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LOCAL FINITE COHESION(1)

BY

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ABSTRACT. Local finite cohesion is a new condition which provides a general topological setting for some useful theorems. Moreover, many spaces, such as the product of any two nondegenerate generalized Peano continua, have the local finite cohesion property. If X is a locally finitely cohesive, locally compact metric space, then the complement in X of a totally disconnected set has connected quasicomponents; connectivity maps from X into a regular T_1 space are peripherally continuous; and each connectivity retract of X is locally connected. Local finite cohesion is weaker than finite coherence [4], although these conditions are equivalent among planar Peano continua. Local finite cohesion is also implied by local cohesiveness [12] in locally compact T_2 spaces, and a converse holds if and only if the space is also rim connected. Our study answers a question of Whyburn about local cohesiveness.

1. Basic properties.

Definition. A topological space is rim connected at a point p if, for any open set U containing p, there is an open connected set V containing p, such that $\overline{V} \subset U$ and the boundary of V (Fr V) is connected.

Definition. If X is a connected subset of a topological space, the statement that $X = A + B$ is a representation will mean that A and B are closed (in X) connected sets whose union is X.

Definition. A topological space X is locally finitely cohesive at a point p if for any open neighborhood U of p, there is a connected open set V about p with $\overline{V} \subset U$, and an integer n such that, for any representation $\overline{V} = A + B$, the set $A \cap B$ has no more than n components which do not meet Fr V. V is termed a k-canonical region, where k is assumed to be the least of all integers n which meet the above requirement. A space which is locally finitely cohesive at each of its points has local finite cohesion.

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By definition, local finite cohesion implies local connectedness. However, even a rim connected Peano continuum may fail to have local finite cohesion.

Example 1.1. If $W = \{(x, y) \in E^2: x \geq |y| \text{ and } 0 \leq x \leq 1\}$ and the set $D_n = \{(x, y) \in E^2: (x - 1/n)^2 + y^2 < 1/8^n\}$, then $W - \bigcup_{n=1}^{\infty} D_n$ is a planar rim connected Peano continuum which fails to have local finite cohesion at the point (0, 0).

Lemma 1.2. *If U is a k-canonical region in X, $\overline{U}$ is compact and T_2, and V is a connected open set with $\overline{V} \subset U$, then V is an n-canonical region, $n \leq k$.*

Proof. Let $\overline{V} = E + F$ be a representation such that $E \cap F$ has more than k components which miss Fr V. If a component of $\overline{U} - V$ does not meet Fr V, then a separation of $\overline{U}$ will result between $\overline{V}$ and that component of $\overline{U} - V$. Therefore, both $A = E \cup \{$union of all components of $\overline{U} - V$ which meet $E\}$ and $B = F \cup \{$union of all components of $\overline{U} - V$ which meet $F\}$ are connected sets, and $\overline{U} = \overline{A} + \overline{B}$ is a representation. The set $\overline{A} \cap \overline{B}$ contains as components all those components of $E \cap F$ which miss Fr V, which by assumption are more numerous than k, contradicting the fact that U is a k-canonical region.

Definition. A topological space X is said to be k-cohesive at p if $k = \inf\{n: p$ is in an n-canonical region$\}$ and $k < \infty$.

Lemma 1.3. *If X is locally compact, locally connected, T_2, and k-cohesive at p, then for any neighborhood U of p, there is a k-canonical region V about p, with $\overline{V} \subset U$ such that* Fr V *has only a finite number of components.*

Proof. It may be assumed that U itself is a k-canonical region, with $\overline{U}$ compact, by Lemma 1.2. Because U is semilocally connected at p, there is a neighborhood W of p, $\overline{W} \subset U$, and such that $\overline{U} - W$ has only a finite number of components. Thus if V is the component of W containing p, V is open and $\overline{U} - V$ can have no more components than $\overline{U} - W$ has.

Therefore, suppose that $D_1, D_2, \cdots, D_N$ are the components of $\overline{U} - V$. Then D_1 can contain no more than k components of Fr V. Setting $A = D_2 \cup \cdots \cup D_N \cup \overline{V}$ and $B = D_1$, it follows that $\overline{U} = A + B$ is a representation and $A \cap B = (\text{Fr } V) \cap D_1$ does not meet Fr U and thus can have no more than k components. Similarly, the other sets $D_2, \cdots, D_N$ each meet Fr V in k or fewer components. But Fr $V \subset D_1 \cup \cdots \cup D_N$, so the set Fr V has no more than $N \cdot k$ components, and by Lemma 1.2, V is a k-canonical region.

Definition. If X is locally finitely cohesive and $k = \inf\{n: p \in X$ implies that p is in an m-canonical region, $m \leq n\}$ and $k < \infty$, then X is locally k-cohesive.

The following theorem shows that local finite cohesion occurs naturally.

Theorem 1.4. *If X and Y are each nondegenerate generalized metric Peano continua, then $X \times Y$ is locally* 1-*cohesive and rim connected.*

Proof. For any point (p, q) in $X \times Y$, and any open set R about (p, q) there is a product set $U \times V$ about (p, q) which is open and connected, and such that $\overline{U} \times \overline{V} \subset R$, Fr U and Fr V are each nonempty, and both $\overline{U}$ and $\overline{V}$ are Peano continua. $\mathrm{Fr}(U \times V)$ is connected, since, for $x \in \mathrm{Fr}\, U$ and $y \in \mathrm{Fr}\, V$, $\mathrm{Fr}(U \times V) = \overline{U} \times \mathrm{Fr}\, V \cup \mathrm{Fr}\, U \times \overline{V} = \{\overline{U} \times \mathrm{Fr}\, V \cup \{x\} \times \overline{V}\} \cup \{\overline{U} \times \{y\} \cup \mathrm{Fr}\, U \times \overline{V}\}$, and these two sets are connected and both contain (x, y).

