HANKEL TRANSFORMS AND GASP

BY

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ABSTRACT. The inversion of the classical Hankel transform is considered from three viewpoints. The first approach is direct, and a theorem is given which allows inversion in the $(C, 1)$ sense under fairly weak hypotheses. The second approach is via Abel summability, and it is shown that inversion is possible if it is known that the Hankel transform is Abel summable and if certain critical growth conditions are satisfied. The third approach rests on the observation that Abel means of Hankel transforms satisfy a variant of the GASP equation in two arguments. In this setting the inversion problem becomes a boundary value problem for GASP in a quadrant of the plane with boundary values on one of the axes; a uniqueness theorem for this problem is proved which is best possible in several respects.

1. **Introduction.** We shall use the notations $S_\nu(z) = z^{1/2} J_\nu(z)$ and

$$P_\nu(s, x, t) = \int_0^\infty e^{-ty} S_\nu(xy) S_\nu(sy) dy$$

for $\nu > -1/2$ and z, s, x and t positive; here J_ν denotes the Bessel function of first kind and order ν. If the function f has domain $E \times (0, \infty)$ we use the notation $\|f(x, t)\| = \sup |f(x, t)|$, the supremum being taken over all x in E for fixed positive t; thus $\|f(x, t)\|$ is a function of t. The main results of this investigation are contained in the following three theorems:

Theorem 1. H. (1) $\nu > -1/2$ *and* g *and* h *are locally integrable over* $[0, \infty)$. (2) h *is finite valued and*

$$\lim_{R \to \infty} \int_0^R g(y) S_\nu(xy) dy = h(x)$$

for each positive x.

C. $g(x) = \lim_{\lambda \to \infty} \lambda^{-1} \int_0^\lambda du \int_0^u h(y) S_\nu(xy) dy$ *a.e.*

Theorem 2a. H. (1) $-1/2 < \nu < 1/2$ *and* g *is locally integrable over* $[0, \infty)$.

(2) $g(y) = o(y^{\nu+1/2})$ *as* $y \to \infty$. (3) h *is a finite valued member of* $L^1(0, \infty)$ *and*

$$\lim_{t \to 0+} \int_0^\infty e^{-ty} g(y) S_\nu(xy) \, dy = h(x)$$

for each positive x.

C. $g(x) = \int_0^\infty h(y) S_\nu(xy) \, dy$ *a.e.*

Theorem 2b. H. *The same as Theorem* 2a *except that* $\nu \geq 1/2$ *and* $g(y) = o(y)$ *as* $y \to \infty$.

C. *The same as Theorem* 2a.

Theorem 3. H. $\nu > 1/2$ *and* $f(x, t)$ *is a* C^2 *solution of the equation* $u_{xx} + u_{tt} = (\nu^2 - 1/4)x^{-2}u$ *in the quadrant* $(0, \infty) \times (0, \infty)$. (2) $\|f(x, t)\|$ *is bounded on compact subsets of* $(0, \infty)$, *is* $o(t)$ *as* $t \to \infty$ *and is* $o(t^{-2})$ *as* $t \to 0$. (3) $f(x)$ *is a finite valued member of* $L^1(0, \infty)$ *and* $\lim_{t \to 0} f(x, t) = f(x)$ *for each positive* x.

C. $f(x, t) = \int_0^\infty f(s) P_\nu(s, x, t) \, ds$ *for all positive* x *and* t.

The second hypothesis of Theorem 1 expresses that h is the (νth) Hankel transform of g. When $\nu = \pm 1/2$ the Hankel transform is a Fourier sine or cosine transform; in this case Theorem 1 is well known and is proved in [7]. Theorem 1 can be proved by adapting the argument given in [7]; since the proof is cumbersome and fairly routine, we shall omit it. If $h(x) \equiv 0$ and $-1/2 < \nu < 1/2$, then Theorem 1 fails if its second hypothesis is interpreted in the $(C, 1)$ sense (for an example, take $g(y) = y^{\nu+1/2}$); however, as shown in [3], this is no longer the case when $\nu = 1/2$; what happens in this regard when $\nu > 1/2$ seems to be unknown. A conjecture is that $(C, \nu + 1/2)$ is appropriate in place of $(C, 1)$ and that this should fail if $(C, \nu + 1/2 + \epsilon)$ is used (I am indebted to the referee for this conjecture).

Theorem 2 is a partial generalization of Theorem 1 and is a variant of known theorems about Fourier series and transforms ([4], [10], [11]). Interestingly, Theorem 2 is best possible with respect to the growth condition on $g(y)$; that is, if o is replaced by O in its hypotheses, the theorem fails. To see this, take $g(y) = y^{\nu+1/2}$ in Theorem 2a and $g(y) = yS_\nu'(y)$ in Theorem 2b; then $h \equiv 0$ but it is not the case that $g(x) = 0$ a.e. We shall obtain Theorem 2 as a corollary of Theorem 3 and its proof.

We shall denote by E_ν the partial differential equation which occurs in Theorem 3; and we shall use the abbreviation $I = (0, \infty) \times (0, \infty)$. It is easily seen that if $f(x, t)$ satisfies E_ν on I then $u(x, t) = x^{-\nu-1/2} f(x, t)$ satisfies the equation

$$u_{xx} + u_{tt} + (2\nu + 1)x^{-1}u_x = 0$$

on I. This latter equation is the equation of generalized axially symmetric potentials (GASP).

