TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 178, April 1973

FREE PRODUCTS OF VON NEUMANN ALGEBRAS(1)

BY

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ABSTRACT. A new method of constructing factors of type II_1, called free product, is introduced. It is a generalization of the group construction of factors of type II_1 when the given group is a free product of two groups. If A_1 and A_2 are two von Neumann algebras with separating cyclic trace vectors and ortho-unitary bases, then the free product $A_1 * A_2$ of A_1 and A_2 is a factor of type II_1 without property Γ.

1. **Introduction.** In the study of von Neumann algebras, factors of type II_1 have been constructed by the so-called measure-construction, the group-construction, and the infinite tensor product (see Murray and von Neumann [10], [11], von Neumann [20]). A more general construction, called crossed product, which includes the measure-construction and the group-construction as special cases has been studied by Nakamura and Takeda [12], Suzuki [17], Turumaru [19], and Ching [3]. All hyperfinite factors of type II_1 are isomorphic to each other (Murray and von Neumann [11]), and a hyperfinite factor of type II_1 can be constructed by the measure-construction, the group-construction, or the infinite tensor product. This indicates that factors of type II_1 produced from quite different methods may actually be the same, i.e. isomorphic to each other. On the other hand, all nonhyperfinite factors of type II_1 so far discovered and classified, are constructed by the group-construction (see Ching [3], [4], Dixmier and Lance [5], McDuff [8], [9], Murray and von Neumann [11], Sakai [14], [15], Schwartz [16], Zeller-Meier [21]). The question that arises then is whether all factors of type II_1 can be produced by the group construction; or put in another way, is any factor of type II_1 isomorphic to $\mathfrak{A}(G)$ for some discrete group G, where $\mathfrak{A}(G)$ is the von Neumann algebra generated by the regular representation of G. This paper resulted from an attempt to give the above question hopefully a negative answer. We introduce a new method to construct a factor of type II_1, called free product, which is modeled after the free product of groups. It is, in fact, a generalization of the group-construction, with the given group being a free product of two groups. For example, we would see that the free product

Received by the editors February 29, 1972.

AMS (MOS) subject classifications (1970). Primary 46L10.

Key words and phrases. von Neumann algebra, group construction factor of type II_1, free product, property Γ.

(1) This research was supported by National Science Foundation grant GN-28517.

$\mathfrak{A}(Z) * \mathfrak{A}(Z)$ is isomorphic to $\mathfrak{A}(\Phi_2)$, where Z is the integer group and Φ_2 is the free group with two generators. We shall show that a free product of two finite von Neumann algebras with ortho-unitary bases, one of linear dimension at least two, the other of dimension at least three, is always a nonhyperfinite factor without property Γ. Although the motivation of this paper is to construct a factor of type II_1 nonisomorphic to any group-von Neumann algebra $\mathfrak{A}(G)$, we are not able, at present, to show that such a factor actually exists. For instance, it is not known whether $M_2 * M_3$, the free product of the 2 by 2 matrix algebra and the 3 by 3 matrix algebra, is isomorphic to $\mathfrak{A}(\Phi_2)$.

Another unsolved problem concerning von Neumann algebras is whether a factor of type II_1 necessarily has an outer automorphism. All factors of type I_n (i.e. $n \times n$ matrix algebra) do not have any outer automorphism (Dixmier [5, Proposition 4, p. 255]). Recently, Takesaki [18] proved that a factor of type III always has an outer automorphism. Any locally compact group can be represented as a group of outer automorphisms of the hyperfinite factor of type II_1 (Blattner [2]). Behncke [1] showed that this result is still true if the hyperfinite factor is replaced by $\mathfrak{A}(\Phi_2)$. We shall see that these known methods of finding an outer automorphism of a factor of type II_1 cannot yield any outer automorphism of $M_2 * M_3$ for us. However, the task to determine all outer automorphisms of $M_2 * M_3$ (probably none) must be left to further study of free products of von Neumann algebras.

All Hilbert spaces in this paper are complex and separable and all groups in this paper are countable. The identity operator in all von Neumann algebras is denoted by 1, and we shall call a scalar multiple of identity simply a scalar.

