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θ-MODULAR BANDS OF GROUPS

BY

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ABSTRACT. The class of θ-modular bands of groups is defined by means of a type of modularity condition on the lattice of congruences on a band of groups. The main result characterizes θ-modularity as a condition on the multiplication in the band of groups. This result is then applied to the classes of normal bands of groups and orthodox bands of groups.

Introduction. The class of θ-modular bands of groups is defined in [11], where it is shown that θ-modularity is necessary and sufficient for a certain function to embed the lattice of congruences $\Lambda(S)$ in a product lattice. In this paper, we investigate some properties of θ-modular bands of groups, and find several conditions equivalent to θ-modularity. The main result gives a condition on the multiplication in a band of groups S which is necessary and sufficient for S to be θ-modular. This result is then applied to the classes of normal bands of groups and orthodox bands of groups, to obtain a very simple characterization of θ-modularity in these classes.

1. **Preliminaries.** We use the notation and terminology of Clifford and Preston [2], with the following exceptions:

x^{-1}: the inverse of x in H_x, in a band of groups.
$B(S)$: the lattice of band congruences on S.
$M(S)$: the lattice of idempotent-separating congruences on S.
$\{(x, y)\}^*$: the congruence generated by the relation whose only element is the pair (x, y) (see [2, Theorem 1.8]).

By a *band of groups*, we mean a union of groups on which Green's $\mathcal{H}$-relation is a congruence. Following [12], a band B will be called *normal* if $axya = ayxa$ for all $a, x, y \in B$. Theorem 8.2.9 of [6] lists several necessary and sufficient conditions for a band to be normal. A band of groups S is called a *normal band of groups* if the band $S/\mathcal{H}$ is normal. It is known that a band of groups S is a normal band of groups if and only if S is a strong semilattice (mapping semigroup) of completely simple semigroups, namely the $\mathcal{D}$-classes of S. This result

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appears in [7], and in [10]. Some necessary ingredients to obtain the result also appear in [6].

A regular semigroup S is called *orthodox* if its set E_S of idempotents forms a subsemigroup. Orthodox semigroups have been studied in [4] and elsewhere. If A is a subset of a semigroup S, then $\langle A \rangle$ will denote the subsemigroup of S generated by A. Of special interest is the subsemigroup $\langle E_S \rangle$, which has been studied in [3].

If S is any regular semigroup, then the θ-relation on $\Lambda(S)$ is defined by $(\rho, \tau) \in \theta$ if and only if $\rho \cap (E_S \times E_S) = \tau \cap (E_S \times E_S)$ (see [8]). It is shown in [9] that if S is any regular semigroup, then θ is a complete lattice congruence on $\Lambda(S)$. The θ-relation provides an enlightening means of viewing $\Lambda(S)$, particularly in the case that S is a band of groups (see [11]).

Now let S be an arbitrary semigroup. If $\rho, \gamma \in \Lambda(S)$, and $\gamma \subseteq \rho$, then the relation ρ/γ on S/γ defined by $\rho/\gamma = \{(\gamma^{\natural}(x), \gamma^{\natural}(y)) | (x, y) \in \rho\}$ is a congruence. Moreover, the lattice $\gamma \vee \Lambda(S)$ is isomorphic with $\Lambda(S/\gamma)$ under the map $\gamma \vee \tau \to (\gamma \vee \tau)/\gamma$. In particular, if $\gamma \subseteq \rho, \tau$, then $(\rho \wedge \tau)/\gamma = (\rho/\gamma) \wedge (\tau/\gamma)$, and $(\rho \vee \tau)/\gamma = (\rho/\gamma) \vee (\tau/\gamma)$. These facts are readily verified, as is pointed out in [8].

2. **θ-modularity.** The following generalization of the concept of modularity is defined in [11]:

Definition 2.1 [11, Definition 3.11]. Let L be a lattice, and ζ a lattice congruence on L. We say that L is *ζ-modular* if the conditions $a \geq b$, $(a, b) \in \zeta$, $a \wedge c = b \wedge c$, and $a \vee c = b \vee c$, for elements $a, b, c \in L$, imply that $a = b$.

The strength of the ζ-modularity condition depends, of course, on the lattice congruence ζ. For instance, if ζ is the congruence $L \times L$, then ζ-modularity reduces to modularity. If ζ is the diagonal congruence on L, we obtain a trivial condition.

We agree to call a semigroup S ζ-modular, provided that $\Lambda(S)$ is ζ-modular. In particular, taking ζ to be the congruence θ on $\Lambda(S)$ yields the notion of θ-modularity. Several basic classes of semigroups are θ-modular. It is pointed out in [11] that bands and groups are θ-modular. Also, we will see below that completely simple semigroups are θ-modular.

The class of θ-modular bands of groups is particularly interesting, in view of the following result:

Proposition 2.2 [11, Theorems 3.14 and 3.15]. *Let S be a band of groups, and define ψ: $\Lambda(S) \to B(S) \times M(S)$ by $\psi(\rho) = (\rho \vee \mathcal{H}, \rho \wedge \mathcal{H})$. Then ψ is a lattice embedding if and only if S is θ-modular.*

In view of this result, we regard θ-modular bands of groups as being those whose lattice of congruences can be embedded in a natural way into the product of certain sublattices. With this in mind, the main theorem of this paper (Theorem 4.9) can be interpreted as providing information about the structure of S, from information about the structure of the lattice of congruences on S.

We will now show that θ-modularity is preserved under homomorphisms, but first we need some preliminary results. The following lemma is due to Lallement.

