these algebras and cylindric algebras are easily derived from Theorem 2.7.

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