

## JACOBSON STRUCTURE THEORY FOR HESTENES TERNARY RINGS<sup>(1)</sup>

BY

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ABSTRACT. The principal results are an extension of the density theorem to Hestenes ternary rings and a characterization of primitive ternary rings.

**Introduction.** M. R. Hestenes [2] defines a *complex ternary algebra* to be a linear space  $S$  over the complex field such that to each triple of elements of  $S$  there is an element  $AB^*C$  in  $S$  called the triple product of  $A$ ,  $B$  and  $C$  satisfying the following conditions for all  $A, B, C, D$  and  $E$  in  $S$ :

1.  $AA^*A = 0$  if and only if  $A = 0$ ;
2.  $(AB^*C)DE = A(DC^*B)^*E = AB^*(CD^*E)$ ;
3. for every scalar  $a$ ,  $(aA)B^*C = a(AB^*C) = AB^*(aC)$ ;
4.  $(A + B)C^*D = AC^*D + BC^*D$ ,  $DC^*(A + B) = DC^*A + DC^*B$ .

Two important examples of complex ternary algebras are the class of rectangular matrices,  $*$  being the conjugate transpose, and the class of all closed operators from one complex Hilbert space to another,  $*$  being the adjoint. Both of these examples also illustrate the concept of a ternary ring whose elements are homomorphisms from one abelian group to another (see §1). In this paper we shall begin the extension of the classical Jacobson structure theory for binary rings to Hestenes ternary rings, obtaining first the Chevalley-Jacobson density theorem. Next we extend the notion of primitivity and obtain two characterizations: one in terms of ternary modules and the other as conditions imposed on maximal modular one-sided ideals. Following the theory, two examples will be discussed. We conclude with a statement of some unsolved problems, but since the topic has scarcely been approached by strictly algebraic methods, the reader will no doubt easily extend our list.

**1. The density theorem.** To avoid confusion with other kinds of ternary rings, we shall define a *Hestenes ternary ring* to be a set  $R$  together with a binary operation  $+$  such that  $(R, +)$  is an abelian group and a ternary operation denoted by

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juxtaposition satisfying the following for all  $a, b, c, d$  and  $e$  in  $R$ :

$$(1.1) \quad (abc)de = a(dcb)e = ab(cde);$$

$$(1.2) \quad (a + b)cd = acd + bcd, \quad a(b + c)d = abd + acd, \quad ab(c + d) = abc + abd.$$

Now let  $R$  be a Hestenes ternary ring and suppose  $U$  and  $V$  are abelian groups.

We call  $(U, V)$  an  $R$ -module in case there are mappings  $U \times R \rightarrow V$  and  $R \times V \rightarrow U$  satisfying

$$(1.3) \quad (x_1 + x_2)a = x_1a + x_2a, \quad x(a + b) = xa + ab,$$

$$(1.4) \quad a(y_1 + y_2) = ay_1 + ay_2, \quad (a + b)y = ay + by,$$

$$(1.5) \quad (b(xa))c = x(abc), \quad a((cy)b) = (abc)y,$$

for all  $x_1, x_2, x$  in  $U$ , all  $y_1, y_2, y$  in  $V$  and all  $a, b, c$  in  $R$ .

Let us also define

$$UR = \left\{ \sum_{\alpha \in A} x_{\alpha} a_{\alpha} \mid A \text{ is finite, } x_{\alpha} \in U, a_{\alpha} \in R \right\}$$

and

$$RV = \left\{ \sum_{\beta \in B} a_{\beta} y_{\beta} \mid B \text{ is finite, } y_{\beta} \in V, a_{\beta} \in R \right\}.$$

Let  $W$  and  $Z$  be abelian subgroups of  $U$  and  $V$  respectively. We called  $(W, Z)$  an  $R$ -submodule of  $(U, V)$  in case  $WR \subseteq Z$  and  $RZ \subseteq W$ . The module  $(U, V)$  is said to be *irreducible* in case  $(RV, UR) \neq (0, 0)$  and the only submodules of  $(U, V)$  are itself and  $(0, 0)$ .

The following lemmas will be useful for the proof of the density theorem.

**Lemma 1.1.**  $(U, V)$  is irreducible if and only if  $xR = V$  for each nonzero  $x \in U$  and  $Ry = U$  for each nonzero  $y \in V$ .