Theorem 3 is best possible in several respects. For example, it fails if $\nu = 1/2$, as can be seen by taking $f(x, t) = e^{-x} \sin t$. If $f(x)$ is not finite at one point, Theorem 3 fails by virtue of the example $f(x, t) = P_\nu(1, x, t)$. If $f(x, t) = O(t^{-2})$ as $t \to 0$, the function $f(x, t) = P_{\nu s}(1, x, t)$ provides a counterexample. It is not known whether the hypothesis $\|f(x, t)\| = o(t)$ as $t \to \infty$ can be relaxed.

Theorem 3 was motivated by the work of V. L. Shapiro in [5] and [6]. We remark without proof that Theorem 3 is valid in the case $\nu = 1/2$ if the first quadrant is replaced by the upper half plane and the kernel P_ν is replaced by the usual kernel for harmonic functions in the upper half plane.

2. **Lemmas.** This section is devoted to thirteen lemmas which yield Theorems 2 and 3. Of these lemmas, the first four are easy and the fifth is known. Lemmas 6–11 have more or less direct analogues in [6], but their correct formulation and proof offer some challenge; accordingly, we have tried to emphasize the more difficult steps in proving Lemmas 6–11, leaving the easier steps to the reader. Lemma 10 is proved in detail in order to illustrate the role of Lemmas 8 and 9. Lemmas 12 and 13, which together imply Theorem 3, are less straightforward than the rest.

We introduce some facts and notations. The letter A will be used to denote constants. Legendre's function of second kind and order α is given by

(1)
$$Q_\alpha(w) = \pi^{1/2} \Gamma(\alpha + 1) \Gamma(\alpha + 3/2)^{-1} (2w)^{-\alpha - 1}$$
$$\cdot F(\alpha/2 + 1/2, \alpha/2 + 1; \alpha + 3/2; w^{-2})$$

for $\alpha > -1$ and $w > 1$; here F denotes the hypergeometric series of Gauss. For positive s, x and t we write

(2)
$$z = z(s, x, t) = (s^2 + x^2 + t^2)/(2sx).$$

It is proved in [8] that

(3)
$$P_\nu(s, x, t) = -t(\pi s x)^{-1} Q'_{\nu - 1/2}(z).$$

It is a consequence of (1) that there exist constants A depending only on ν and n such that

(4)
$$|Q_{\nu - 1/2}(w)| \leq A w^{-\nu - 1/2} |\log(w - 1)|$$

and

(5)
$$|Q^{(n)}_{\nu - 1/2}(w)| \leq A w^{-\nu - 1/2} (w - 1)^{-n}$$

hold for $\nu > -1/2$, $w > 1$ and $n = 1, 2, \cdots$. It is clear from (1) that, when $\nu > -1/2$,

(6)
$$Q_{\nu - 1/2}(w) = O(w^{-\nu - 1/2}) \quad \text{as} \quad w \to \infty.$$

Lemma 1. H. (1) $f(x, t)$ *is continuous for* $x > 0$, $t \geq 0$ *and satisfies* E_ν *on*

I, where $\nu > -1/2$. (2) *For each* $T > 0$, $f(x, t)$ *is bounded on the strip* $0 \leq t \leq T$, $x > 0$. (3) $\|f(x, t)\| = o(t)$ *as* $t \to \infty$.

C. *If* $f(x, 0) = 0$ *for each positive* x, *then* $f(x, t) = 0$ *on I*.

We omit the proof of Lemma 1, which is easily carried out with the aid of the comparison function $h_\epsilon(x, t) = f(x, t) - \epsilon(x^{\nu+1/2}x^{-\nu+1/2})(1 + t)$, which is a solution of E_ν on I for each $\epsilon > 0$.

Lemma 2. H. (1) $\nu > -1/2$ *and* g *is continuous on* $(0, \infty)$. (2) *There exist positive numbers* α *and* β *such that* $g(s) = O(s^{\nu+3/2-\beta})$ *as* $s \to \infty$ *and* $g(s) = O(s^{-\nu-3/2+\alpha})$ *as* $s \to 0$. (3) $f(x, t) = \int_0^\infty g(s) P_\nu(s, x, t) ds$ *on I*.

C. (1) $f(x, t)$ *is a* C^2 *solution of* E_ν *on I*. (2) *For each* $x_0 > 0$, $f(x, t) \to g(x_0)$ *as* $(x, t) \to (x_0, 0)$. (3) *If* g *is bounded on* $(0, \infty)$ *then* $f(x, t)$ *is bounded on I*.

Since the proof of Lemma 2 is a straightforward application of (2)–(6), we omit it. An immediate consequence of Lemmas 1 and 2 is

Lemma 3. *Under the hypotheses of Lemma 1,* $f(x, t) = \int_0^\infty f(s, 0) P_\nu(s, x, t) ds$ *on I*.