Lemma 2.3 [5, Lemma 2.2]. *Let S be a regular semigroup, and $\varphi : S \to T$ a surjective homomorphism. Then $E_T = \{\varphi(e) | e \in E_S\}$.*

Lemma 2.4. *Let S be a regular semigroup, and $\rho, \tau, \alpha \in \Lambda(S)$ such that $\alpha \subseteq \rho, \tau$. Then $\rho\, \theta\, \tau$ if and only if $\rho/\alpha\, \theta\, \tau/\alpha$.*

Proof. By Lemma 2.3, we have $E_{S/\alpha} = \{\alpha^\natural(e) | e \in E_S\}$. Then if $\rho\, \theta\, \tau$, we have $\alpha^\natural(e)\, \rho/\alpha\, \alpha^\natural(f) \Leftrightarrow e\, \rho\, f \Leftrightarrow e\, \tau\, f \Leftrightarrow \alpha^\natural(e)\, \tau/\alpha\, \alpha^\natural(f)$, so that $\rho/\alpha\, \theta\, \tau/\alpha$. The converse argument is similar.

Lemma 2.5. *Suppose S is a band of groups, and $\varphi: S \to T$ is a surjective homomorphism. Then $(\varphi \times \varphi)[\mathcal{H}_S] = \mathcal{H}_T$.*

Proof. We first note that $T = \bigcup_{x \in S} \varphi[H_x]$, so that T is a union of groups. Thus the $\mathcal{H}$-classes of T are groups. Now it is easily seen that for any semigroup, $(\varphi \times \varphi)[\mathcal{H}_S] \subseteq \mathcal{H}_T$. Conversely, suppose $t_1\, \mathcal{H}_T\, t_2$, and let $p_1, p_2 \in S$ such that $\varphi(p_i) = t_i$. Let e_i be the identity of H_{p_i}, for $i = 1, 2$. Then $\varphi(e_i)\, \mathcal{H}_T\, \varphi(p_i) = t_i$. But $\varphi(e_i)$ is an idempotent, so since $H_{t_1} = H_{t_2}$ is a group, we must have $\varphi(e_1) = \varphi(e_2)$ is the identity of $H_{t_1} = H_{t_2}$. Now let $s_1 = p_1 e_2$, and $s_2 = e_1 p_2$. Then since $p_1\, \mathcal{H}\, e_1$, and $p_2\, \mathcal{H}\, e_2$, and since $\mathcal{H}$ is a congruence, we obtain $s_1\, \mathcal{H}\, s_2$. Moreover, we have $\varphi(s_1) = \varphi(p_1)\varphi(e_2) = t_1\varphi(e_2) = t_1$, and $\varphi(s_2) = \varphi(e_1)\varphi(p_2) = \varphi(e_1)t_2 = t_2$, completing the proof.

Corollary 2.6. *Suppose S is a band of groups, and $\varphi: S \to T$ is a surjective homomorphism. Then T is a band of groups.*

Proof. Having already noted that T is a union of groups, it remains to show that $\mathcal{H}_T$ is a congruence. So suppose $t_1\, \mathcal{H}_T\, t_2$, and $t_3 \in T$. By Lemma 2.5, there are elements $s_1, s_2 \in S$ such that $s_1\, \mathcal{H}\, s_2$, and $\varphi(s_i) = t_i$. Let $s_3 \in S$ such that $\varphi(s_3) = t_3$. Then since $\mathcal{H}_S$ is a congruence, and again by Lemma 2.5, we have $t_1 t_3 = \varphi(s_1 s_3)\, \mathcal{H}_T\, \varphi(s_2 s_3) = t_2 t_3$. Thus $\mathcal{H}_T$ is a right congruence. Dually, $\mathcal{H}_T$ is a left congruence.

We can now prove

Proposition 2.7. *If S is a θ-modular semigroup, and $\gamma \in \Lambda(S)$, then S/γ is a θ-modular semigroup. If S is also a band of groups, then S/γ is also a band of groups.*

Proof. The second statement follows from Corollary 2.6. Now suppose that $\rho/\gamma \subsetneq \tau/\gamma$ are θ-related congruences on S/γ. Then $\rho \subsetneq \tau$, and by Lemma 2.4, these are θ-related congruences on S. Now suppose there is some congruence α/γ on S/γ such that $\rho/\gamma \vee \alpha/\gamma = \tau/\gamma \vee \alpha/\gamma$ and $\rho/\gamma \wedge \alpha/\gamma = \tau/\gamma \wedge \alpha/\gamma$. Rewriting this, we have $(\rho \vee \alpha)/\gamma = (\tau \vee \alpha)/\gamma$ and $(\rho \wedge \alpha)/\gamma = (\tau \wedge \alpha)/\gamma$, so that $\rho \vee \alpha = \tau \vee \alpha$ and $\rho \wedge \alpha = \tau \wedge \alpha$. By θ-modularity of S, we conclude that $\rho = \tau$, and hence $\rho/\gamma = \tau/\gamma$. Thus S/γ is θ-modular.

The property of θ-modularity is unfortunately not preserved under direct product, as the following example shows.

Example 2.8. Let $S = \{e, a, f, b\}$ be the semigroup given by the table

	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f

Then S is isomorphic with the direct product of the two-element semilattice with the cyclic group of order two. Each of the factors is clearly θ-modular; and it is not difficult to show that S is not θ-modular (see [11, Example 3.7]).

We shall see in Example 4.15 below that the property of θ-modularity need not be inherited by subsemigroups.

3. **Conditions equivalent to θ-modularity.** In our first result, we use Proposition 2.2. We will also have occasion to use the following results from [11], concerning bands of groups:

(a) $(\rho \vee \mathcal{H}, \rho) \in \theta$ for all $\rho \in \Lambda(S)$.