The author would like to thank Professor C. Heinrich for some helpful discussions.

2. **Construction of the free product.** Let A_i be a von Neumann algebra on Hilbert space H_i with a cyclic separating trace vector ξ_i (of norm 1), $i = 1, 2$. Let $A = A_1 \otimes A_2$ be the algebraic tensor product of A_1 and A_2. Let $A^n = A \otimes \cdots \otimes A$ be the nth algebraic tensor power of A. Let $\tilde{A} = \Sigma \bigotimes_{n=1}^{\infty} A^n$ be the tensor algebra of A without the summand of the field of constants. A general element of $\tilde{A}$ is of the form

$$\sum_{k=1}^{m} \sum_{i=1}^{n_k} x_{1,i} \otimes y_{1,i} \otimes \cdots \otimes x_{k,i} \otimes y_{k,i},$$

where $x_{j,i} \otimes y_{j,i} \in A_1 \otimes A_2$, $1 \le i \le n_k$, $1 \le j \le k$, $1 \le k \le m$. Let I be the ideal of $\tilde{A}$ generated by elements $\{x_1 \otimes y_1 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes y_{i+1} \otimes \cdots \otimes x_n \otimes y_n - x_1 \otimes y_1 \otimes \cdots \otimes x_{i-1} \otimes y_{i-1} \otimes x_i x_{i+1} \otimes y_{i+1} \otimes \cdots \otimes x_n \otimes y_n, x_1' \otimes y_1' \otimes \cdots \otimes x_{i-1}' \otimes y_{i-1}' \otimes 1 \otimes y_i' \otimes \cdots \otimes x_m' \otimes y_m' - x_1' \otimes$

$y_1' \otimes \cdots \otimes x_{i-1}' \otimes y_{i-1}'y_i' \otimes x_{i+1}' \otimes y_{i+1}' \otimes \cdots \otimes x_m' \otimes y_m'\}$. Let $\overline{A} = \tilde{A}/I$ be the quotient algebra. An element a in $\tilde{A}$ of the form $x_1 \otimes y_1 \otimes \cdots \otimes x_n \otimes y_n$ is called a *monomial*, which we shall write simply $x_1y_1 \cdots x_ny_n$ hereafter. n is called the *length* of the monomial, and is denoted by $\rho(a) = n$. Every element of $\tilde{A}$ is a sum of monomials.

Any element x of A_i can be written as $x = c1 + x'$, where $c = \mathrm{tr}(x) = (x\xi_i|\xi_i)$, and $x' = x - c1$ (hence $\mathrm{tr}(x') = 0$), $i = 1, 2$. A monomial $x_1y_1 \cdots x_ny_n$ is said to be *irreducible* if $\mathrm{tr}(x_i) = 0$ and $\mathrm{tr}(y_i) = 0$ for all i with the possible exception of x_1 and y_n, each of which is either of trace zero or a scalar. Every monomial $x_1y_1 \cdots x_ny_n$ (and hence every element of $\tilde{A}$) is equivalent to a finite sum $\overline{x_1y_1 \cdots x_ny_n}$, unique up to a rearrangement of summation, of irreducible monomials each having length no more than n. The unique sum $\overline{x_1y_1 \cdots x_ny_n}$ of irreducible monomials is called the *canonical form* of $x_1y_1 \cdots x_ny_n$. This can be readily proved by induction.

Lemma 1. *There exists a rewriting process to reduce every monomial $x_1y_1 \cdots x_ny_n$ into a unique canonical form $\overline{x_1y_1 \cdots x_ny_n}$.*