**Proof.** Suppose  $(U, V)$  is irreducible. Define  $W = \{x/x \in U \text{ and } xR = 0\}$ , clearly  $(W, WR)$  is an  $R$ -submodule of  $(U, V)$  different from  $(U, V)$ . It follows that  $W = 0$  and  $WR = 0$ . For the rest of the proof let  $x \in U, x \neq 0$ . By what we have just shown  $xR \neq 0$  and hence  $(R(xR), xR) = (U, V)$  since it is an  $R$ -submodule. We conclude  $R(xR) = U$  and  $xR = V$ . Similarly  $Ry = U$  for each nonzero  $y \in V$ .

Conversely, suppose  $xR = V$  and  $Ry = U$  for each nonzero  $x$  and  $y$ . Then  $R(xR) = U$  and  $(Ry)R = V$  from which it follows that the only nonzero submodule of  $(U, V)$  is  $(U, V)$  itself.

**Lemma 1.2.** If there is a faithful  $R$ -module  $(U, V)$ , then  $R$  is isomorphic to a ternary ring  $S$  each of whose elements is a homomorphism of  $U$  into  $V$  and  $R$  is anti-isomorphic to a ternary ring  $T$  each of whose elements is a homomorphism from  $V$  into  $U$ .

**Proof.** For each  $a, b \in R$  consider the maps  $a_{\tau}: x \rightarrow xa$  for each  $x \in U$ ,  $b_{\tau}: y \rightarrow by$  for each  $y \in V$  and define  $R_1 = \{a_{\tau} \mid a \in R\}$  and  $R_2 = \{b_{\tau} \mid b \in R\}$ .  $R_1$  is a

ternary ring with multiplication defined by  $a_r b_r c_r = a_r b_l c_r$ . The maps  $a \rightarrow a_r$  and  $b \rightarrow b_l$  are easily seen to be respectively an isomorphism and an anti-isomorphism.

For the remainder of this section we assume that  $(U, V)$  is a faithful, irreducible  $R$ -module. By Lemma 1.2, we may as well think of  $R$  as a ring whose elements are homomorphisms of  $U$  into  $V$  anti-isomorphic to a ring  $R^*$  whose elements are homomorphisms of  $V$  into  $U$ . Here we can actually realize  $*$  as the anti-isomorphism  $a \rightarrow a_r$ , and the triple product  $abc$  as  $ab^*c$ . Now let us form two binary rings:

$$R_1 = \left\{ \sum_{\alpha \in A} r_\alpha s_\alpha^* \mid r_\alpha, s_\alpha \in R, A \text{ is finite} \right\}$$

and

$$R_2 = \left\{ \sum_{\beta \in B} r_\beta^* s_\beta \mid r_\beta, s_\beta \in R, B \text{ is finite} \right\}.$$

By Lemma 1 and the classical theory,  $R_1$  and  $R_2$  are dense rings of linear transformations of  $U$  and  $V$  respectively. By Shur's lemma, the underlying division rings are given by

$$D_1 = \{d \in \text{Hom}(U, U) \mid dab^* = ab^*d \text{ for all } a, b \in R\}$$

and

$$D_2 = \{e \in \text{Hom}(V, V) \mid ea^*b = a^*be \text{ for all } a, b \in R\}.$$

**Lemma 1.3.**  $D_1 \cong D_2$  (as rings).

**Proof.** Let  $x \in U, x \neq 0$  and  $y \in V, y \neq 0$ . For each  $d \in D_1$ , each  $e \in D_2$  and each  $a \in R$  define the maps  $d'$  and  $e'$  as follows:

$$(1.6) \quad (xa)d' = (xd)a,$$

$$(1.7) \quad (ya^*)e' = (ye)a^*.$$

Since  $xR = V$  and  $Ry = U, d'$  and  $e'$  are indeed maps. It is easy to show that  $d' \in D_2$  and  $e' \in D_1$ . The mappings  $d \rightarrow d'$  and  $e \rightarrow e'$  are actually inverse ring isomorphisms. It is trivial to show they are homomorphisms. We now verify  $d'' = d$  for any  $d \in D_1$ . Since  $(U, V)$  is irreducible, there is an  $a \in R$  such that  $xa = y$  and there is a  $b \in R$  such that  $yb^* = x$ . It follows that  $xd'' = (yb^*)d'' = (yd')b^* = (xad')b^* = x dab^* = xab^*d = xd$  and hence  $d'' = d$ .