Lemma 4. *If* $\nu \geq 1/2$ *there is a constant* A, *depending only on* ν, *such that* $\int_0^\infty |P_{\nu x}(s, x, t)| ds \leq A/t$ *on I*

Proof. Using (3) and (5) we obtain that

$$\int_0^\infty |P_{\nu x}(s, x, t)| ds$$

$$\leq Ax^{-2}t \int_0^\infty [sz(z-1)]^{-1} ds + Atx^{-3} \int_0^\infty [s^2 z(z-1)^2]^{-1}|s^2 + t^2 - x^2| ds$$

$$\equiv I_1 + I_2.$$

It is easy to see that $z \geq (x^2 + t^2)^{1/2}/x$; and it follows that

$$I_1 \leq At(x^2 + t^2)^{-1/2} \int_0^\infty [(s - x)^2 + t^2]^{-1} ds \leq A/t.$$

A similar estimate shows that $I_2 \leq A/t$ and Lemma 4 follows.

We introduce the difference quotient

(7)
$$G_2(x, h, F)$$
$$= \{[x + (\nu + 1/2)h] F(x + h) + [x - (\nu + 1/2)h] F(x - h) - 2xF(x)\}/h^2$$

and denote its lim sup and lim inf as $h \to 0$ by $G_2^* F(x)$ and $G_{2*} F(x)$ respectively. In general, for functions $f(x, t)$ we abbreviate

$$f^*(x) = \lim_{t \to 0+} \sup f(x, t) \quad \text{and} \quad f_*(x) = \lim_{t \to 0+} \inf f(x, t).$$

Lemma 5 is proved in [4].

Lemma 5. H. (1) $\nu > -1/2$, $0 < a < b$ and $F(x)$ is continuous on (a, b). (2) On (a, b), $f(x)$ is finite valued and integrable and $G_{2*}F(x) \leq f(x) \leq G_2^* F(x)$.

C. There are constants p and q such that

$$F(x) = px^{-2\nu} + q + \int_a^x r^{-2\nu-1}\, dr \int_a^r s^{2\nu} f(s)\, ds$$

holds on (a, b); when $\nu = 0$ the term $px^{-2\nu}$ should be replaced by $p \log x$.

Lemma 6. H. (1) $s^{\nu+1/2} F(s)$ is continuous and bounded on $(0, \infty)$, where $\nu > -1/2$. (2) $F(x, t) = x^{-\nu-1/2} \int_0^\infty s^{\nu+1/2} F(s)\, P_\nu(s, x, t)\, ds$ on I.

C. For each positive x, $G_{2*}F(x) \leq x \lim \sup_{t \to 0} [- F_{tt}(x, t)]$ and $G_2^* F(x) \geq x \lim \inf_{t \to 0} [- F_{tt}(x, t)]$.

Proof. We observe that it suffices to prove the first inequality of the conclusion. For brevity we denote $G(t) = F(x_0, t)$, $f(x, t) = -x^{\nu+1/2} F_{tt}(x, t)$ and $R_\nu(w) = (w^2 - 1)^2 Q''_{\nu-1/2}(w)$. Fix δ in the interval $(0, x_0/4)$. It is a routine matter to establish that

$$(8) \qquad R_\nu(w) = 2 + (9/4 - \nu^2)(w - 1) + O((w - 1)^{3/2}) \quad \text{as } w \to 1+.$$

In this proof we use $o(1)$ to denote any function which tends to zero with t. The key to this proof lies in establishing that

$$
\begin{aligned}
(9) \qquad & [G(t)/t]' + 4(\pi t^2)^{-1} F(x_0) \int_0^{\delta/t} (1 + s^2)^{-1}\, ds \\
& + 2t (\pi x_0)^{-1} \int_0^\delta u^2(t^2 + u^2)^{-2} G_2(x_0, u, F)\, du = o(1).
\end{aligned}
$$

To prove (9) we use (1)–(8) to write

$$
\begin{aligned}
[G(t)/t]' &= -tx_0^{-\nu-5/2} \pi^{-1} \int_0^\infty s^{\nu-3/2} F(s)(z^2 - 1)^{-2} R_\nu(z)\, ds \\
&= o(1) + tx_0^{-\nu-5/2}(4\pi)^{-1} \int_{x_0-\delta}^{x_0+\delta} s^{\nu-3/2} F(s) R_\nu(z)[(z-1)^{-1} - (z-1)^{-2}]\, ds \\
&= o(1) + tx_0^{-\nu-5/2}(4\pi)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
(10) \qquad & \int_{x_0-\delta}^{x_0+\delta} s^{\nu-3/2} F(s)\, [(\nu^2 - 1/4)(z - 1)^{-1} - 2(z - 1)^{-2}]\, ds \\
&= o(1) + x_0^{-\nu-3/2}(\nu^2 - 1/4) \\
& (2\pi)^{-1} \int_{-\delta/t}^{\delta/t} (x_0 + ut)^{\nu-1/2}(u^2 + 1)^{-1} F(x_0 + ut)\, du \\
& - 2tx_0^{-\nu-1/2}\pi^{-1} \int_{-\delta}^\delta (x_0 + u)^{\nu+1/2}(u^2 + t^2)^{-2} F(x_0 + u)\, du.
\end{aligned}
$$

Now (9) is obtained from (10) by applying the binomial theorem to $(x_0 + ut)^{\nu-1/2}$ and $(x_0 + u)^{\nu+1/2}$.

Now assume, to obtain a contradiction, that

(11) $G_{2*}F(x_0) > m > x_0^{-\nu+1/2} f^*(x_0)$ for some real m.