The proof will be completed by a demonstration that $U \times V$ is a 1-canonical region. Suppose that $\overline{U} \times \overline{V} = A + B$ is a representation and that $A \cap B$ has two or more components which miss $\mathrm{Fr}(U \times V)$. Then let $\overline{U}$ and $\overline{V}$ be subsets of U^* and V^* respectively, where U^* and V^* are unicoherent Peano continua in the Hilbert cube. With $x \in \mathrm{Fr}\, U$ and $y \in \mathrm{Fr}\, V$, let $N = \overline{U} \times \overline{V} \cup \{x\} \times V^* \cup U^* \times \{y\}$. It will be shown below that N is unicoherent.

Setting $A' = A \cup \{$either of $\{x\} \times V^*$, $U^* \times \{y\}$ which meets $A\}$ and $B' = B \cup \{$either of $\{x\} \times V^*$, $U^* \times \{y\}$ which meets $B\}$, then $N = A' + B'$ is a representation, but $A' \cap B'$ is not a connected set since $A \cap B$ has two or more components which miss $\mathrm{Fr}(U \times V)$ and hence miss both $\{x\} \times V^*$ and $U^* \times \{y\}$. The unicoherence of N will imply that no such A' and B', and therefore no such A and B, can exist.

The proof that N is unicoherent uses the exponential representation methods of [8]. Briefly, a continuous map $f\colon X \longrightarrow S^1$ is exponentially representable provided that $f = e^{ig}$, where $g\colon X \longrightarrow R^1$ is continuous. A Peano continuum is unicoherent if and only if every continuous map $f\colon X \longrightarrow S^1$ is exponentially representable.

Let $f\colon N \longrightarrow S^1$ be a continuous map. Then $f|\{x\} \times V^*$ and $f|U^* \times \{y\}$ can be written as e^{ig_1} and e^{ig_2} respectively, and because these restricted domains intersect in a single point, it may be assumed that $g = g_1 \cup g_2$ is a continuous map into R^1. Thus $f|\{\overline{U} \times \{y\} \cup \{x\} \times \overline{V}\}$ is equal to e^{ig}, and by a theorem of Whyburn [8, p. 224], the map g can be extended to all of $\overline{U} \times \overline{V}$, so that $f = e^{ig}$ on N is an exponential representation, and N must be unicoherent.

Corollary. *Local finite cohesion is a productive property for locally compact metric spaces.*

Example 1.5. Let X be the union of a sequence of successively tangent circles which converge to a point p. Then $\overline{X}$ is a Peano continuum, so that $\overline{X} \times \overline{X}$ is locally 1-cohesive. However, the fundamental group as well as the first homology group of $\overline{X} \times \overline{X}$ is infinitely generated.

Example 1.6. If $Y = W - \bigcup_{n=1}^{\infty} D_n$, the space of Example 1.1, then $Y \times [0, 1]$ is locally 1-cohesive.

A natural question suggested by these examples is this, "Is every locally finitely cohesive space locally 1-cohesive?" A continuum which is k-cohesive at one point, for any positive integer k, can be constructed as follows.

Example 1.7. Let $\{T_n: n = 1, 2, \cdots\}$ be a sequence of mutually disjoint indecomposable continua which converge to a point p. Then k distinct points of T_n are selected and each is identified with a distinct point of T_{n+1}: $n = 1, 2, \cdots$. Care is taken never to choose two points from a single composant of any T_n. Then let X be the union of all these T_n so identified, together with p. X is a continuum which is k-cohesive at p.

Of course, the space X in Example 1.7 is not locally finitely cohesive at every point since it is not locally connected. We proceed to answer the above question affirmatively for locally compact metric spaces.

Lemma 1.8. *Suppose that U is a k-canonical region, $\overline{U}$ a metric Peano continuum. Then if there is a k-canonical region V, $\overline{V} \subset U$ such that* Fr V *has only a finite number of components, there exists a representation $\overline{U} = A + B$ in which $A \cap B$ has k components and misses* Fr U *altogether.*

Proof. By assumption there is a representation $\overline{V} = C + D$, and $C \cap D$ has k components which miss Fr V. If there are only N components of Fr V, then there are N or fewer components of $\overline{U} - V$. If $A = C$ together with the components of $\overline{U} - V$ that meet C, and $B = D$ together with the components of $\overline{U} - V$ that do not meet C, then $\overline{U} = A + B$ is the required representation.

Lemma 1.9. *In Lemma* 1.8, *for any $\epsilon > 0$, the sets A, B, and $A \cap B$ may be assumed to be the union of a finite number of Peano continua of diameter $< \epsilon$.*

Proof. Let ϵ be smaller than half the distance between any distinct pair of components of $A \cap B$, and also smaller than half the distance from $A \cap B$ to Fr U. $\overline{U}$ can be the union of a finite number of Peano continua each of diameter $< \epsilon$. If A^* is the union of all these continua which meet A, and B^* is the union of all these continua which meet B, then $\overline{U} = A^* + B^*$ is the needed representation.

Theorem 1.10 *If X is locally compact, locally finitely cohesive, and metric, then X is locally* 1-*cohesive.*

Proof. If X is not 1-cohesive at x, there is an integer $k > 1$ and a k-canonical region U about x, with $\overline{U}$ a Peano continuum, and every region V containing x, $\overline{V} \subset U$, is also a k-canonical region. Thus the hypothesis of Lemma 1.8 is satisfied. Therefore there is a representation $\overline{U} = A + B$ in which $A \cap B$ misses Fr U, $A \cap B$ has k components, and the point x is interior to $A \cap B$. (If x fails to be interior to $A \cap B$, there is a region R about x whose closure does not meet Fr U and either meets exactly one component of

$A \cap B$, or is joined to $A \cap B$ by an arc α in U which meets $A \cap B$ in only one point. Then $A^* = A \cup \alpha \cup \overline{R}$ and $B^* = B \cup \alpha \cup \overline{R}$ meet the claim above.) By Lemma 1.9, if d is the distance from x to Fr U, then A, B, and $A \cap B$ are each the union of a finite subcollection of Peano continua T_n each of diameter $< d/2$.

The set $D \equiv \{$union of all these T_n which do not contain $x\}$ is closed and contains Fr U. Then if V is the component of $\overline{U} - D$ containing x, V is a k-canonical region, $\overline{V}$ meets exactly one component of $A \cap B$ and $\overline{V} \subset U$. Also $A - V$ and $B - V$ have only a finite number of components. There must be a representation $\overline{V} = C + D$ in which $C \cap D$ has k components which do not meet Fr V.