We can now regard both  $U$  and  $V$  as  $D$ -vector spaces where  $D$  is a division ring. Furthermore, (1.6) and (1.7) say that  $R$  is a ring of  $D$ -linear transformations of  $U$  into  $V$  and  $R^*$  is a ring of  $D$ -linear transformations of  $V$  into  $U$ . It is now possible to prove a generalization of the Chevalley-Jacobson density theorem. As Professor Smiley has pointed out in [8], Jacobson's original proof can be extended. In order to make the exposition complete, we take the liberty of presenting this argument in its entirety.

**Theorem.** Let  $(U, V)$  be a faithful, irreducible  $R$ -module where  $R$  is a ternary ring. Let  $x_1, \dots, x_n$  be linearly independent in  $U$  and suppose  $y_1, \dots, y_n$  are arbitrary in  $V$ ; then there is an element  $a$  of  $R$  such that  $x_i a = y_i$  for  $i = 1, \dots, n$ . That is,  $R$  is a dense ring of linear transformations of  $U$  into  $V$ .

**Remark.** Of course, the theorem has an analog for  $R^*$ .

**Proof.** We use induction on  $n$ , noting that Lemma 1.1 proves the theorem for  $n = 1$ . We now assume the theorem for any linearly independent set of  $n - 1$  elements from  $U$ . The following lemma contains the meat of the theorem.

**Lemma 1.4.** There is some element  $c$  of  $R$  such that  $x_i c = 0$  for all  $i$  less than  $n$  but  $x_n c \neq 0$ .

**Proof.** Suppose the lemma is false and define  $S$  by  $S = \{a \in R \mid x_i a = 0 \text{ for } 1 < i < n\}$ . If  $a$  is in  $S$  and  $x_1 a = 0$  then  $x_n a = 0$ . Our induction hypothesis tells us that  $x_1 S \neq 0$ , and by a previous theorem,  $x_1 S = V$ . We can therefore define a map  $f: V \rightarrow V$  by  $(x_1 a)f = x_n a$  for  $a$  in  $S$ . It is trivial to show  $f$  is a map and easy to verify that  $f$  is actually an element of  $D_2$ . By the proof of Lemma 1.3, there is a unique  $e$  in  $D_1$  such that  $x_1 e a = x_1 a f$  for all  $a$  in  $R$ . But this implies  $(x_1 e - x_n)S = 0$ . Now I say that if  $x_0$  is linearly independent of  $x_2, \dots, x_n$  and  $y_0$  is arbitrary in  $V$ , then there is an element  $b$  of  $S$  such that  $x_0 b = y_0$ . Applying the induction hypothesis to  $x_0, x_2, \dots, x_{n-1}$  we see that  $x_0 S \neq 0$ ; it follows that  $x_0 S = V$  and the claim is proved. In fact,  $x_0 S = x_0 R$ . If we set  $x_0 = x_1 e - x_n$  then we must conclude  $x_0 R = 0$  which means  $x_1 e = x_n$  and linear independence has been contradicted. Thus, the lemma is proved.

Using the lemma and the case when  $n = 1$ , there is an  $a_n \in R$  such that  $x_i a_n = 0$  for  $i \neq n$  and  $x_n a_n = y_n$ . Replacing  $x_n$  by  $x_k$ , we obtain  $a_k \in R$  such that  $x_i a_k = 0$  for  $i \neq k$  and  $x_k a_k = y_k$ . Finally,  $a = \sum_{k=1}^n a_k$  has the property that  $x_i a = y_i$  for  $i = 1, \dots, n$  and the theorem is proved.

**2. Primitive ternary rings.** In §1 we made implicit use of the concept of a ternary ring homomorphism and its corresponding fundamental theorem. We now make these notions explicit. An additive subgroup of a Hestenes ternary ring  $R$  is called a *right (left) ideal* if  $ars \in A$  ( $rsa \in A$ ) for all  $a \in A$  and all  $r, s \in R$ ;  $A$  is an *ideal* if it is both a right and left ideal and  $ras \in A$  for all  $a \in A$  for all  $r, s \in R$ . A mapping  $\rho$  from a ternary ring  $R$  to another  $S$  satisfying  $(a + b)\rho = a\rho + b\rho$  and  $(abc)\rho = a\rho b\rho c\rho$  for all  $a, b, c \in R$  is called a *ternary ring homomorphism*.