It follows from (11) that there exist positive numbers ϵ, t_0 and δ such that $0 < \delta < x_0/4$ and

(12) $G_2(x_0, u, F) \geq m + \epsilon > m - \epsilon \geq x_0^{-\nu+1/2} f(x_0, t)$

for $0 < u \leq \delta$ and $0 < t \leq t_0$. Because $G''(t) = -x_0^{-\nu-1/2} f(x_0, t)$, (12) yields that $G''(t) \geq (\epsilon - m)/x_0$ for $0 < t < t_0$ and it follows that

(13) $G'(t) + mt/x_0$ is a strictly increasing function of t in the interval $(0, t_0)$.

By Lemma 2, $G(t)$ is continuous on $[0, t_0]$ if we define $G(0) = F(x_0)$; so, using the mean value theorem twice and (13), we conclude that for each t in $(0, t_0)$ there exists a number ρ in $(0, t)$ such that

(14) $$\frac{G(t) - G(0)}{t} + \frac{mt}{2x_0} > \frac{G(\rho) - G(0)}{\rho} + \frac{m\rho}{2x_0}.$$

But a straightforward computation based on (9) and the first inequality in (12) shows that

(15) $$\limsup_{t \to 0} \frac{d}{dt}\left[\frac{G(t) - G(0)}{t} + \frac{mt}{2x_0}\right] < 0.$$

Now (14) and (15) are contradictory, so Lemma 6 is proved.

Lemma 7. H. (1) $f(x, t)$ is a C^2 solution of E_ν on I, with $\nu > 1/2$. (2) $\|f(x, t)\|$ is bounded on compact subsets of $(0, \infty)$, is $o(t^{-2})$ as $t \to 0$ and is $O(t^{-5})$ as $t \to \infty$. (3) $F(x, t) = -x^{-\nu-1/2} \int_t^\infty dr \int_r^\infty f(x, s)\,ds$ on I. (4) $0 < a < b$ and $\lim_{t \to 0} F(x, t) = F(x)$ exists and is continuous on (a, b).
    C. (1) $G_{2*} F(x) \leq x^{-\nu+1/2} f^*(x)$ and $G_2^* F(x) \geq x^{-\nu+1/2} f_*(x)$ on (a, b). (2) If $a < \alpha < \beta < b$ then $F(x, t) \to F(x)$ uniformly on $[\alpha, \beta]$ as $t \to 0$.

Proof. By Lemma 3,

(16) $$f(x, s) = \int_0^\infty f(u, s/2)\,P_\nu(u, x, s/2)\,du$$

holds for (x, s) in I. Substitution of (16) into the formula of our third hypothesis and application of Lemma 4 allows us to differentiate with respect to x under the integral sign to conclude that

                $x^{\nu+1/2}F(x, t)$ is a continuous solution of E_ν and

(17) $x^{\nu+1/2}F_{tt}(x, t) = -f(x, t)$ on I.

Next we define

(18) $$h(x, t) = -\int_t^\infty dr \int_r^\infty x^{\nu+1/2} F(x, s) ds$$

on I. An argument similar to that which establishes (17) yields that

(19) $h(x, t)$ is a continuous solution of E_ν and
         $h_{tt}(x, t) = -x^{\nu+1/2} F(x, t)$ on I.

It follows easily from our growth assumptions on $f(x, t)$ that $h(x, t)$ is bounded on I and that, for t_1 and t_2 in an interval $(0, t_0)$, $|h(x, t_1) - h(x, t_2)| \le A|t_1 - t_2|$, where A depends on t_0 but not on x. Therefore there is a bounded continuous function h on $(0, \infty)$ such that

(20) $h(x, t) \to h(x)$ uniformly on $(0, \infty)$ as $t \to 0$.

So, by Lemma 3, we have that on I,

(21) $$h(x, t) = \int_0^\infty h(s) P_\nu(s, x, t) ds.$$

Next we conclude from (21), Lemma 6 and (19) that

(22) $$G_{2*}[x^{-\nu-1/2} h(x)] \le xF(x) \le G_2^*[x^{-\nu-1/2} h(x)].$$

Now (22) and Lemma 5 yield that there exist constants p_2 and q_2 such that

(23) $$x^{-\nu-1/2} h(x) = p_2 x^{-2\nu} + q_2 + \int_\alpha^x r^{-2\nu-1} dr \int_\alpha^r s^{2\nu+1} F(s) ds$$

holds for $\alpha \le x \le \beta$.

Let $F_1(x)$ be a bounded continuous function on $(0, \infty)$ which agrees with $F(x)$ on $[\alpha, \beta]$ and vanishes outside (a, b). For (x, t) in I define

$$F_1(x, t) = \int_0^\infty F_1(s)(s/x)^{\nu+1/2} P_\nu(s, x, t) ds \quad \text{and}$$

$$h_1(x, t) = x^{\nu+1/2} \int_t^\infty dr \int_r^\infty F_1(x, s) ds.$$

Then arguments paralleling those of the preceding paragraph yield that (19)–(23) hold with h_1 in place of h, F_1 in place of F and p_1, q_1 in place of p_2, q_2. Next we define $h_2(x, t) = h(x, t) - h_1(x, t)$, $h_2(x) = h(x) - h_1(x)$, $p = p_2 - p_1$, $q = q_2 - q_1$. It follows that $h_2(x)$ is bounded and continuous on $(0, \infty)$,

(24) $$h_2(x) = px^{-\nu+1/2} + qx^{\nu+1/2} \quad \text{on } [\alpha, \beta],$$

and

(25) $$h_2(x, t) = \int_0^\infty h_2(s) P_\nu(s, x, t) ds \quad \text{on } I.$$

We propose to establish that, for $\alpha < x < \beta$,

(26) $$\partial^4 h_2(x, t)/\partial t^4 \to 0 \quad \text{as } t \to 0.$$

Note that once (26) is proved the first conclusion of Lemma 7 follows easily.