(b) The function $\psi\colon \Lambda(S) \to B(S) \times M(S)$ defined by $\psi(\rho) = (\rho \vee \mathcal{H}, \rho \wedge \mathcal{H})$ is one-to-one and $\wedge$-preserving (without the assumption of θ-modularity).

As usual, the idempotents E_S of any regular semigroup form a partially ordered set, under the ordering $f \leq e$ if and only if $ef = fe = f$. We can now prove

Proposition 3.1. *Let S be a band of groups. Then S is θ-modular if and only if the following condition is satisfied*:

($*$) *For each pair of idempotents e, f with $f < e$, and for each $x \in H_e$, $y \in H_f$, then $x \,\rho\, e$ for every congruence ρ such that $x \,\rho\, y$.*

Proof. Suppose S is θ-modular, but that condition ($*$) fails to hold. Then there exist idempotents e, f with $f < e$, and $x \in H_e$, $y \in H_f$, and some congruence ρ such that $x \,\rho\, y$, but $(x, e) \notin \rho$. We do, however, have $e \,(\rho \vee \mathcal{H})\, f$, so that $e \,\rho\, f$ since $\rho \vee \mathcal{H} \,\theta\, \rho$. Let τ be the congruence on S generated by the relation $\{(f, y)\}$. Since $D_f < D_e$ in the natural partial ordering of the $\mathcal{D}$-classes, we have

$\tau \cap (D_e \times D_e) = \Delta D_e^2$; that is, τ identifies no two distinct elements of D_e. Now $(e, x) \in \mathcal{H}$, and $e \,\rho\, f \,\tau\, y \,\rho\, x$, so that $(e, x) \in (\rho \vee \tau) \wedge \mathcal{H}$. But $(e, x) \notin \rho \wedge \mathcal{H}$ by assumption, and so by the above remark about τ, we have $(e, x) \notin (\rho \wedge \mathcal{H}) \vee (\tau \wedge \mathcal{H})$. Thus $(\rho \vee \tau) \wedge \mathcal{H} \neq (\rho \wedge \mathcal{H}) \vee (\tau \wedge \mathcal{H})$, so that the function $\psi: \Lambda(S) \to B(S) \times M(S)$ is not $\vee$-preserving. But this contradicts Proposition 2.2.

Conversely, suppose $(*)$ holds. By Proposition 2.2, it will suffice to show that the function $\psi: \Lambda(S) \to B(S) \times M(S)$ is an embedding; and since ψ is always one-to-one and $\wedge$-preserving, it will suffice to show that $(\rho \wedge \mathcal{H}) \vee (\tau \wedge \mathcal{H}) = (\rho \vee \tau) \wedge \mathcal{H}$ for each $\rho, \tau \in \Lambda(S)$. Now it is obvious that $(\rho \wedge \mathcal{H}) \vee (\tau \wedge \mathcal{H}) \subseteq (\rho \vee \tau) \wedge \mathcal{H}$. On the other hand, suppose $(x, y) \in (\rho \vee \tau) \wedge \mathcal{H}$. Then $x \,\mathcal{H}\, y$, so that x and y lie in the same maximal subgroup, which we will label G. Let e be the identity of this group. Since $(x, y) \in \rho \vee \tau$, there exist elements $z_1, z_2, \cdots, z_n$ such that $x \,\rho\, z_1 \,\tau\, z_2 \,\rho \cdots \rho\, z_{n-1} \,\tau\, z_n = y$. For each i, let $v_i = ez_ie$. Then clearly $x \,\rho\, v_1 \,\tau\, v_2 \,\rho \cdots \rho\, v_{n-1} \,\tau\, v_n = y$. Moreover, each v_i lies either in G, or in some $\mathcal{D}$-class below D_e in the natural partial ordering of the $\mathcal{D}$-classes. If all v_i are in G, it is obvious that $(x, y) \in (\rho \wedge \mathcal{H}) \vee (\tau \wedge \mathcal{H})$. Otherwise, let $j = \min\{i \mid v_i \notin G\}$. Then $v_j \notin G$, but for $i < j$, $v_i \in G$. Thus, letting α alternately denote ρ and τ, we have $x \,(\alpha \wedge \mathcal{H})\, v_1 \,(\alpha \wedge \mathcal{H})\, v_2 \,(\alpha \wedge \mathcal{H}) \cdots (\alpha \wedge \mathcal{H})\, v_{j-1} \,\alpha\, v_j$. Let f be the identity of H_{v_j}. Then $f = v_j^{-1}v_j = (ez_je)^{-1}(ez_je)$, so that $fe = (ez_je)^{-1}(ez_je)e = (ez_je)^{-1}(ez_je) = f$. Similarly writing f as $v_jv_j^{-1}$, we see that $ef = f$, and so $f \leq e$. But since $v_j \notin G$, we must have $f < e$. Then by $(*)$, we have $v_{j-1} \,\alpha\, e$, and thus $v_{j-1} \,(\alpha \wedge \mathcal{H})\, e$. Now let $k = \max\{i \mid v_i \notin G\}$. Then $v_k \notin G$, but for $j > k$, $v_j \in G$. Thus $v_k \,\alpha\, v_{k+1} \,(\alpha \wedge \mathcal{H})\, v_{k+2} \,(\alpha \wedge \mathcal{H}) \cdots (\alpha \wedge \mathcal{H})\, v_n = y$. Letting g be the identity of the group H_{v_k}, and proceeding as above, we see that $g < e$, and hence $v_{k+1} \,(\alpha \wedge \mathcal{H})\, e$. Thus we have $x \,(\alpha \wedge \mathcal{H})\, v_1 \,(\alpha \wedge \mathcal{H}) \cdots (\alpha \wedge \mathcal{H})\, v_{j-1} \,(\alpha \wedge \mathcal{H})\, e \,(\alpha \wedge \mathcal{H})\, v_{k+1} \,(\alpha \wedge \mathcal{H}) \cdots (\alpha \wedge \mathcal{H})\, v_n = y$. Thus $(x, y) \in (\rho \wedge \mathcal{H}) \vee (\tau \wedge \mathcal{H})$, which completes the proof.