Now if K is a component of $A \cap B$ that does not intersect $\overline{V}$, then K must lie in $A_1 \cap B_1$, where A_1 is a component of $A - V$ and B_1 is a component of $B - V$. With no loss of generality A_1 can be required to meet C, and if B_1 does not meet D, then an arc β is constructed in $U - K$ which meets B_1 in a single point and D in a single point. Such an arc β exists because the set $\overline{V}$, together with the union of all T_n which meet $\overline{V}$, is a Peano continuum in $U - K$, and B_1 meets $\overline{V}$ in Fr V.

Thus the sets $E = C \cup A_1 \cup \{$union of all components of $A - V$ and $B - V$ which meet C, except $B_1\}$ and $F = D \cup \beta \cup B_1 \cup \{$union of all components of $A - V$ and $B - V$ which meet D, except $A_1\}$ are closed and connected, and $\overline{U} = E + F$ is a representation such that $E \cap F$ misses Fr U. The set $E \cap F$ contains $\beta \cup K \cup \{k$ components of $C \cap D$ that miss Fr $V\}$. K, as well as each of the $k - 1$ components of $E \cap F$ that miss Fr V, adds a component to $E \cap F$, so that $E \cap F$ has a minimum of $k + 1$ components, which is a contradiction.

Corollary 1.11. *If X is locally compact, metric, and locally finitely cohesive, then X is rim connected except at its local separating points.*

Proof. For any point $p \in X$ which is not a local separating point, and any 1-canonical region U about p, $U - p$ is connected. Thus there is a region V about p, with $\overline{V} \subset U$, and $U - V$ connected. (See [8, p. 50].) The representation $\overline{U} = (\overline{U} - V) + \overline{V}$ has as its intersection Fr V, which misses Fr U and therefore is connected.

2. Relation of local finite cohesion to other conditions.

Definition. A set X is m-coherent if X is connected and m is the least integer k such that in each representation $X = A + B$, the set $A \cap B$ has $\leq k + 1$ components. If no such m exists, then X is ∞-coherent.

Finite coherence is studied in [3], [4], and [7].

Theorem 2.1. *If X is locally connected, connected, and m-coherent, $m < \infty$, then X is locally finitely cohesive.*

Proof. For $p \in X$, let V be a region about p with Fr V nonempty. Then V is an n-canonical region, $n \leq m + 1$. This fact is verified as follows: first, let $\overline{V} = E + F$ be a representation. Now the components of $X - \overline{V}$ are open, so each one must have limit points in Fr V to avoid being open and closed. Following the proof of Lemma 1.2, with $X = \overline{U}$, it is easy to see that $E \cap F$ has $\leq m + 1$ components that miss Fr V.

Corollary 2.2. *If X is locally connected and has a cover of finitely coherent regions, then X is locally finitely cohesive.*

We now record the fact that local finite cohesion can be made into the global condition of finite coherence if X is a planar continuum.

Theorem 2.3. *A Peano continuum X in the plane E^2 is finitely coherent if and only if it is locally finitely cohesive.*

Proof. The necessity is a special case of Theorem 2.1; the sufficiency is argued as follows. X may be assumed to lie interior to I^2, the unit square in E^2. If X has only a finite number, $N + 1$, of complementary domains, then X is a retract of $I^2 - P_1 \cup P_2 \cup \cdots \cup P_N$ where each P_n is a simple region chosen from a distinct bounded complementary domain of X [1, p. 138]. Because the set $I^2 - P_1 \cup \cdots \cup P_N$ is N-coherent, X, as a retract of this set, must be k-coherent, $k \leq N$ by [3].

Suppose that $E^2 - X$ has infinitely many components $C_1, C_2, \cdots$. The sequence $\{C_n\}$ is a null collection [8, p. 113] of sets which must cluster at some $p \in X$. We may assume that there is a 1-canonical region V about p, with V containing an infinite subsequence $\{\text{Fr } C_{n_k}: k = 1, 2, \cdots\}$ of the sets $\{\text{Fr } C_n\}$, and $\overline{V}$ a Peano continuum. Whyburn has proved [8, p. 177] that if A is a nonseparating Peano continuum in E^2, and Y is any 2-cell containing A in its interior, there is a monotone retraction $r: Y \to A$ which sends $Y - A$ onto Fr A. This fact will be used repeatedly to define a monotone retraction $r: I^2 - \bigcup_{k=1}^{\infty} C_{n_k} \to \overline{V}$, and this retraction will contradict the fact that V is 1-canonical.

Set $\overline{V}_1 = \overline{V} \cup \bigcup_{k=1}^{\infty} C_{n_k}$; then $\overline{V}_1$ is a nonseparating Peano continuum, so there is a monotone retraction $r_1: I^2 \to \overline{V}_1$. For each $k > 1$, take $\overline{V}_k = \overline{V}_1 - C_{n_k}$. Let $c: E^2 \cup \omega \to S^2$ be the one-point compactification of E^2 and let t_k be a rotation of the sphere $c(E^2 \cup \omega)$ which makes a uniformly locally connected open 2-cell $c(P_k) \subset c(C_{n_k})$ into a neighborhood of $c(\omega)$ in S^2. Defining $e_k = c^{-1} \circ t_k \circ c$, the set $e_k(\overline{V}_k)$ is a nonseparating Peano continuum in E^2. There is a monotone retraction $f_k: E^2 - e_k(P_k) \to e_k(\overline{V}_k)$, since the set $E^2 - e_k(P_k)$ is a closed 2-cell containing $e_k(\overline{V}_k)$ on its interior. Finally, the map $r_k = e_k^{-1} \circ f_k \circ e_k$, when restricted to $I^2 - P_k$, is a monotone retraction of that set onto $I^2 - C_{n_k}$ which moves only points of C_{n_k}, for $k > 1$. Once r_k is defined in this way for each positive integer k, set $r = \lim_{k\to\infty}(r_k \circ r_{k-1} \circ \cdots \circ r_1)$. Notice that a point $p \in I^2$ is moved by r_k only if $p \in C_{n_k}$, and even then, the point p cannot be moved a distance larger than diam C_{n_k}, since r_k maps the points of $C_{n_k} - P_k$ onto Fr C_{n_k}. (These observations hold for $k > 1$.) Hence the limit which defines r is uniformly convergent, as diam $C_{n_k} \longrightarrow 0$ as $k \longrightarrow \infty$, so r is continuous. Moreover, r is monotone [8, p. 174], so r is a continuous, monotone retraction of $I^2 - \bigcup_{k=1}^{\infty} P_k$ onto $\overline{V}$.