To prove (26), fix x_0 in (α, β). Differentiation of (3) yields that

$$-\pi \frac{\partial^4}{\partial t^4} P_\nu(s, x, t) = t^5 (sx)^{-5} Q^{(5)}_{\nu-1/2}(z) + 10 t^3 (sx)^{-4} Q^{(4)}_{\nu-1/2}(z)$$

(27)

$$+ 15 t (xs)^{-3} Q^{(3)}_{\nu-1/2}(z).$$

Choose δ so that $\alpha < x_0 - \delta < x_0 + \delta < \beta$. Then it follows from (27) and (5) that

$$\int_{|s-x_0| \geq \delta} b_2(s) \frac{\partial^4}{\partial t^4} P_\nu(s, x_0, t) \, ds \quad \text{and}$$

(28)

$$\int_{|s-x_0| \geq \delta} (ps^{-\nu+1/2} + qs^{\nu+1/2}) \frac{\partial^4}{\partial t^4} P_\nu(s, x_0, t) \, ds$$

tend to zero with t.

By Lemma 1,

$$e^{-tr} S_\nu(rx_0) r^{-\nu-1/2} = \int_0^\infty S_\nu(rs) r^{-\nu-1/2} P_\nu(s, x_0, t) \, ds$$

for positive r and t; letting $r \to 0$ we obtain by dominated convergence that

(29) $$\int_0^\infty s^{\nu+1/2} P_\nu(s, x_0, t) \, ds = x_0^{\nu+1/2}.$$

We obtain by definition of P_ν, Fubini's theorem and dominated convergence that

(30) $$\int_0^\infty s^{-\nu+1/2} P_\nu(s, x_0, t) \, ds = 2^{1-\nu} \Gamma(\nu)^{-1} \int_0^\infty e^{-ty} y^{\nu-1} J_\nu(x_0 y) \, dy.$$

It follows from (30) and [8, p. 386] that

(31) $$\frac{\partial^4}{\partial t^4} \int_0^\infty s^{-\nu+1/2} P_\nu(s, x_0, t) \, ds = o(1) \quad \text{as } t \to 0.$$

Finally, (26) follows from (28), (29), (31) and the fact that four differentiations with respect to t are permissible under the integral sign in (25).

To prove the second conclusion of Lemma 7, it suffices to prove that if $\alpha < \alpha_1 < \beta_1 < \beta$ then $F_1(x, t) - F(x, t) \to 0$ uniformly on $[\alpha_1, \beta_1]$ as $t \to 0$. Since $F_1(x, t) - F(x, t) = x^{-\nu-1/2} b_{2tt}(x, t)$ on I, we see from (25) that we shall be done once we show that $b_{2tt}(x, t) \to 0$ uniformly on $[\alpha_1, \beta_1]$ as $t \to 0$; the proof of this last assertion is very similar to the proof of (26), so we omit it. This completes our proof of Lemma 7.

Smoothness is defined in [11].

Lemma 8. H. (1) *The first three hypotheses of Lemma 7.*

(2) $$G(x, t) = - \int_t^\infty dr \int_r^\infty x^{2\nu+1} F_x(x, s) \, ds \quad \text{on } I.$$

C. $\lim_{t \to 0} G(x, t) = G(x)$ *exists and is continuous and smooth on* $(0, \infty)$.

Proof. Fix $x_0 > 0$ and define $\theta(s) = \|f(x, s)\| + \|f_x(x, s)\|$, $\beta(x, s) = x^{2\nu+1}F_x(x, s)$ and

$$\delta(s, h) = h^{-1}[\beta(x_0 + h, s) + \beta(x_0 - h, s) - 2\beta(x_0, s)] \quad \text{for } s > 0, \; x > 0$$

and $0 < h < x_0/2$. Using (16), Lemma 4 and our growth assumptions on $f(x, t)$ we obtain that

$$(32) \qquad \theta(s) = o(s^{-3}) \quad \text{as } s \to 0 \quad \text{and} \quad \theta(s) = O(s^{-5}) \quad \text{as } s \to \infty.$$

Using (16) again, we see that if K is a compact subset of $(0, \infty)$ then there is a constant A, independent of x and s, such that

$$(33) \qquad |\beta(x, s)| \leq A \int_s^\infty dr \int_r^\infty \theta(u)\, du$$

for x in K and $s > 0$. It follows easily from (32) and (33) that $G(x, t) \to G(x)$ uniformly on K as $t \to 0$, where

$$(34) \qquad G(x) = - \int_0^\infty dr \int_r^\infty \beta(x, s)\, ds$$

is continuous on $(0, \infty)$.