Proof. It is trivially true for the case $n = 1$. Suppose that $\overline{x_1y_1 \cdots x_ky_k}$ has been defined for $k < n$. Let $x_1y_1 \cdots x_ny_n$ be given. Suppose

$$\overline{x_1y_1 \cdots x_{n-1}y_{n-1}} = \sum_{k=1}^{n-1} \sum_{i=1}^{n_k} x_{1,i}y_{1,i} \cdots x_{k,i}y_{k,i},$$

where $x_{1,i}y_{1,i} \cdots x_{k,i}y_{k,i}$ is an irreducible monomial, $1 \le i \le n_k$, $1 \le k \le m$. Define

$$\overline{x_1y_1 \cdots x_ny_n} = \sum_{k=1}^{n-1} \sum_{i=1}^{n_k} \overline{x_{1,i}y_{1,i} \cdots x_{k,i}y_{k,i}x_ny_n},$$

where each $\overline{x_{1,i}y_{1,i} \cdots x_{k-1,i}y_{k-1,i}x_ny_n}$ is defined as the following: first of all,

$$\overline{x_{1,i}y_{1,i} \cdots x_{k-1,i}y_{k-1,i}x_ny_n}$$
$$= \mathrm{tr}(y_n)\overline{x_{1,i}y_{1,i} \cdots x_{k-1,i}y_{k-1,i}x_n1} + \overline{x_{1,i}y_{1,i}x_{k-1,i}y_{k-1,i}x_ny_n'},$$

where $y_n' = y_n - \mathrm{tr}(y_n)1$. Hence, we can assume either $\mathrm{tr}(y_n) = 0$ or $y_n = c1$. Now,
(i) if $\mathrm{tr}(x_n) = 0$ and $\mathrm{tr}(y_{k-1,i}) = 0$, then

$$\overline{x_{1,i}y_{1,i} \cdots x_{k-1,i}y_{k-1,i}x_ny_n} = x_{1,i}y_{1,i} \cdots x_{k-1,i}y_{k-1,i}x_ny_n;$$

(ii) if $x_n = c1$ is a scalar, then

$$x_{1,i}y_{1,i} \cdots x_{k-1,i}y_{k-1,i}x_ny_n = cx_{1,i}y_{1,i} \cdots x_{k-1,i}(y_{k-1,i}y_n),$$

which is already defined by our assumption; (iii) if $y_{n-1,i} = c1$, then

$$\overline{x_{1,i}y_{1,i}\cdots x_{k-1,i}y_{k-1,i}x_ny_n} = c\overline{x_{1,i}y_{1,i}\cdots (x_{k-1,i}x_n)y_n},$$

which is already defined. For general x_n, $x_n = \operatorname{tr}(x_n)1 + x_n'$, with $\operatorname{tr}(x_n') = 0$. And we define

$$\overline{x_{1,i}y_{1,i}\cdots x_{k-1,i}y_{k-1,i}x_ny_n}$$

$$= \operatorname{tr}(x_n)\overline{x_{1,i}y_{1,i}x_{k-1,i}(y_{k-1,i}y_n)} + \overline{x_{1,i}y_{1,i}\cdots x_{k-1,i}y_{k-1,i}x_n'y_n}.$$

This proves the lemma.

For every element $a = \Sigma a_i$ in $\tilde{A}$, where the a_i's are monomials, let $\overline{a} = \Sigma \overline{a}_i \cdot \overline{a}$ is a unique representative of the equivalence class of a in $\overline{A} = \tilde{A}/I$. So, we can regard $\overline{a}$ as the element in $\overline{A}$ it represents. Define $c\cdot\overline{a} = \overline{c\cdot a}$ for complex number c, $\overline{a}\cdot\overline{b} = \overline{a\cdot b}$, $\overline{a} + \overline{b} = \overline{a+b}$, where $a\cdot b$ is the multiplication in $\tilde{A}$, in the case of the two monomials

$$(x_1y_1\cdots x_ny_n)\cdot(x_1'y_1'\cdots x_m'y_m') = x_1y_1\cdots x_ny_nx_1'y_1'\cdots x_m'y_m',$$

and in the case of the two general elements it is defined by the linear extension of the above operation. It is easy to see that $\overline{A}$ is the same complex associative algebra as defined by quotient $\tilde{A}/I$. We can regard A as consisting of linear combinations of irreducible monomials. Furthermore, we define $(x_1y_1\cdots x_ny_n)^* = \overline{1y_n^*x_n^*y_{n-1}^*\cdots x_2^*y_1^*x_1^*1}$ for an irreducible monomial $x_1y_1\cdots x_ny_n$. Extending the conjugate linearly to all elements of $\overline{A}$, $*$ is an involution of $\overline{A}$ such that $(c\cdot a)^* = \overline{c}\cdot a^*$, and $(ab)^* = b^*a^*$ for complex number c, and $a, b \in \overline{A}$.