There is obviously a representation $I^2 - \bigcup_{k=1}^{\infty} P_k = A + B$ with $B \subset V$ such that $A \cap B$ is not connected. Thus $\overline{V} = r(A) + r(B)$ is a representation and $r(A) \cap r(B)$ is not connected, as r is closed and monotone. But $r(A) \cap r(B)$ fails to meet Fr V, contradicting the fact that V is 1-canonical.

Definition. A space X is locally cohesive at p if for each neighborhood U of p, there is a region $V \subset U$, $p \in V$ with Fr V connected, and for each representation $\overline{V} = A + B$ such that Fr V is interior to A, and p is interior to B, the set $A \cap B$ is connected. Such a V is a canonical region for p. If X is locally cohesive at each of its points, then X is locally cohesive.

Results involving local cohesiveness are found in [9] and [12].

Theorem 2.4. *If X is locally cohesive, normal, and T_1, then X is locally k-cohesive, $k \leq 2$.*

Proof. A canonical region W about $p \in X$ is also a k-canonical region for $k \leq 2$. If $W = A + B$ is a representation such that $A \cap B$ has components C_1, C_2, and C_3 that do not meet Fr W, then p can be assumed not to lie in C_1 or C_2, and p can be assumed to lie in B. There is a region R about p with $\overline{R} \subset W$ and $\overline{R} \cap C_1 = \varnothing = \overline{R} \cap C_2$. Also in X there is a connected open set Y about Fr W whose closure misses $C_1 \cup C_2 \cup C_3 \cup \overline{R}$. Then if Z is the union of the components of $Y -$ Fr W which meet W, the set $Z \cup$ Fr W is connected, because each component of $Y -$ Fr W meets the connected set Fr W.

For each point $x \in W$, there is a region L_x whose closure meets at most one of the sets C_1, C_2, C_3, $Z \cup$ Fr W. A finite chain joining x to $Z \cup$ Fr W will be a collection $L_1, \cdots, L_N$ with $L_1 \cap (Z \cup \text{Fr } W) \neq \varnothing$, $L_k \cap L_{k+1} \neq \varnothing$ and $x \in L_N$.

If U_x denotes the union of all points x in W that can be joined to $Z \cup$ Fr W by a finite chain, then U_x is clearly open. However, U_x is also closed; for if y is a limit point of U_x, there is a region L_y about y whose closure meets at most one of the sets C_1, C_2, C_3. In addition, L_y must meet some point $x \in U_x$, and there is a chain $L_1, \cdots, L_N$ joining x to $Z \cup$ Fr W. The chain $L_1, \cdots, L_N$, L_y then joins y to $Z \cup$ Fr W, so $y \in U_x$. Because W is connected and U_x is open and closed, $U_x = W$ and there is a chain L from $Z \cup$ Fr W to A in which the closure of only one link meets A. We identify L with the union of its links.

Now define $E = A \cup L \cup Z \cup \operatorname{Fr} W$ and $F = B \cup \overline{R}$. Then E and F are connected, p is interior to F, Fr W is interior to E, and $\overline{W} = \overline{E} + F$ is a representation. Since W is a canonical region for p, the set $\overline{E} \cap F$ must be connected. It is easy to see that C_1 and C_2 lie in different components of $\overline{E} \cap F$, however.

We note that the proof still works if X is locally cohesive, locally compact, and T_2.

Theorem 2.5. *If X is locally compact, rim connected, metric, and locally finitely cohesive, then X is locally cohesive.*

Proof. For $p \in X$, let U be a 1-canonical region about p with Fr U connected and $\overline{U}$ compact. Then U is a canonical region for p; otherwise there is a representation $\overline{U} = A + B$ with Fr U interior to A, p interior to B, and $A \cap B = C + D$, a separation. Since B is compact, there is a subcontinuum K of B that is irreducible from C to D. Then the set $K' = K - K \cap (C \cup D)$ is connected and misses A, so it is contained in a component M of $(\overline{U} - A)$. M is open and evidently $\operatorname{Fr} M \subset A \cap B = C \cup D$, and because $\overline{K}' \subset \overline{M}$, and $\overline{K}'$ meets both C and D, the set Fr M is separated. One notices that Fr M is in U because Fr U is interior to A. Finally, $\overline{U} = \overline{M} + \overline{U} - M$ is a representation, and the intersection of these sets, which equals Fr M, lies in U and is separated, which is a contradiction.

Corollary 2.6. *If X is locally compact, T_2, metric, and rim connected, then X is locally cohesive if and only if X is locally finitely cohesive.*

Corollary 2.7. *If X is locally compact, metric, and locally finitely cohesive, then X is locally cohesive except at its local separating points.*

Corollary 2.8. *If X is finitely coherent, locally connected, metric, and locally compact, then X is locally cohesive except at its local separating points.*

Corollary 2.9. *If X and Y are each nondegenerate generalized metric Peano continua, then $X \times Y$ is locally cohesive.*

A question raised by Whyburn in [9] is whether each locally cohesive space necessarily has a cover of unicoherent regions. A negative answer is now easily obtained, even for Peano continua, by Example 1.5 or 1.6, together with Corollary 2.9.

Viewed as a generalization of local cohesiveness, local finite cohesion relaxes the requirement of rim connectedness, and otherwise the conditions are equivalent (for locally compact, T_2 spaces). In the process of studying local finite cohesion, new information about local cohesiveness was gained. An example of a space which is locally 1-cohesive, but neither finitely coherent nor locally cohesive, follows.