We include Propositions 2.2 and 3.1 in the following theorem.

Theorem 3.2. *The following conditions on a band of groups S are equivalent*:

(i) *S is θ-modular.*

(ii) *The function $\psi: \Lambda(S) \to B(S) \times M(S)$ defined by $\psi(\rho) = (\rho \vee \mathcal{H}, \rho \wedge \mathcal{H})$ is an embedding.*

(iii) *Condition $(*)$ of Proposition 3.1.*

(iv) *For each pair of idempotents e, f with $f < e$, and each $x \in H_e$, $y \in H_f$, we have $(f, y) \in \{(y, x)\}^*$.*

(v) *For each pair of idempotents e, f with $f < e$, and each $x \in H_e$, we have $(f, fx) \in \{(e, f)\}^*$.*

Proof. The equivalence of (i), (ii), and (iii) has already been demonstrated. We will now show that (iii), (iv), and (v) are equivalent.

(iii) *implies* (iv): Suppose (iii) holds, and let e, f, x, and y be as in (iv). Let $\rho = \{(y, x)\}^*$. Then $x \rho y$, so by (iii), $x \rho e$. But $(\rho \vee \mathcal{H}, \rho) \in \theta$, and $(e, f) \in \rho \vee \mathcal{H}$, so we have $(e, f) \in \rho$. Thus $f \rho e \rho x \rho y$, and hence (iv) holds.

(iv) *implies* (v): We first note that for e, f, and x as in (v), we have $\{(x, fx)\}^* = \{(e, f)\}^*$; for $(e, f) = (e, fe) = (xx^{-1}, fxx^{-1}) \in \{(x, fx)\}^*$, and $(x, fx) = (ex, fx) \in \{(e, f)\}^*$. Now taking $y = fx$ in (iv) yields $(f, fx) \in \{(fx, x)\}^* = \{(e, f)\}^*$, whence (v) holds.

(v) *implies* (iii): Let e, f, x, and y be as in (iii). By (v), we have $(f, fx) \in \{(e, f)\}^*$. Let ρ be as in (iii); that is, $x \rho y$. Then $e (\rho \vee \mathcal{H}) f$, so that $e \rho f$ since $(\rho \vee \mathcal{H}, \rho) \in \theta$. Thus $(f, fx) \in \{(e, f)\}^* \subseteq \rho$. But then since $x \rho y$, we have $y = fy \rho fx \rho f$. Thus $x \rho y \rho f \rho e$, and (iii) holds.

Since every completely simple semigroup vacuously satisfies conditions (iii), (iv), and (v) of Theorem 3.2, it follows immediately that such a semigroup is θ-modular.

The next theorem shows that in order to check a band of groups for θ-modularity, it suffices to check each subsemigroup of the form $D_e \cup D_f$, where e and f are idempotents with $f < e$. We first need a lemma.

Lemma 3.3. *Let S be a band of groups, and let e, f be idempotents with $f \leq e$. Then f commutes with every element of H_e.*

Proof. Let $x \in He$, and consider the element $x^{-1}fx$. We first note that $x^{-1}fx$ is idempotent, for $x^{-1}fxx^{-1}fx = x^{-1}fefx = x^{-1}fx$. Also, since $\mathcal{H}$ is a congruence and $x \mathcal{H} e$, we have $x^{-1}fx \mathcal{H} efe = f$. Then since each $\mathcal{H}$-class contains only one idempotent, we must have $x^{-1}fx = f$. Then $xf = x(x^{-1}fx) = efx = fx$, completing the proof.

It will also be convenient to introduce some notation. Theorem 1.8 of [2] determines the congruence ρ generated by (i.e., the smallest congruence containing) a given relation ρ_0 on a semigroup S. We will be particularly interested in this in the case when ρ_0 is a singleton pair $\{(e, f)\}$ of idempotents. In this case, it follows that elements x and y of S are ρ-related if and only if there exists a nonnegative integer n, and elements a_i, b_i $(i = 1, 2, \cdots n)$ of S such that

$$\begin{aligned} x &= a_1gb_1 \\ a_1gb_1 &= a_2gb_2 \\ a_2gb_2 &= a_3gb_3 \\ a_3gb_3 &= \cdots \\ &\vdots \\ a_ngb_n &= y \end{aligned}$$

where g denotes either e or f. If this can be done, we will say that there is a *ρ-chain* (or simply a *chain*) *from x to y,* and we will call the number n its *length*. Of course, there is no loss of generality in assuming that such a chain has minimal length, and this will always be the assumption. In this case, we may assume that the g in the chain alternately denotes e and f in its vertical steps, lest there be a chain of shorter length. It is possible, however, that g could denote the same element in horizontal steps of the chain. For ease of notation, we will agree to write the above chain as

$$\begin{aligned}
x &= a_1 {}^{e}_{f} b_1 \\
a_1 {}^{f}_{e} b_1 &= a_2 {}^{e}_{f} b_2 \\
a_2 {}^{f}_{e} b_2 &= a_3 {}^{e}_{f} b_3 \\
a_3 {}^{f}_{e} b_3 &= \cdots \\
&\vdots \\
a_n {}^{f}_{e} b_n &= y.
\end{aligned}$$

We can now proceed with the theorem.