Define a function f by $f(x_1y_1\cdots x_ny_n) = 0$ for an irreducible monomial $x_1y_1\cdots x_ny_n$ in A, except for the case $n = 1$ and $x_1y_1 = c1$, a scalar, for which we define $f(x_1y_1) = c$. Extend this function linearly to all elements of A. Note that for a monomial of length one (not necessarily irreducible), $f(x_1y_1) = \operatorname{tr}(x_1)\operatorname{tr}(y_1)$, where $\operatorname{tr}(x_1)$ and $\operatorname{tr}(y_1)$ are the traces of A_1 and A_2 respectively. We have

Lemma 2. *The linear functional f defined on $\overline{A}$ has the following properties*:
(a) $f(ab) = f(ba)$ *for any* $a, b \in \overline{A}$,
(b) $f(a^*a) > 0$ *for all nonzero element* a *in* $\overline{A}$,
(c) $f(a^*) = \overline{f(a)}$ *for any* $a \in \overline{A}$.

Proof of (a). It is clear that we only need to prove (a) for $a = x_1y_1\cdots x_ny_n$, and $b = s_1t_1\cdots s_mt_m$ being two irreducible monomials in $\overline{A}$.

Case 1. Neither y_n nor s_1 is a scalar. Hence

$$ab = x_1y_1\cdots x_ny_ns_1t_1\cdots s_mt_m, \text{ and } f(ab) = 0.$$

(i) If neither x_1 nor t_m is a scalar, then

(1) $$ba = s_1 t_1 \cdots s_m t_m x_1 y_1 \cdots x_n y_n, \text{ and } f(ba) = 0.$$

(ii) If x_1 is a scalar, but t_m is not, then

$$\begin{aligned} f(ba) &= \operatorname{tr}(x_1) f(\overline{s_1 t_1 \cdots s_m (t_m y_1) x_2 y_2} \, \overline{\cdots x_n y_n}) \\ &= \operatorname{tr}(x_1) \operatorname{tr}(t_m y_1) f(\overline{s_1 t_1 \cdots s_{m-1} t_{m-1} (s_m x_2) y_2 \cdots x_n y_n}); \end{aligned}$$

(2) $$f(ba) = \operatorname{tr}(x_1) \operatorname{tr}(t_m y_1) \operatorname{tr}(s_m x_2) \cdots \operatorname{tr}(t_{m-1+1} x_{i+1}) \cdots \operatorname{tr}(t_1 y_n) \operatorname{tr}(s_1) = 0,$$

in case $m = n$;

(3) $$f(ba) = \operatorname{tr}(x_1) \operatorname{tr}(t_m y_1) \operatorname{tr}(s_m x_2) \cdots \operatorname{tr}(t_1 y_{n-1}) \operatorname{tr}(s_1 x_n) \operatorname{tr}(y_n) = 0,$$

in case $n = m + 1$. In all other cases, ba is a sum of irreducible monomials all of length at least two, hence $f(ba) = 0$.

(iii) If t_m is a scalar, but x_1 is not, then

(4) $$\begin{aligned} f(ba) &= \operatorname{tr}(t_m) f(\overline{s_1 t_1 \cdots s_{m-1} t_{m-1} (s_m x_1) y_1 \cdots x_n y_n}) \\ &= \operatorname{tr}(t_m) \operatorname{tr}(s_m x_1) \operatorname{tr}(t_{m-1} y_1) \cdots \operatorname{tr}(s_{m-1+1} x_i) \operatorname{tr}(t_{m-1} y_i) \\ &\qquad \cdots \operatorname{tr}(t_1 y_{n-1}) \operatorname{tr}(s_1 x_n) \operatorname{tr}(y_n) = 0, \end{aligned}$$

in case $m = n$;

(5) $$f(ba) = \operatorname{tr}(t_m) \operatorname{tr}(s_m x_1) \operatorname{tr}(t_{m-1} y_1) \cdots \operatorname{tr}(s_2 x_2) \operatorname{tr}(t_1 y_n) \operatorname{tr}(s_1) = 0$$

in case $m = n + 1$. In all other cases, ba is clearly a sum of irreducible monomials all of length at least two, hence $f(ba) = 0$.