Example 2.10. Let $T = \bigcup_{n=1}^{\infty} T_n$ be the union of a sequence of punctured hollow cones of unit altitude which are mutually disjoint save at a common vertex point p. The base of T_n has diameter $1/2^n$ and from T_n a simple region of diameter $1/4^n$, which is at a distance $1/2^n$ from p, is removed. (Thus none of the sets T_n are unicoherent.) T is locally 1-cohesive. At points other than p, T is a 2-manifold so there is nothing to prove. For any $\epsilon > 0$, there is a region V containing p of diameter $< \epsilon$ such that $\overline{V}$ is homeomorphic to the union of a finite number of 2-cells together with an infinite number of punctured 2-cells, with a sequence of points, exactly one from each cell, identified to p. If C_n is a 2-cell (possibly punctured) in $\overline{V}$, then for any representation $C_n = E + F$, $E \cap F$ is either connected or each component of $E \cap F$ meets Fr C_n. This is proved by considering the monotone quotient map of C_n whose only nondegenerate point inverse is Fr C_n.

Thus in a representation $\overline{V} = A + B$, if B meets only one cell C_n, there is nothing to prove. If, however, B meets more than one cone, B must contain p. Therefore, $p \in A \cap B$, and as is easily verified, $A_n = A \cap C_n$ and $B_n = B \cap C_n$ are connected for all n. Either each component of $A_n \cap B_n$ meets Fr V or else $A_n \cap B_n$ is connected and contains p. Hence the union of all components of $A \cap B$ that miss Fr V is connected, and V is a 1-canonical region.

Question 2.11. Is there a compact metric space X which is locally finitely cohesive, and yet fails to be either finitely coherent or locally cohesive?

3. Applications of local finite cohesion.

Quasicomponent theorems. A new theorem establishing an equivalence of components with quasicomponents is obtained with a sharpened form of Lemma 2.3 of [8, p. 90].

Lemma 3.1. *Let X be a locally connected metric space, and U a 1-canonical region in X with $\overline{U}$ compact. If p and q are distinct points of U which are separated in U by a totally disconnected subset D of U, then p and q are separated in U by a single point of D.*

Proof. If $U - D = M + N$, a separation with $p \in M$, $q \in N$, and V is the component of $U - U \cap \overline{M}$ containing q, then V is nonempty because q is not a limit point of M, and V is open. Now let K be the component of $\overline{U} - V$ containing p, and set $L = \overline{V} \cup \{$union of components of $\overline{U} - V$ other than $K\}$. If C is a component of $\overline{U} - V$, then C must meet $\overline{V}$ or else $\overline{U}$ could be separated between C and $\overline{V}$ [8, p. 15]. Thus each component of $\overline{U} - V$ meets $\overline{V}$, so that $\overline{L}$ is connected.

The set $K \cap \overline{L}$ separates p and q in $\overline{U}$ and is a subset of $(\overline{V} \cap K) \cap \mathrm{Fr}\, U$. If this were not true, there would be a point $x \in K$, x neither in $\overline{V}$ nor in Fr U, such that x is a limit point of a sequence of components of $\overline{U} - V$. Since each such component meets Fr V, no neighborhood of x which does not meet $\mathrm{Fr}\, U \cup \overline{V}$ could be connected, contradicting the local connectedness of U.

Now $K \cap \overline{V} \subset K \cap \mathrm{Fr}\, V$ since $K \cap V = \emptyset$, and $\mathrm{Fr}\, V \subset D \cup \mathrm{Fr}\, U$, so that $K \cap \overline{L} \subset (\overline{V} \cap K) \cup \mathrm{Fr}\, U \subset D \cup \mathrm{Fr}\, U$. Because U is 1-canonical, and $\overline{U} = K + \overline{L}$ is a representation, $K \cap \overline{L}$ can contain at most one point of D. This follows from the observation that since $K \cap \overline{L}$ is compact, each point of D in $K \cap \overline{L}$ is a component of $K \cap \overline{L}$ which misses Fr U. Since p and q are separated in $\overline{U}$ by a point of D together with some subset of Fr U, then p and q are separated in U by a single point of D.

Lemma 3.2. *Let X be a connected, locally compact metric space with local finite cohesion, and suppose that $\{W_n: n = 1, 2, \cdots\}$ is a collection of mutually disjoint open connected sets in X with $\bigcup_{n=1}^{\infty} \mathrm{Fr}\, W_n = D$, a totally disconnected set. If L is the limit set of some convergent subsequence of the $\{W_n\}$, then L is a single point.*

Proof. If L is not degenerate, let p and s be distinct points of L, and select a subsequence of the W_n, say $W_{n_1}, W_{n_2}, \cdots$, such that W_{n_k} contains a point p_k which is a distance less than $1/k$ from p, and a point s_k which is a distance less than $1/k$ from s. Since X is a generalized Peano continuum, each W_{n_k} is arcwise connected, so there is in particular an arc $\widehat{p_k s_k}$ in W_{n_k} with endpoints p_k and s_k. (It is clear that the accumulation points of $\bigcup_{k=1}^{\infty} \widehat{p_k s_k}$ are contained in L.) Now let U be a 1-canonical region about p with $\overline{U}$ compact and $s \notin \overline{U}$. For each arc $\widehat{p_n s_n}$, let K_n be the component of $\widehat{p_n s_n} \cap U$ which contains p_n. Notice that $\overline{K}_n$ must meet Fr U, or else the compact set $\widehat{p_n s_n}$ could be separated between $\overline{K}_n$ and $\widehat{p_n s_n} \cap (X - U)$ [8, p. 16]. (Since $\widehat{p_n s_n}$ meets Fr U for almost all n, we assume that $\widehat{p_n s_n}$ meets Fr U for every n.)

Now if R is a region about p, $\overline{R} \subset U$, then Fr R meets almost all the sets K_n, and the sequence of sets $\{K_n \cap \mathrm{Fr}\, R\}$ must have a cluster point $q \in U \cap L$. Thus there are distinct points p and q in $U \cap L$, and about p and q, disjoint regions O_p and O_q may be taken in U.