**Theorem 2.1 (Fundamental Theorem).** If  $\rho: R \rightarrow S$  is a ternary ring homomorphism, then  $\text{Ker } \rho$  is an ideal of  $R$  and  $R\rho \cong R/\text{Ker } \rho$ .

We call a ternary ring  $R$  *primitive* in case there exists a faithful, irreducible  $R$ -module. A ternary ring whose elements are homomorphisms of  $U$  into  $V$  is called

irreducible in case  $(U, V)$  is irreducible as an  $R$ -module. As an immediate consequence of Lemma 1.2 and the density theorem we remark that a ternary ring  $R$  is primitive if and only if  $R$  is isomorphic to a dense ring of  $D$ -linear transformations of  $U$  into  $V$  and anti-isomorphic to a dense ring of  $D$ -linear transformations of  $V$  into  $U$ .

If  $R$  is a Hestenes ternary ring and  $r \in R$  we define  $RRr = \{str \mid s \in R, t \in R\}$  and  $RrR = \{srt \mid s \in R, t \in R\}$ . If  $A$  is a right ideal of  $R$ , we define  $(A : R) = \{r \in R \mid RRr \subseteq A\}$  and  $[A : R] = \{r \in R \mid RrR \subseteq A\}$ . We call  $A$  maximal in case  $A \neq R$  and, for any right ideal  $B$  such that  $A \subset B \subseteq R$ , we have  $B = R$ . We call  $A$  modular in case there are  $e, f$  in  $R$  such that  $r - efr \in A$  for every  $r \in R$ . We make analogous definitions for left ideals.

**Theorem 2.2.** *The following are equivalent:*

- (1)  $R$  is primitive.
- (2) There is a maximal modular right ideal  $A$  such that  $(A : R) = 0 = [A : R]$ .
- (3) There is a maximal right ideal  $A$ , a maximal left ideal  $B$  and  $e, f \in R$  such that

- (2.1)  $r - efr \in A$  for all  $r \in R$ ,
- (2.2)  $r - ref \in B$  for all  $r \in R$ ,
- (2.3)  $eBR \subseteq A$ ,
- (2.4)  $RAf \subseteq B$ ,
- (2.5)  $(A : R) = 0 = (B : R)$ .

**Proof.** We shall prove  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ .

Let  $(U, V)$  be a faithful, irreducible  $R$ -module. Let  $x \in U, x \neq 0$  and define  $A = \{a \in R \mid xa = 0\}$ . Since  $(U, V)$  is irreducible, there are  $e, f \in R$  such that  $f(xe) = x$ . Clearly  $A$  is a modular right ideal and  $A \neq R$ . Suppose  $A \subset B \subseteq R$  and  $B$  is a right ideal; then  $xB$  is a nonzero  $R$ -submodule of  $V$  and so  $xB = xR$ . It follows that for each  $r \in R$ , there is a  $b \in B$  such that  $r - b \in A$ . Since  $A \subset B, R = B$  and  $A$  is maximal. It is easy to verify that  $(A : R) = (0 : U)$  and  $[A : R] = (V : 0)$ . Hence (2) must follow from the fact that  $(U, V)$  is faithful.

Suppose (2) is the case and  $r - efr \in A$  for every  $r \in R$ ; define  $B = \{b \in R \mid ebR \subseteq A\}$ . It is trivial to verify that  $B$  is a left ideal and  $r - ref \in B$  for every  $r \in R$ . Since  $A$  is maximal and modular, it is clear that  $f \notin B$ . Now suppose  $C$  is a left ideal and  $B \subset C \subseteq R$ . Then there is some  $c \in C$  such that  $ecR \not\subseteq A$ ; it follows that there is some  $a \in A$  and some  $s \in R$  such that  $e = a + ecs$ . Now for each  $r \in R, ref = raf + r(ecs)f$  and, since one can easily show  $RAf \subseteq B$ , it follows that  $r - rsc \in B$  for each  $r \in R$ . But  $B \subset C$  and  $c \in C$  so  $R = C$ . Hence,  $B$  is maximal. To prove (3) we need only show that  $eBR \subseteq A$  and  $(B : R) = 0$ . Since  $A$  and  $B$  are maximal,  $A = \{a \in R \mid Raf \subseteq B\}$ . But  $R(eBR)f \subseteq (RRB)ef \subseteq Bef \subseteq B$  and so  $eBR \subseteq A$ . If  $rRR \subseteq B$ , then  $e(rRR)R = eR(RrR) \subseteq A$  and so  $(B : R) \subseteq [A : R]$ . From (2) it follows that  $(A : R) = 0 = (B : R)$ .