Theorem 3.4. *Let S be a band of groups, and for each pair of idempotents e, f of S with $f < e$, let $T(e, f) = D_e \cup D_f$. Then S is θ-modular if and only if each $T(e, f)$ is θ-modular.*

Proof. Suppose first that each $T(e, f)$ is θ-modular. Let e and f be idempotents with $f < e$. Then by Theorem 3.2(v), it suffices to show that if $x \in H_e$, then $(f, fx) \in \{(e, f)\}_S^*$. But by θ-modularity of $T = T(e, f)$, we have $(f, fx) \in \{(e, f)\}_T^*$. But certainly $\{(e, f)\}_T^* \subseteq \{(e, f)\}_S^*$, so it follows that S is θ-modular. Conversely, suppose that S is θ-modular, and that e and f are idempotents with $f < e$. Let $T = T(e, f)$, and let $g < k$ be idempotents of T. Then for $x \in H_k$, since S is θ-modular, there is a chain in S from g to gx:

$$\begin{aligned}
g &= a_1 {}^{g}_{k} b_1 \\
a_1 {}^{k}_{g} b_1 &= a_2 {}^{g}_{k} b_2 \\
a_2 {}^{k}_{g} b_2 &= \cdots \\
&\vdots \\
a_n {}^{k}_{g} b_n &= gx.
\end{aligned}$$

It clearly suffices to show that these a_i, b_i can be chosen from T. Now $gxg = gx$ by Lemma 3.3, so that multiplying each entry in the above chain on each side by g, we obtain the chain

$$g = ga_{1k}{}^{g}b_1 g$$

$$ga_{1g}{}^{k}b_1 g = \cdots$$

$$\vdots$$

$$ga_{ng}{}^{k}b_n g = gx.$$

Thus we may assume that in the original chain, $a_i \in gS$ and $b_i \in Sg$. With this assumption, since $g = a_{1k}{}^{g}b_1 \in a_1 S \cap Sb_1$, we have $a_1 \mathcal{R}\, g\, \mathcal{L}\, b_1$. Since D_g is completely simple, we then have $a_1 g b_1 \in H_g$. But also, $a_1 k b_1 \in D_g$, so $a_1 k b_1 = g a_1 k b_1 g \in g D_g g \subseteq H_g$. Thus $a_{1g}{}^{k}b_1 = a_{2k}{}^{g}b_2 \in H_g$. So there is an element $y \in H_g$ such that $g = a_{2k}{}^{g}b_2 y = y a_{2k}{}^{g}b_2$. Thus $g \in a_2 S \cap Sb_2$, so we obtain $a_2 \mathcal{R}\, g\, \mathcal{L}\, b_2$. Hence certainly $a_2, b_2 \in T$. Repeating the argument, we see that all a_i, b_i are in T. Thus $(g, gx) \in \{(k, g)\}_T^*$, so that T is θ-modular.

4. The main result. In this section we obtain a condition on the multiplication in a band of groups S which is necessary and sufficient for S to be θ-modular. We then apply this condition to θ-modular orthodox bands of groups and θ-modular normal bands of groups to obtain information about their structure. We begin with a preliminary discussion of the idempotents in a band of groups.

Lemma 4.1. *Let S be a band of groups, and let e, f, and g be idempotents with $f, g < e$. Let $f * g$ denote the idempotent in H_{fg}, and $g * f$ the idempotent in H_{gf}. Then $f * g$, $g * f < e$.*

Proof. Since $f \mathcal{R}\, f * g\, \mathcal{L}\, g$, we have $f * g = f(f * g)g$. Then $e(f * g)e = ef(f * g)ge = f(f * g)g = f * g$, so that $f * g \leq e$. But obviously $f * g \neq e$, since $f * g$ and e are in different $\mathcal{D}$-classes, and so $f * g < e$. The result for $g * f$ is dual.

Lemma 4.2. *Let S be a band of groups, and let e, f be idempotents with $f < e$. Let $G = E_{D_f} \cap eSe$, and let $T = \bigcup_{g \in G} H_g$. Then T is a completely simple subsemigroup of D_f.*

Proof. Let $a, b \in T$, and let g, h be the idempotents in H_a, H_b respectively. Then $ab \mathcal{H}\, gh\, \mathcal{H}\, g * h$, and by Lemma 4.1, $g * h < e$. Thus $g * h \in G$, so $ab \in T$, establishing that T is a subsemigroup. Moreover, $a \mathcal{R}_S\, ab\, \mathcal{L}_S\, b$; and since T is a union of groups, and hence regular, it then follows from [1, Proposition 2] that $a \mathcal{R}_T\, ab\, \mathcal{L}_T\, b$. Thus $a \mathcal{D}_T\, b$, implying that T is bisimple. Since the idempotents of T are also idempotents in D_f, and are thus primitive, it follows that they are primitive in T. Thus, T is completely simple.