Some member K_m of the sequence $\{K_n\}$ must meet both O_p and O_q. It follows that K_m is in a different quasicomponent of $X - D$ from L, since K_m lies in one of the sets W_{n_k}, and W_{n_k} is open and closed in $X - D$. Thus if $x \in K_m \cap U$, then x and p are separated in U by $U \cap D$, since x and p are separated in X by D. An application of Lemma 3.1 to U indicates that a single point $d \in U \cap D$ separates x from p in U. The regions O_p and O_q are disjoint, so that d has to miss one of them, say O_q. Then x and q are in the same component of $U - \{d\}$ because $K_m \cup O_q$ is connected. Also p and q are in the same quasicomponent of

$U - \{d\}$ since otherwise $U - \{d\}$ would be the union of two disjoint proper open subsets, each of which would meet almost all of the connected sets $\{K_n'\}$. We conclude that x and p are not separated in $U - \{d\}$, and hence L must be degenerate.

Lemma 3.3. *Let X be a connected, locally compact metric space with local finite cohesion, and let D be a totally disconnected subset of X. Then if C is any quasicomponent of $X - D$ and $p \in C$, and R is a neighborhood of p, there exists a neighborhood V of p, $(R \cap C) \subset V \subset R$, and $\mathrm{Fr}\, V \cap (X - D) \subset C \cap \mathrm{Fr}\, R$.*

Proof. Let B be the collection of quasicomponents Q of $X - D$ different from C which meet $\mathrm{Fr}\, R$. For each such Q, there is a pair of mutually separated sets M_q and N_q, whose union is $X - D$, containing C and Q respectively. If G_q = all points nearer to N_q than to M_q, then G_q is open in X, G_q contains Q, and $\mathrm{Fr}\, G_q \subset D$. There is a countable refinement $G_1, G_2, \cdots$ of the G_q whose union is a cover for $E = \bigcup\{Q: Q \in B\}$. Next set $U_1 = G_1$, $U_2 = G_2 - \overline{G}_1, \cdots, U_n = G_n - (\overline{G}_1 \cup \overline{G}_2 \cup \cdots \cup \overline{G}_{n-1})$ and so on. Note that $\mathrm{Fr}\, U_n \subset \mathrm{Fr}\, G_1 \cup \cdots \cup \mathrm{Fr}\, G_n \subset D$.

The union of all the U_n covers E and is open. Let $W_1, W_2, \cdots$ be the components of $U = \bigcup_{n=1}^{\infty} U_n$ which meet $\mathrm{Fr}\, R$. Each W_k is open, since X is locally connected. Because the U_n are disjoint, for each k there is an n such that $W_k \subset U_n$ so $\mathrm{Fr}\, W_k \subset D$. With $W = \bigcup_{n=1}^{\infty} W_n$, it will follow that $\mathrm{Fr}\, W \cap (X - D) \subset (X - D) \cap \mathrm{Fr}\, R$. For, if $p \in \mathrm{Fr}\, W$, then either $p \in D$ or p is a limit point of a convergent subsequence $\{W_{n_k}\}$ of W. We claim that the sets $\{W_{n_k}: k = 1, 2, \cdots\}$ form a null sequence. If not, let Y be a neighborhood about p whose closure is compact, such that for infinitely many k, W_{n_k} meets both Y and $X - \overline{Y}$. A subsequence $S_1, S_2, \cdots$ of the $\{W_{n_k}\}$ which meet both Y and $X - \overline{Y}$ may be chosen so that S_n contains a point p_n whose distance from p is $< 1/n$, and S_n contains a point $q_n \in S_n \cap \mathrm{Fr}\, Y$ ($S_n \cap \mathrm{Fr}\, Y \neq \emptyset$, because S_n is connected and meets both Y and $X - \overline{Y}$). Let the arc $\widehat{p_n q_n}$ be taken in S_n; then $p \in \liminf_{n\to\infty} \widehat{p_n q_n}$ and $K = \liminf_{n\to\infty} \widehat{p_n q_n}$ is compact, so K is a nondegenerate continuum. Now if $K - p$ were totally disconnected, it would be 0-dimensional, since $K - p$ is locally compact. But K is connected, so $K - p$ is not totally disconnected, and hence $K - p$ is not a subset of D, so there is a point $q \in (K - p) \cap (X - D)$. Since the set K is a subset of $L = \lim_{k\to\infty} W_{n_k}$, p and q are in L, which contradicts Lemma 3.2. We conclude that $\{W_{n_k}\}$ is a null sequence of sets, and since each one meets $\mathrm{Fr}\, R$, it follows that $p = L = \lim_{k\to\infty} W_{n_k} \subset \mathrm{Fr}\, R$.

Thus $\mathrm{Fr}\, W \cap (X - D) \subset (X - D) \cap \mathrm{Fr}\, R \cap \mathrm{Fr}\, W$, and this set is actually $C \cap \mathrm{Fr}\, R$, as $(X - D) \cap \mathrm{Fr}\, R \cap (X - C) \subset E \subset W$. Now let $V = R - R \cap \overline{W}$. Then $R \cap C \subset V \subset R$, and $\mathrm{Fr}\, V \cap (X - D) \subset C \cap \mathrm{Fr}\, R$ since $\mathrm{Fr}\, V \subset \mathrm{Fr}\, R \cup \mathrm{Fr}\, W$, and $(X - D) \cap \mathrm{Fr}\, R = (C \cap \mathrm{Fr}\, R) \cup (E \cap \mathrm{Fr}\, R)$ while $E \cap \mathrm{Fr}\, R \subset W$.