To prove that $G(x)$ is smooth at x_0 we write

$$h^{-1}|G(x_0 + h) + G(x_0 - h) - 2G(x_0)| \leq \int_0^\infty \int_r^\infty |\delta(s, h)|\, ds\, dr$$

$$(35) \qquad = \int_{t_0}^\infty \int_r^\infty + \int_h^{t_0} \int_{t_0}^\infty + \int_h^{t_0} \int_r^{t_0} + \int_0^h \int_r^h + \int_0^h \int_h^{t_0} + \int_0^h \int_{t_0}^\infty$$

$$\equiv I_1 + \cdots + I_6,$$

where $0 < h < t_0 < x_0/2$. By (17) we have that

$$(36) \qquad \delta(s, h) = \int_0^h dy \int_{x_0-y}^{x_0+y} \frac{\partial}{\partial r}[r^{\nu+1/2}f(r, s)]\, dr.$$

And (36) yields that

$$(37) \qquad |\delta(s, h)| \leq Ah\theta(s)$$

for $s > 0$ and $0 < h < x_0/2$, where A is independent of s and h. It is now a routine matter to use (32), (33), (35) and (37) to prove that $G(x)$ is smooth at x_0; we omit the details of this and consider Lemma 8 proved.

Lemma 9. H. (1) *The hypotheses of Lemma 8.* (2) $x_0 > 0$ *and* $\lim_{t \to 0} f(x_0, t)$ *exists.*

C. $\lim_{h \to 0}[G(x_0 + h) - G(x_0 - h)]/(2h) = x_0^{2\nu+1}F(x_0)$, *where* $F(x_0)$ *denotes* $\lim_{t \to 0} F(x_0, t)$.

Proof. We keep the notation of the proof of Lemma 8. Let $\Delta(s, h) =$

$x_0^{\nu+1/2} f(x_0, s) - (2h)^{-1}[\beta(x_0 + h, s) - \beta(x_0 - h, s)]$. Then

$$(2h)^{-1}[G(x_0 + h) - G(x_0 - h)] - x_0^{2\nu+1}F(x_0) = \lim_{t \to 0} \int_t^\infty dr \int_r^\infty \Delta(s, h)\,ds.$$

It suffices to prove that

(38) $$\int_0^\infty dr \int_r^\infty |\Delta(s, h)|\,ds = o(1) \quad \text{as } h \to 0.$$

We write

$$\int_0^\infty dr \int_r^\infty |\Delta(s, h)|\,ds$$

(39)
$$\leq \left(\int_{t_0}^\infty \int_r^\infty + \int_h^{t_0} \int_r^{t_0} + \int_0^{t_0} \int_{t_0}^\infty\right)(|\Delta(s, h)|\,ds\,dr)$$

$$+ \int_0^h dr \int_r^{t_0} x_0^{\nu+1/2}|f(x_0, s)|\,ds$$

$$+ \int_0^h dr \int_r^{t_0} (2h)^{-1}|\beta(x_0 + h, s) - \beta(x_0 - h, s)|\,ds$$

$$\equiv J_1 + \cdots + J_5.$$

By the mean value theorem there exist numbers x and r in the interval $(x_0 - h, x_0 + h)$ such that $\Delta(s, h) = x_0^{\nu+1/2}f(x_0, s) - \beta_x(x, s) = x_0^{\nu+1/2}f(x_0, s) - x^{\nu+1/2}f(x, s) = (x_0 - x)\partial[r^{\nu+1/2}f(r, s)]/\partial r$; and it follows that

(40) $$|\Delta(s, h)| \leq Ah\theta(s).$$

Again by the mean value theorem there exists a number y in $(x_0 - h, x_0 + h)$ such that

(41)
$$(2h)^{-1}|\beta(x_0 + h, s) - \beta(x_0 - h, s)| = |\beta_x(y, s)|$$
$$= |y^{\nu+1/2}f(y, s)| \leq A \|f(y, s)\|.$$

Then, as in the proof of Lemma 8, it is a routine matter to establish (38), using (32), (33), (40) and (41); we again omit the details and consider Lemma 9 proved.

Lemma 10. H. (1) *The first three hypotheses of Lemma* 7. (2) $f(s)$ *is locally integrable on* $[0, \infty)$ *and* $\lim_{t \to 0} f(x, t) = f(x)$ *exists for each positive* x. (3) $\lim_{t \to 0} F(x, t) = F(x)$ *is continuous on* (a, b) *where* $0 < a < b$.
    C. $F(x)$ *is continuous on* $[a, b]$.

    Proof. By Lemma 5 and the first conclusion of Lemma 7, there exist constants p and q such that

$$F(x) = px^{-2\nu} + q + \int_a^x r^{-2\nu-1}\,dr \int_a^r s^{\nu+1/2}f(s)\,ds \quad \text{on } (a, b).$$

Therefore $F(a+) = \lim_{x \to a+} F(x)$ exists. It is a consequence of (19) that $G_x(x, t)$ $= x^{2\nu+1} F(x, t)$ on I; therefore $G(\beta, t) - G(\alpha, t) = \int_\alpha^\beta x^{2\nu+1} F(x, t) \, dx$ holds for $a < \alpha < \beta < b$ and $t > 0$. Letting t tend to zero we obtain that $G(\beta) - G(\alpha) =$ $\int_\alpha^\beta x^{2\nu+1} F(x) \, dx$ by Lemma 8 and the second conclusion of Lemma 7. Since $G(x)$ is continuous at a (by Lemma 8) and $F(a+)$ exists, we may let $\alpha \to a+$ to obtain that $G(\beta) - G(a) = \int_a^\beta x^{2\nu+1} F(x) \, dx$ for $a \leq \beta < b$. Therefore $[G(\beta) - G(a)]/(\beta - a)$ $\to a^{2\nu+1} F(a+)$ as $\beta \to a+$; since $G(x)$ is smooth at a (Lemma 8), it follows that $G'(a) = a^{2\nu+1} F(a+)$. By the preceding sentence and Lemma 9 we see that $F(a+) = F(a)$. The proof that $F(b-) = F(b)$ is similar. So Lemma 10 is established.