Assume that (3) holds; define  $U = R - B$  and  $V = R - A$ . By (2.3) and (2.4) the following are maps:  $(r + B)s = ers + A$ ,  $r(s + A) = rsf + B$ . Using (2.1) and (2.2) one can easily verify that  $(U, V)$  is an  $R$ -module.  $(U, V)$  is irreducible because of the correspondence between  $R$ -submodules of  $U(V)$  and right (left) ideals of  $R$  containing  $A(B)$ . We now prove  $(0 : U) = 0$ ; a similar argument shows  $(V : 0) = 0$ . If  $a \in (0 : U)$  then  $eRa \subseteq A$ . It follows that  $e(RRR)a = eR(RRa) \subseteq A$  and so  $RRa \subseteq A$ . Hence,  $(0 : U) \subseteq (A : R) = 0$ .

**3. Examples of primitive ternary rings.** Consider the class  $\mathfrak{M}$  of all infinite matrices with elements taken from a division ring  $D$ . Look at the subclass  $\mathfrak{Q}$  of all such matrices which are both row and column finite (all but a finite number of elements are zero). If  $F, G, H \in \mathfrak{Q}$  we define  $FGH$  to be  $FG^T H$  using ordinary matrix multiplication; with this multiplication  $\mathfrak{Q}$  becomes a Hestenes ternary ring. Let  $A$  be the set of all elements of  $\mathfrak{Q}$  with the entire first row zero and  $B$  the set of all elements of  $\mathfrak{Q}$  with the entire first column zero. It is easy to show that  $A$  and  $B$  are respectively maximal right and left ideals of  $\mathfrak{Q}$ . At this point we should mention a possible source of confusion. There are actually two structures on the ring  $\mathfrak{Q}$ , one binary and one ternary. If we restrict ourselves to the binary structure, we need only say that  $A$  is a maximal modular (binary) ideal and therefore by the classical theory  $\mathfrak{Q}$  is primitive (binary). However, in this section we are concerned with the ternary structure of  $\mathfrak{Q}$ , and we now proceed to show that  $\mathfrak{Q}$  is also a primitive ternary ring. Let both  $e$  and  $f$  denote the element of  $\mathfrak{Q}$  with one in the intersection of the first row and column and zero elsewhere. It is not difficult to prove that (2.1)–(2.5) are satisfied. By Theorem 2.2,  $\mathfrak{Q}$  is a primitive ternary ring and the proof of the theorem tells us  $(\mathfrak{Q} - B, \mathfrak{Q} - A)$  is a faithful irreducible  $\mathfrak{Q}$ -module. Now let  $U = \mathfrak{Q} - B$  and  $V = \mathfrak{Q} - A$ ; it is clear that for our example  $U$  and  $V$  are of infinite  $D$ -dimension. Shur's lemma and Lemma 1.3 tell us that there is a division ring  $E$  such that  $U$  and  $V$  are  $E$  vector spaces. A natural question arises: What is the  $E$ -dimension of  $U$  and  $V$ ? In order to provide an answer, let us digress for a moment.