Theorem 3.4. *If X is a locally finitely cohesive, locally compact metric space, and D is any totally disconnected subset of X, then the quasicomponents of $X - D$ are connected.*

Proof. If Y is any component of X, then we shall apply Lemma 3.3 to the sets Y and $Y \cap D$. Suppose that C is a quasicomponent of $Y - Y \cap D$, and $C = A + B$ is a separation of C, where $A = C \cap U$, U is open in Y, and $\mathrm{Fr}\, U \cap B = \emptyset$. By Lemma 3.3 there is a neighborhood M of A such that $A = U \cap C \subset M \subset U$ and $\mathrm{Fr}\, M \cap (Y - Y \cap D) \subset C \cap \mathrm{Fr}\, U = \emptyset$. Therefore $Y - Y \cap D = M \cap (Y - Y \cap D) + (Y - M) \cap (Y - Y \cap D)$ is a separation of $Y - Y \cap D$ between A and B, so that C is not a quasicomponent of Y, and hence C is not a quasicomponent of X, since Y is open in X.

Corollary 3.5. *If a metric space X is either* (a) *a generalized Peano continuum which is finitely coherent, or* (b) *locally cohesive and locally compact; and D is a totally disconnected subset of X, then the quasicomponents of $X - D$ are connected.*

Proof. All such spaces are locally finitely cohesive.

Example 3.6. Let $X_1 = \{(x, y): y = \sin(1/x)$ for $0 < x \leq 1\}$, $X_2 = \{(0, y): -1 \leq y \leq 1\}$ and $X_3 = \bigcup_{n=1}^{\infty} E_n$, where $E_n = \{(x, y): 0 \leq x \leq 1/n$ and $y = k/2^n$, $k = \pm 1, \pm 3, \cdots, \pm(2^n - 1)\}$. Then $X = X_1 \cup X_2 \cup X_3$ is the $\sin(1/x)$ continuum "made locally connected"; it is a Peano continuum to which Theorem 2.4 does not apply. If K is a component of X_3, then K is an interval $\approx [0, 1/n]$. Let the set $K \cap X_1 = \{a_1, a_2, \cdots\}$ be ordered in the natural order of K from $1/n$ to 0. Let $D_K = \{b_1, b_2, \cdots\}$ be a subset of K, so chosen that $b_k > a_k > b_{k+1}$ for every k. Then $D = \bigcup\{D_K: K$ is component of $X_3\} \cup \{(0, 0)\}$ is a totally disconnected subset of X, and $X - D$ has the quasicomponent $X_2 - \{(0, 0)\}$, which is not connected. Thus a cohesion condition of some sort is needed to prove Theorem 3.4.

Theorem 3.7. *If X is a finitely coherent generalized metric Peano continuum and S is a semiclosed* [8, p. 131] *subset of X, then the quasicomponents of $X - S$ are connected.*

Proof. The quotient map $F: X \to M$ whose only nondegenerate point inverses are the components of S is a closed, monotone map. M is a finitely coherent generalized Peano continuum and $F(S)$ is a totally disconnected subset of M. By Corollary 3.6, the quasicomponents of $M - F(S)$ are connected. Because $F|X - S$ is a homeomorphism, the quasicomponents of $X - S$ must be connected.

4. Connectivity functions.

Definition. A function $f: X \to Y$ is a connectivity function provided that for each connected set C in X, the set $\{(x, f(x)): x \in C\}$ is connected in $X \times Y$.

Connectivity functions are a generalization of continuous functions and are discussed generally in [6].

Lemma 4.1. *If $f: X \to Y$ is a connectivity function, so also is its graph function $g: X \to X \times Y$, and conversely.*

Proof. If f is a connectivity function, then so is g because the projection map p_X: $X \times X \times Y \longrightarrow X \times Y$ maps the graph of g topologically onto the graph of f.

Conversely, if g is a connectivity, then $f = p_X \circ g$ is also, as the composition of a connectivity map followed by a continuous map is itself a connectivity map.

Lemma 4.2. *If X is a metric Peano continuum and f is a connectivity function from X into a regular, T_1 space Y, then for any closed set C in Y, the set $f^{-1}(C)$ is semiclosed in X.*

Proof. This is proved in [2].

Lemma 4.3. *If $f: X \longrightarrow Y$ is a connectivity map and $M \subset X$ is a Peano continuum with $U \subset M$, U open, and V open in Y, the set $g^{-1}(\mathrm{Fr}(U \times V))$ is semiclosed in M.*

Proof. This proposition follows from Lemmas 4.1 and 4.2.

Definition. A function F: $X \longrightarrow Y$ is peripherally continuous if at each point $p \in X$, and for each neighborhood U of p, and for each neighborhood V of $F(x)$ in Y, there is a neighborhood $W \subset U$ containing p such that $F(\mathrm{Fr}\ W) \subset V$.

The property of peripheral continuity has been closely related to that of connectivity in [9] and [12]. The next theorem extends results of [5], [9], and [12] to a larger class of domain spaces.

Theorem 4.4. *If X is locally compact, metric, and has local finite cohesion, then any connectivity map $f: X \longrightarrow Y$ is peripherally continuous when Y is regular and T_1.*

Proof. For any point p in X, and any $\epsilon > 0$, there is a region U about p of diameter $< \epsilon$ such that $\overline{U}$ is a Peano continuum. Let R be a 1-canonical region about p with $\overline{R} \subset U$, and with the added property that $\mathrm{Fr}\ R$ has only a finite number of components, $B_1, B_2, \cdots, B_N$. Then if V is a neighborhood of $f(p)$ in Y, an open set $W \times V_1$ can be chosen in $X \times Y$ to contain $(p, f(p))$ with $\overline{W} \subset R$ and $\overline{V}_1 \subset V$.

The set $D = g^{-1}\{\mathrm{Fr}(W \times V_1)\}$ is a semiclosed subset of $\overline{W} \subset \overline{U}$ because g, the graph function of f, is a connectivity function, $\mathrm{Fr}(W \times V_1)$ is closed in $X \times Y$, and Lemma 4.3 applies. (D is a subset of $\overline{W}$ because $\mathrm{Fr}(W \times V_1) = \mathrm{Fr}\ W \times \overline{V}_1 \cup \overline{W} \times \mathrm{Fr}\ V_1$, and for x to be in D, the point $(x, f(x))$ must be in $\mathrm{Fr}(W \times V_1)$; hence x is in $\overline{W}$.) The decomposition of $\overline{U}$ into the components of D, the components $B_1, \cdots, B_N$ of $\mathrm{Fr}\ R$, and individual points of $\overline{U} - (D \cup \mathrm{Fr}\ R)$ is upper semicontinuous and the associated quotient map q: $\overline{U} \longrightarrow M$ is closed and monotone. The set $q(\overline{R})$ is a k-coherent Peano continuum, $k \leq N$. To see that $q(\overline{R})$ is k-coherent in M, let $q(\overline{R}) = A + B$ be a representation, and note that at most N components of $A \cap B$ meet the finite set $q(\mathrm{Fr}\ R)$. Therefore, in the representation

$\bar{R} = q^{-1}(A) + q^{-1}(B)$, because R is 1-canonical, $q^{-1}(A) \cap q^{-1}(B)$ can have at most one component that misses $\mathrm{Fr}\, R$, so it has at most $N + 1$ components since q is closed and monotone. Therefore $A \cap B$ can have at most $N + 1$ components.