Lemma 11. *Under the first two hypotheses of Lemma* 10, $\lim_{t \to 0} F(x, t) =$ $F(x)$ *is continuous on* $(0, \infty)$.

Proof. Let $0 < a < b$ and let Z be the set of points in (a, b) at which $F(x)$ is not continuous. Assume, to get a contradiction, that Z is nonempty. We choose a sequence $t_1 > t_2 > \cdots$ such that $t_n \to 0$ and $|f(x, t) - f(x, t_n)| \leq 1$ whenever $x \in [a, b]$ and $t \in [t_{n+1}, t_n]$. By a familiar consequence of Baire's category theorem there exist an open subinterval J of (a, b) and a number M such that $J\overline{Z}$ is nonempty and $|f(x, t_n)| \leq M$ holds for x in $J\overline{Z}$ and $n = 1, 2, \cdots$. It follows that if x is in $J\overline{Z}$ then $|F(x, t_n) - F(x, t_m)| \leq A|t_n - t_m|$. Therefore $F(x, t_n)$ converges uniformly on $J\overline{Z}$ as $n \to \infty$; so $F(x)$ is continuous on the subspace $J\overline{Z}$ of $(0, \infty)$.

Next fix $\alpha > 0$ and define $F_1(x) = \int_\alpha^x r^{-2\nu-1} \, dr \int_\alpha^r s^{\nu+1/2} f(s) \, ds$ and $F_2(x) =$ $F(x) - F_1(x)$ on $(0, \infty)$. If K is any subinterval of J which is contiguous to $\overline{Z}$ then it follows from Lemmas 5, 7 and 10 that there exist constants p and q (depending on K) such that $F_2(x) = px^{-2\nu} + q$ for x in $\overline{K}$. And it follows from the preceding sentence and the previously noted fact that $F(x)$ is continuous on the space $J\overline{Z}$ that $F(x)$ is continuous on J, a contradiction. This proves Lemma 11.

Lemma 12. H. (1) *The hypotheses of Theorem* 3. (2) *There exist* C^2 *solutions* $f_1(x, t)$ *and* $f_2(x, t)$ *of* E_ν *such that*
    (i) $f(x, t) = f_1(x, t) + f_2(x, t)$ *on* I;
    (ii) $f_2(x, t)$ *is bounded on* I *and* $\lim_{(x,t) \to (x_0, 0)} f_2(x, t) = f_2(x_0)$ *exists for each* $x_0 > 0$;
    (iii) $\|f_1(x, t)\| = O(t^{-5})$ *as* $t \to \infty$.
    C. *The conclusion of Theorem* 3.

Proof. Define $g(x, t) = -\int_t^\infty dr \int_r^\infty f_1(x, s) \, ds$ and $h(x, t) = -\int_t^\infty dr \int_r^\infty g(x, s) \, ds$ on I. Let $g(x)$, $h(x)$ and $f_1(x)$ denote the limits as $t \to 0$ of $g(x, t)$, $h(x, t)$ and $f_1(x, t)$. Let α be fixed and positive. Then by Lemmas 11, 7, and 5 there exist constants p and q such that

$$(42) \quad g(x) = px^{\nu+1/2} + qx^{-\nu+1/2} + x^{\nu+1/2} \int_\alpha^x r^{-2\nu-1} \, dr \int_0^r s^{\nu+1/2} f_1(s) \, ds$$

holds for $x > 0$. By the proof of Lemma 7 we have that $b(x)$ is bounded and continuous on $(0, \infty)$ and that

(43) $$b''(s) - (\nu^2 - 1/4)\, s^{-2} b(s) = g(s) \quad \text{for } s > 0$$

and

(44) $$g(x, t) = - b_{tt}(x, t) = - \int_0^\infty b(s)\, P_{\nu tt}(s, x, t)\, ds \quad \text{on } I.$$

Using (43), (44), the fact that P_ν satisfies E_ν on I, and integration by parts we obtain that

$$g(x, t) = \int_0^\infty g(s)\, P_\nu(s, x, t)\, ds$$

(45)
$$+ \int_0^\infty [b(s)\, P_{\nu ss}(s, x, t) - b''(s)\, P_\nu(s, x, t)]\, ds$$

$$= \int_0^\infty g(s)\, P_\nu(s, x, t)\, ds.$$

Now define

$$F_2(s) = s^{\nu + 1/2} \int_a^s r^{-2\nu - 1}\, dr \int_0^r u^{\nu + 1/2} f_2(u)\, du$$

and

$$F(s) = s^{\nu + 1/2} \int_a^s r^{-2\nu - 1}\, dr \int_0^r u^{\nu + 1/2} f(u)\, du$$

for $s > 0$. Since $f_1(x, t) = - g_{tt}(x, t)$, it follows from (42) and (45) that

$$f_1(x, t) = -q \frac{\partial^2}{\partial t^2} \int_0^\infty s^{-\nu + 1/2} P_\nu(s, x, t)\, ds$$