Proposition 4.3. *Let S be a band of groups, and let e, f be idempotents with $f < e$. Let $F = \langle E_{D_f} \cap eSe \rangle \cap H_f$. Then F is a subgroup of H_f, but is not necessarily normal in H_f.*

Proof. It is obvious that F is closed under product. To see that F is also closed under taking inverses, let $p \in F$. Let T be the subsemigroup of Lemma 4.2 which is associated with (e, f). Then $p \in T$, so by Lemma 1.1 of [3], p has an inverse q in T which is also a product of idempotents in T; that is, $q \in \langle E_{D_f} \cap eSe \rangle$. But then fqf is also an inverse of p. For $(fqf)p(fqf) = fq(fpf)qf = fqpqf = fqf$; and $p(fqf)p = (pf)q(fp) = pqp = p$. But $fqf \in fTf \subseteq fD_f f \subseteq H_f$, and so $fqf = p^{-1}$. But $q \in \langle E_{D_f} \cap eSe \rangle$, so $fqf \in \langle E_{D_f} \cap eSe \rangle \cap H_f = F$. Thus F is a subgroup of H_f. We will see in Example 4.15 that F need not be normal.

Our next proposition relates θ-modularity with the normal subgroup generated by the subgroup F above. We begin with some lemmas.

Lemma 4.4. *Let S be a band of groups, and let e, f be idempotents with $f < e$. Let $a, b \in S$ such that $a \mathcal{R} f \mathcal{L} b$. Then $a^{-1}ae \in E_{R_f} \cap eSe$, and $ebb^{-1} \in E_{L_f} \cap eSe$.*

Proof. Since $fe = f$, and $a^{-1}a \mathcal{D} f$, we have $a^{-1}ae \in D_f$. Then since $a^{-1} \mathcal{R} f$, we have $fa^{-1} = a^{-1}$, so that $f(a^{-1}ae) = a^{-1}ae$. Thus $a^{-1}ae \in R_f$. To see that $a^{-1}ae$ is idempotent, note that $(a^{-1}ae)(a^{-1}ae) = (a^{-1}ae)f(a^{-1}ae) = (a^{-1}a)(ef)(a^{-1}ae) = (a^{-1}a)f(a^{-1}ae) = (a^{-1}a)(a^{-1}ae) = a^{-1}ae$. Moreover, $e(a^{-1}ae)e = ea^{-1}ae = e(fa^{-1}ae) = (ef)(a^{-1}ae) = f(a^{-1}ae) = a^{-1}ae$. Thus $a^{-1}ae \in E_{R_f} \cap eSe$. The result for ebb^{-1} is dual.

Lemma 4.5. *Let S be a union of groups. Let f, g, h be idempotents, and $a, b \in S$ such that $a \mathcal{H} g \mathcal{R} f \mathcal{L} h \mathcal{H} b$. Suppose that N is a normal subgroup of $H = H_f$ containing gh. Then $ab \in N$ if and only if $afb \in N$.*

Proof. Let $x = af$, $y = fb$, and $p = gh$. Then $x, y \in H$, and $p \in N$. Furthermore, $xy = affb = afb$, and $xpy = afghfb = aghb = ab$. Thus, it will suffice to show that $xy \in N$ if and only if $xpy \in N$. But now $(xpy)(xy)^{-1} = xpyy^{-1}x^{-1} = xpfx^{-1} = xpx^{-1} \in N$ by normality of N. Thus xy and xpy lie in the same right coset of N in H, and the result follows.

Proposition 4.6. *Let S be a θ-modular band of groups. Then whenever e, f are idempotents with $f < e$, the product $f \cdot H_e$ is contained in the normal subgroup N_f of H_f generated by $F = \langle E_{D_f} \cap eSe \rangle \cap H_f$.*

Proof. Let $x \in H_e$. By Theorem 3.2(v), we have $(f, fx) \in \rho = \{(e, f)\}^*$. Hence there exist elements a_i, b_i of S such that

$$
\begin{aligned}
&f = a_1 {}^{e}_{f} b_1\\
&\quad a_1 {}^{f}_{e} b_1 = a_2 {}^{e}_{f} b_2\\
&\qquad\qquad a_2 {}^{f}_{e} b_2 = \cdots\\
&\qquad\qquad\qquad\qquad \vdots\\
&\qquad\qquad\qquad\qquad a_n {}^{f}_{e} b_n = fx.
\end{aligned}
$$

Following the proof of Theorem 3.4, we may assume that $a_i \mathcal{R} f \mathcal{L} b_i$ for all i. Now letting $g_i = a_i^{-1} a_i e$, and $h_i = e b_i b_i^{-1}$, we have $a_i e b_i = (a_i g_i)(h_i b_i)$, and $a_i f b_i = (a_i g_i) f (h_i b_i)$. Thus, letting $a_i' = a_i g_i$, and $b_i' = h_i b_i$, we can rewrite the above ρ-chain as

$$
\begin{aligned}
&f = a'_1 {}^{-}_{f} b'_1\\
&\quad a'_1 {}^{f}_{-} b'_1 = a'_2 {}^{-}_{f} b'_2\\
&\qquad\qquad a'_2 {}^{f}_{-} b'_2 = \cdots\\
&\qquad\qquad\qquad\qquad \vdots\\
&\qquad\qquad\qquad\qquad a'_n {}^{f}_{-} b'_n = fx,
\end{aligned}
$$

where the symbol $-$ indicates that no element appears. Moreover, we have $a_i' \mathcal{H} g_i \mathcal{R} f \mathcal{L} h_i \mathcal{H} b_i'$, for each i. So applying Lemma 4.5 to each vertical step in the above chain, and using the fact that $f \in N_f$, we see that every entry in the chain is in N_f. In particular, $fx \in N_f$. Thus $f \cdot H_e \subseteq N_f$, as was to be shown.

The converse of this proposition is also true. Before we prove it, however, we need another lemma.