Let  $R$  be any ternary ring. There are two binary rings  $R_1$  and  $R_2$  which arise naturally from  $R$  in such a way that  $R$  (considered as an abelian group) forms a left  $R_1$ -module and a right  $R_2$ -module. We shall make use of this module structure in determining the  $E$ -dimension of  $U$  and  $V$ , but first let us define  $R_1$  and  $R_2$ . Let  $R_1$  or  $R \otimes_1 R$  denote the set of all elements of the form  $\sum_{i \in I} r_i \otimes_1 s_i$  where  $I$  is finite,  $r_i, s_i \in R$ ,  $\otimes_1$  is biadditive and satisfies

$$(3.1) \quad abc \otimes_1 d = a \otimes_1 dcb$$

for all  $a, b, c, d \in R$ . An analogous definition is made for  $R_2$  or  $R \otimes_2 R$  by replacing condition (3.1) by

$$(3.2) \quad abc \otimes_2 d = c \otimes_2 bad \quad \text{for all } a, b, c, d \in R.$$

Using a rather obvious modification of Theorem 1.1 in Chapter Five of Mac Lane [5, pp. 135–139] one can prove that

$$(3.3) \quad \left( \sum_i r_i \otimes_1 s_i \right) \left( \sum_j t_j \otimes_1 u_j \right) = \sum_{i,j} r_i s_i t_j \otimes_1 u_j$$

defines a multiplication in  $R_1$  and

$$(3.4) \quad \left( \sum_i r_i \otimes_2 s_i \right) \left( \sum_j t_j \otimes_2 u_j \right) = \sum_{i,j} r_i \otimes_2 s_i t_j u_j$$

defines a multiplication in  $R_2$ . It is routine to verify that these operations make  $R_1$  and  $R_2$  into binary rings,  $R$  into a left  $R_1$ -module and  $R$  into a right  $R_2$ -module.

Suppose now that  $R$  is a primitive ring of  $E$ -linear transformations of  $U$  into  $V$  and the  $E$ -dimension of  $U$  and  $V$  is finite. Let  $X_1, \dots, X_n$  be an  $E$ -basis for  $U$  and define a sequence of right ideals in  $R$  by  $A_i = \{a \mid a \in R \text{ and } x_j a = 0 \text{ for } j = 1, \dots, i\}$ . From the previous paragraph it becomes clear that the  $A_i$  and  $R$  are left  $R \otimes_1 R$  modules. Since  $R$  is a dense ring, we see that  $0 = A_n \subset A_{n-1} \subset \dots \subset A_1 \subset R$  is a composition series for  $R$ . By the Jordan-Hölder theorem for modules [3, p. 141] we see that the  $E$ -dimension of  $U$  is the length of a maximal chain of right ideals in  $R$ . Of course, a similar result holds for  $V$  and a maximal chain of left ideals. Returning to our particular example, if the  $E$ -dimension of  $U$  were finite, there would be a finite maximal chain of right ideals of  $\mathcal{U}$ . But no such chain exists in  $\mathcal{U}$  and so the  $E$ -dimension of  $U$  is infinite. Similarly, the  $E$ -dimension of  $V$  is infinite. Finite-dimensional examples are easily obtained by restricting our attention to the ring of rectangular  $m \times n$  matrices.

As a final example, let us consider any unitary or euclidean space  $\mathcal{U}$ . Define the triple multiplication to be  $(a, b)c$  for all  $a, b, c \in \mathcal{U}$ . Let  $e$  be any nonzero element of  $\mathcal{U}$  and define  $f = e/\|e\|$ . Let  $A = 0$  and  $B = \{a \mid a \in \mathcal{U} \text{ and } (a, e) = 0\}$ . It is easy to verify that  $A$  and  $B$  are respectively maximal right and left ideals of  $\mathcal{U}$  and satisfy (2.1)–(2.5). By Theorem 2.2,  $\mathcal{U}$  is a primitive ternary ring. Note that  $A$  is always of dimension one over  $E$  whereas  $B$  is finite or infinite dimensional depending upon the dimension of  $\mathcal{U}$ .

4. **Unanswered questions.** I shall propose basically two unanswered questions. What is the radical of a ternary ring? Is there a rectangular matrix representation theorem for primitive ternary rings? One can define the right radical of a Hestenes ternary ring to be the intersection of the maximal modular right ideals and the left radical to be the intersection of the maximal modular left ideals. In the primitive case, both are zero. However, the question of what the radical is in general seems to be a difficult one, especially if we hope to find an analogue of quasi-regularity. In the case of the representation theorem, the main obstacle seems to be whether

the ternary ring product corresponds to the ternary homomorphism product. A final question is to what extent can the right-left action of  $R$  in  $(U, V)$  be replaced by a one-sided action by means of the anti-automorphism  $v \rightarrow -v$  of  $V$ ?

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