For the remainder of the proof, let $q(\bar{R}) = R'$, $q(p) = p'$, and $q(B_k) = b_k$, $1 \leq k \leq N$. If the totally disconnected set $q(D)$ fails to separate p' from some b_k in R', then the quasicomponent Q' of $R' - q(D)$ containing p' and b_k is connected by Corollary 3.5. Thus $Q = q^{-1}(Q')$ is a connected subset of X, so the graph set $G = \{(x, f(x)) \colon x \in Q\}$ has to be connected in $X \times Y$, which is impossible since G meets $W \times Y_1$ in $(p, f(p))$ and the complement of $W \times V_1$ in $\{(y, f(y)) \colon y \in B_k\}$, while G misses the set $\mathrm{Fr}(W \times V_1)$.

Hence, for each of $b_1, \cdots, b_N$, the set $q(D)$ separates p' from b_k in R'. Thus a maximum of N^2 points of $q(D)$ is needed to separate p' from $\{b_1, \cdots, b_N\}$, as $q(R) = R'$ is k-coherent, $k \leq N$. (See [3].) If F denotes a finite subset of $q(D)$ such that $R' - F$ is separated between p' and $q(\mathrm{Fr}\, R)$, then let Z' be the component of $R' - F$ containing p'. The set $Z = q^{-1}(Z')$ is a region in R about p, and $\mathrm{Fr}\, Z \subset q^{-1}(F) \subset D$. Therefore $f(\mathrm{Fr}\, Z) \subset \bar{V}_1 \subset V$, and Z is the required neighborhood.

These results do not seem likely to extend to the nonlocally connected case, regardless of the cohesion present. Indeed, a hereditarily unicoherent (chainable) continuum affords a counterexample.

Example 4.5. Let

$$L_n = \left\{(\sin(1/y), y) \colon \frac{2}{(2n+1)\pi} \leq y \leq \frac{2}{(2n-1)\pi}\right\}$$

and let $X = \overline{\bigcup_{n=1}^{\infty} L_n}$. Define $f \colon X \to X$ by

$$\begin{aligned} f(x, y) &= (x + (x-1)(x+1)/4,\ 0) \quad \text{if } (x, y) \in L_n \text{ for } n \text{ odd}, \\ &= (x, 0) \quad \text{if } y = 0 \text{ or if } (x, y) \in L_n \text{ for } n \text{ even}. \end{aligned}$$

It is not hard to show that f is a connectivity function, but the peripheral continuity of f fails on the set $\{(x, 0) \colon -1 < x < 1\}$.

Definition. A connectivity function $r \colon X \to X$ which is the identity function on $r(X)$ is a connectivity retraction, and $r(X)$ is a connectivity retract of X.

Theorem 4.6. *If X is locally compact, connected, metric and has local finite cohesion, and $r \colon X \to X$ is a connectivity retraction, then $r(X)$ is locally compact, locally connected, and connected.*

Proof. That $r(X)$ has the required properties follows from results of [6], except for local connectedness. If, at some point p, the space $r(X)$ fails to be locally connected, then there is a neighborhood U of p in X such that $\bar{U} \cap r(X)$

contains an infinite sequence $(C_n: n = 1, 2, \cdots)$ of disjoint components which cluster at p, and each of which intersects $(X - U) \cap r(X)$. Let δ be the distance from p to $(X - U) \cap r(X)$.

Let V be a 1-canonical region about p of diameter $< \delta/2$, such that $\overline{V}$ is compact and Fr V has only a finite number of components. By peripheral continuity, there is a region Q about p, $Q \subset V$ and $r(\text{Fr } Q) \subset U \cap r(X)$. We select a region Q' about p in U with Fr $Q' \subset$ Fr Q and such that Fr Q' has only a finite number of components. If $(E_a: a \in A)$ are the components of $\overline{V} - Q$, then only a finite number of them meet Fr V, since it has only a finite number of components. Set $Q' = Q \cup \{$union of all E_a which do not meet Fr $V\}$; Q' is open, connected, and Fr Q' can have only a finite number of components. This last assertion is argued in the same manner as Lemma 1.3.

Next, it must be proved that Fr $Q' \subset$ Fr Q. Since Q' is open, if Fr $Q' \not\subset$ Fr Q, there must be a point p, not in Q' or Fr Q, which is a limit point of $(E_n: n = 1, 2, \cdots)$, a sequence of components of $\overline{V} - Q$ which are in Q'. The fact that p is a positive distance from $\overline{Q}$ and a limit point of a sequence of E_a's whose boundaries are in $\overline{Q}$ implies that the space X is not locally connected at p. Thus Fr Q' $\subset$ Fr Q.

Finally, because Fr Q' has only a finite number of components and must meet almost all of the components $C_1, C_2, C_3, \cdots$ of $\overline{U} \cap r(X)$, then some component K of Fr Q' meets distinct components C_m and C_n. Thus the set $r(K)$ is connected, lies in $U \cap r(X)$, and meets both C_m and C_n, because r is the identity on $r(X)$. However, no connected subset of $\overline{U} \cap r(X)$ can meet the disjoint components C_m and C_n.

Corollary 4.7. *If a metric space X is a generalized Peano continuum which is either k-coherent, $k < \infty$, or locally cohesive, then any connectivity retract of X is a generalized Peano continuum.*

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