Lemma 4.7. *Let S be a band of groups, and let e, f be idempotents with $f < e$. Let $p \in \langle E_{D_f} \cap eSe \rangle$. Then $(p, f) \in \{(e, f)\}^*$.*

Proof. Let ρ denote $\{(e, f)\}^*$. It will suffice to show that $g \,\rho\, f$, for every $g \in E_{D_f} \cap eSe$. For in this case, writing $p = g_1 g_2 \cdots g_n$, where each g_i is in $E_{D_f} \cap eSe$, we have $p = g_1 g_2 \cdots g_n \,\rho\, ff \cdots f = f$. So suppose $g \in E_{D_f} \cap eSe$. Then $fgf \,\rho\, ege = g$ since $g \in eSe$. But since D_f is completely simple, $fgf \,\mathcal{H}\, f$. Thus $(g, f) \in \rho \vee \mathcal{H}$. Since $(\rho \vee \mathcal{H}, \rho) \in \theta$, we then have $(g, f) \in \rho$, completing the proof.

We can now prove

Proposition 4.8. *Let S be a band of groups such that whenever e, f are idempotents with $f < e$; then $f \cdot H_e$ is contained in the normal subgroup N_f of H_f generated by $F = \langle E_{D_f} \cap eSe \rangle \cap H_f$. Then S is θ-modular.*

Proof. Since by Proposition 4.3, F is a subgroup of H_f, it follows that N_f consists of all finite products of conjugates of F in H_f. That is, the elements of N_f have the form $\Pi_{i=1}^n h_i p_i h_i^{-1}$ for some n, where $h_i \in H_f$, $p_i \in F$. Now letting $\rho = \{(e, f)\}^*$, we have by Lemma 4.7 that $p_i \,\rho\, f$ for each i. Hence $h_i p_i h_i^{-1}$ $\rho\, h_i f h_i^{-1} = f$, and thus $\Pi_{i=1}^n h_i p_i h_i^{-1} \,\rho\, f^n = f$. In particular, since $f \cdot H_e$ is assumed to be contained in N_f, we obtain $fx \,\rho\, f$, for every $x \in H_e$. Then by Theorem 3.2(v), it follows that S is θ-modular.

Combining Propositions 4.6 and 4.8, we obtain the following characterization of θ-modular bands of groups.

Theorem 4.9. *Let S be a band of groups. Then S is θ-modular if and only if whenever e, f are idempotents with $f < e$ then $f \cdot H_e$ is contained in the normal subgroup N_f of H_f generated by $F = \langle E_{D_f} \cap eSe\rangle \cap H_f$.*

We now apply this result to the class of orthodox bands of groups.

Theorem 4.10. *Let S be an orthodox band of groups. Then S is θ-modular if and only if whenever e, f are idempotents with $f < e$ then $f \cdot H_e = \{f\}$.*

Proof. Since S is orthodox, $E_{D_f} \cap eSe$ is a subsemigroup, and hence $\langle E_{D_f} \cap eSe\rangle \cap H_f = E_{D_f} \cap eSe \cap H_f = \{f\}$. The result then follows from Theorem 4.9.

Before we apply Theorem 4.9 to the class of normal bands of groups, we need some additional results.

Lemma 4.11. *Let S be a regular semigroup, and $\varphi: S \to T$ a surjective homomorphism. If $e, f, \in E_T$ with $f \leq e$, then there are elements $e', f' \in E_S$ with $f' \leq e'$, and such that $\varphi(e') = e$, $\varphi(f') = f$.*

Proof. The existence of idempotent pre-images of e and f is the content of Lemma 2.3. The fact that e' and f' can be chosen so that $f' \leq e'$ was first brought to the author's attention by John Selden (unpublished); a proof is given here for completeness. Having chosen $e' \in E_S$ such that $\varphi(e') = e$, restrict φ to the regular subsemigroup $e'Se'$ of S. We have $\varphi[e'Se'] = \varphi(e')\varphi[S]\varphi(e') =$ eTe, and $f = efe \in eTe$. So applying the first part, we obtain $f' \in E_{e'Se'}$, so that $\varphi(f') = f$. But then clearly $f' \leq e'$.

Lemma 4.12 [6, Theorem 8.2.9]. *Let B be a normal band, and denote the $\mathfrak{D}$-classes of B by $\{B_\alpha \mid \alpha \in Y\}$, where Y is the semilattice $B/\mathfrak{D}$. Then for all $\beta \leq \alpha$ in Y, and each $e \in B_\alpha$, there is a unique element $f \in B_\beta$ such that $f \leq e$.*

If S is a band of groups, we will denote the $\mathfrak{D}$-classes of S by $\{S_\alpha \mid \alpha \in Y\}$, where Y is the semilattice $S/\mathfrak{D}$. It follows easily from [1, Proposition 2] that if S is a band of groups, then $\mathcal{H}_{S_\alpha} = \mathcal{H}_S \cap (S_\alpha \times S_\alpha)$. It is also easy to see that the $\mathcal{D}$-classes of $S/\mathcal{H}$ can be identified with the rectangular bands $S_\alpha/\mathcal{H}_{S_\alpha}$. (For the details, see [10].) We then have

Lemma 4.13. *Let S be a normal band of groups, and suppose $e \in E_{S_\alpha}$, and $\beta \le \alpha$. Then there is a unique element $f \in E_{S_\beta}$ such that $f \le e$.*

Proof. Since $S/\mathcal{H}$ is a normal band, we obtain the existence of f from Lemmas 4.11 and 4.12, by considering the homomorphism $\mathcal{H}^\natural: S \longrightarrow S/\mathcal{H}$. Now if there is another element $f' \in E_{S_\beta}$ with $f' \le e$, then $(f, f') \in \mathcal{D}$, so that $(\mathcal{H}^\natural(f), \mathcal{H}^\natural(f')) \in \mathcal{D}_{S/\mathcal{H}}$. By Lemma 4.12, we then have $\mathcal{H}^\natural(f) = \mathcal{H}^\natural(f')$; that is, $f \mathcal{H} f'$. Hence $f = f'$.