(46)
$$+ \frac{\partial^2}{\partial t^2} \int_0^\infty [F_2(s) - F(s)]\, P_\nu(s, x, t)\, ds.$$

From Lemma 3 and partial integration we obtain that

$$f_2(x, t) = \int_0^\infty f_2(s)\, P_\nu(s, x, t)\, ds = - \int_0^\infty F_2(s)\, P_{\nu tt}(s, x, t)\, ds$$

(47)
$$= \frac{-\partial^2}{\partial t^2} \int_0^\infty F_2(s)\, P_\nu(s, x, t)\, ds.$$

Adding (46) and (47) and then integrating by parts, we have that

(48) $$f(x, t) = -q \frac{\partial^2}{\partial t^2} \int_0^\infty s^{-\nu + 1/2} P_\nu(s, x, t)\, ds + \int_0^\infty f(s)\, P_\nu(s, x, t)\, ds.$$

Finally, we conclude from (48), (30) [8, p. 386] and the assumption that $\|f(x, t)\| = o(t^{-2})$ as $t \to 0$ that $q = 0$. And Lemma 12 is immediate from (48) and the fact that $q = 0$.

Lemma 13. *The hypotheses of Theorem 3 imply the second hypothesis of Lemma 12.*

Proof. For (x, t) in I we write $g_0(x, t) = f(x, t)$ and define

$$(49) \qquad g_n(x, t) = e^t \int_t^\infty e^{-s} g_{n-1}(x, s)\, ds, \qquad n = 1, 2, \cdots.$$

By Lemma 3, $f(x, t) = \int_0^\infty f(s, t_0) P_\nu(s, x, t - t_0)\, ds$ holds for $t > t_0 > 0$ and $x > 0$; it follows that $\|f(x, t)\| = O(1)$ as $t \to \infty$. The following facts are easy consequences of this last remark and the hypotheses of Theorem 3:

$g_n(x, t)$ is a C^2 solution of E_ν on I for $n \geq 0$;

$\|g_n(x, t)\| = O(1)$ as $t \to \infty$ for $n \geq 0$;

$\|g_1(x, t)\| = o(t^{-1})$ as $t \to 0$, $\|g_2(x, t)\| = o(\log 1/t)$ as $t \to 0$, and $g_n(x, t)$ is bounded on I for $n \geq 3$;

for $n \geq 3$, $g_n(x, t)$ converges uniformly on compact subsets of $(0, \infty)$ as $t \to 0$;

$g_{nt}(x, t) = g_n(x, t) - g_{n-1}(x, t)$ for $n \geq 1$ and (x, t) in I.

It follows that

$$f(x, t) = \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\partial^j}{\partial t^j} g_n(x, t)$$

and

$$\frac{\partial^j}{\partial t^j} g_n(x, t) = \sum_{k=0}^j (-1)^k \binom{j}{k} g_{n-k}(x, t)$$

for (x, t) in I and $0 \leq j \leq n$. We assume henceforth that $n \geq 7$ and define

$$(50) \qquad f_2(x, t) = \sum_{j=0}^{n-3} (-1)^j \binom{n}{j} \frac{\partial^j}{\partial t^j} g_n(x, t)$$

and $f_1(x, t) = f(x, t) - f_2(x, t)$. It is clear from the above that $f_2(x, t)$ is as required. So we shall be done once we show that

$$(51) \qquad \left\| \frac{\partial^j}{\partial t^j} g_n(x, t) \right\| = O(t^{-j}) \quad \text{as } t \to \infty$$

for $j = 0, 1, 2, \cdots$.

For $j = 0$, (51) is true by Lemma 3. Since $P_{\nu t}(s, x, t) = t^{-1} P_\nu(s, x, t) - \pi^{-1} t^2 (sx)^{-2} Q''_{\nu-1/2}(z)$, (51) will be true by induction on j if we can prove that

$$(52) \qquad \left\| \int_0^\infty |t^m (sx)^{-m} Q^{(m)}_{\nu-1/2}(z)|\, ds \right\| = O(t^{1-m}) \quad \text{as } t \to \infty$$

for $m = 2, 3, 4, \cdots$. But (52) follows easily from (5). This completes our proof of Lemma 13.

3. **Proof of Theorem** 2. Since Theorem 2b is known when $\nu = 1/2$ [10] and follows from Theorem 3 otherwise, we need to prove Theorem 2a.

Let $g_1(y) = 0$ for $0 < y < 1$ and $g_1(y) = g(y)$ for $y \geq 1$. For (x, t) in I define

$$F(x, t) = x^{-\nu-1/2} \int_0^\infty e^{-ty} y^{-2} g_1(y) S_\nu(xy)\, dy \quad \text{and} \quad F(x) = \lim_{t \to \infty} F(x, t).$$

By Theorem 1,

$$(53) \qquad g_1(y) = -y^2 \lim_{\lambda \to \infty} \lambda^{-1} \int_0^\lambda du \int_0^u F(s) s^{\nu+1/2} S_\nu(sy)\, ds \quad \text{a.e.}$$

Integrating twice by parts in (53) and using Lemmas 5 and 6 we obtain that there exists a constant p such that

$$g(y) = p y^{\nu+1/2} + \int_0^\infty h(x) S_\nu(xy)\, dy \quad \text{a.e.;}$$

it follows from our assumptions on g and h that $p = 0$, which proves Theorem 2a.

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