We now have

Theorem 4.14. *Let S be a normal band of groups. Then S is θ-modular if and only if whenever e, f are idempotents with $f < e$ then $f \cdot H_e = \{f\}$.*

Proof. By Lemma 4.13, $E_{D_f} \cap eSe = \{f\}$, and hence $\langle E_{D_f} \cap eSe \rangle \cap H_f = \{f\}$. The result then follows from Theorem 4.9.

We conclude with an example which shows that Theorem 4.9 cannot be strengthened by replacing the condition $f \cdot H_e \subseteq N_f$ with $f \cdot H_e \subseteq F$. (This latter condition, however, is clearly sufficient for θ-modularity.)

Example 4.15. Let G be the dihedral group of eight elements. That is, G is the group generated by two elements a and b, subject to the relations $a^2 = b^4 = 1$, and $ba = ab^3$. The elements of G are $\{1, a, b, b^2, b^3, ab, ab^2, ab^3\}$. Of these, a, b^2, ab, ab^2, and ab^3 have order 2. The center of G is $Z = \{1, b^2\}$.

Let T be the completely simple semigroup $X \times G \times Y$, where $X = \{x, x'\}$, $Y = \{y, y'\}$, and the sandwich function ϕ: $Y \times X \longrightarrow G$ is defined by $\phi(y, x) = a$, $\phi(y', x) = \phi(y, x') = \phi(y', x') = 1$. One then computes that the idempotents of T are (x, a, y), $(x', 1, y)$, $(x', 1, y')$, and $(x, 1, y')$. Furthermore, we have

$$(x, a, y)(x', 1, y') = (x, a, y'), \qquad (x', 1, y)(x, 1, y') = (x', a, y'),$$

$$(x', 1, y')(x, a, y) = (x', a, y), \qquad (x, 1, y')(x', 1, y) = (x, 1, y).$$

Hence, these elements are in $\langle E_T \rangle$. But since the element a has order 2, and ϕ takes on only the values 1 and a, it is evident that $\langle E_T \rangle$ consists precisely of E, together with the four elements computed above. In particular, for each idempotent f of T, $\langle E \rangle \cap H_f$ is the set of elements in H_f having G-coordinate 1 or a. That is, thinking of H_f as a copy of G, $\langle E \rangle \cap H_f$ is a copy of $\{1, a\}$. But $\{1, a\}$ is clearly not normal in G, for $bab^{-1} = bab^3 = ab^3b^3 = ab^2$, and this result carries over to H_f. The normal subgroup of G generated by $\{1, a\}$ is

$\{1, a, b^2, ab^2\}$, and it is readily verified that the appropriate copy of this in H_f is the set of elements in H_f having G-coordinate 1, a, b^2, or ab^2.

Now let $S = T \cup Z$, where Z is the center of G, and extend the multiplication by defining

$$(x_\alpha, g, y_\beta) \cdot h = (x_\alpha, gh, y_\beta), \qquad h \cdot (x_\alpha, g, y_\beta) = (x_\alpha, hg, y_\beta)$$

for $x_\alpha \in X$, $y_\beta \in Y$, $h \in Z$, and $g \in G$. It is straightforward to establish that S is a semigroup. It is then obvious that S is a band of groups having T and Z as its $\mathfrak{D}$-classes. The element 1 of Z is an identity for S, and Z is the group of units of S. (We are now using 1 and b^2 to stand for elements of both S and G; but there should be no confusion.) For each idempotent f of T, we have $f < 1$, and $F = \langle E_{D_f} \cap 1S1\rangle \cap H_f = \langle E_T\rangle \cap H_f$ is a subgroup of H_f, but, as we have seen above, is not normal in H_f. (We continue with the example, but note that at this point, the proof of Proposition 4.3 is complete.)

Write $f = (x_0, g, y_0)$, where f is an idempotent of T. Then g is either 1 or a, as we have seen above. It is then easily seen that $F = \{(x_0, 1, y_0), (x_0, a, y_0)\}$. On the other hand, $f \cdot H_1 = f \cdot Z = (x_0, g, y_0) \cdot \{1, b^2\} = \{(x_0, g, y_0), (x_0, gb^2, y_0)\}$. This set is then either $\{(x_0, 1, y_0), (x_0, b^2, y_0)\}$, or $\{(x_0, a, y_0), (x_0, ab^2, y_0)\}$, depending on whether g is equal to 1 or a. In either case, we see that $f \cdot H_1 \nsubseteq F$.

However, the normal subgroup of H_f generated by F is $N_f = \{(x_0, 1, y_0), (x_0, a, y_0), (x_0, b^2, y_0), (x_0, ab^2, y_0)\}$, so we certainly have $f \cdot H_1 \subseteq N_f$. It then follows from Theorem 4.9 that S is θ-modular; and yet there exist idempotents $f < e$ (namely $e = 1$) in S such that $f \cdot H_e \nsubseteq F$.

This example also shows us that subsemigroups of θ-modular bands of groups need not be θ-modular. For instance, the subsemigroup $\{1, b^2, (x', 1, y'), (x', b^2, y')\}$ of S is isomorphic to the semigroup of Example 2.8, which, as we already remarked, is not θ-modular.

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