(iv) If both x_1 and t_m are scalars, then

(6) $$ba = \operatorname{tr}(x_1) \operatorname{tr}(t_m) s_1 t_1 \cdots s_m y_1 \cdots x_n y_n$$

is an irreducible monomial of length at least two in case $m \neq 1$ or $n \neq 1$. Hence $f(ba) = 0$. In cases $n = 1$ and $m = 1$,

(7) $$ba = \operatorname{tr}(x_1) \operatorname{tr}(t_1) s_1 y_1, \quad \text{and} \quad f(ba) = 0,$$

since s_1 is not a scalar. Therefore $\operatorname{tr}(ba) = \operatorname{tr}(ab)$ in Case 1.

Case 2. s_1 is a scalar, but y_n is not. In case $m = n$,

(8) $$\begin{aligned} f(ab) &= \operatorname{tr}(s_1) \operatorname{tr}(y_n t_1) \operatorname{tr}(x_n s_2) \\ &\qquad \cdots \operatorname{tr}(y_{n-i+1} t_i) \operatorname{tr}(x_{n-i+1} s_{i+1}) \cdots \operatorname{tr}(y_1 t_m) \operatorname{tr}(x_1); \end{aligned}$$

(9) $$f(ab) = \operatorname{tr}(s_1) \operatorname{tr}(y_n t_1) \operatorname{tr}(x_n s_2) \cdots \operatorname{tr}(y_1 t_{m-1}) \operatorname{tr}(x_1 s_m) \operatorname{tr}(t_m),$$

in case $m = n + 1$. In all other cases, ab is a sum of irreducible monomials all of length at least two, hence $f(ab) = 0$.

(i) If neither x_1 nor t_m is a scalar, then $f(ab) = 0$ by (8) and (9); also $ba = s_1 t_1 \cdots s_m t_m x_1 y_1 \cdots x_n y_n$, hence $f(ba) = 0$.

(ii) If x_1 is a scalar but t_m is not, then

$$f(ba) = \operatorname{tr}(x_1)\operatorname{tr}(t_m y_1)\operatorname{tr}(s_m x_2)\cdots\operatorname{tr}(t_1 y_n)\operatorname{tr}(s_1) = f(ab),$$

in case $m = n$. In case $m = n + 1$, $f(ab) = 0$; and ba is a sum of irreducible monomials all of length at least two, hence $f(ba) = 0$. In case $n = m + 1$,

$$f(ba) = \operatorname{tr}(x_1)\operatorname{tr}(t_m y_1)\operatorname{tr}(s_m x_2)\cdots\operatorname{tr}(t_1 y_{n-1})\operatorname{tr}(s_1 x_n)\operatorname{tr}(y_n) = 0;$$

and ab is a sum of irreducible monomials all of length at least two, hence $f(ab) = 0$.

(iii) Suppose t_m is a scalar but x_1 is not. In case $m = n$, $f(ab) = 0$ and

$$f(ba) = \operatorname{tr}(t_m)\operatorname{tr}(s_m x_1)\operatorname{tr}(t_{m-1} y_1)\cdots\operatorname{tr}(s_1 x_n)\operatorname{tr}(y_n) = 0.$$

In case $m = n + 1$,

$$f(ba) = \operatorname{tr}(t_m)\operatorname{tr}(s_m x_1)\operatorname{tr}(t_{m-1} y_1)\cdots\operatorname{tr}(s_2 x_n)\operatorname{tr}(t_1 y_n)\operatorname{tr}(s_1) = f(ab).$$

In all other cases ba is a sum of irreducible monomials all of length at least two, hence $f(ba) = 0 = f(ab)$.

(iv) If both x_1 and t_m are scalars, then $ba = \operatorname{tr}(x_1)\operatorname{tr}(t_m) s_1 t_1 \cdots s_m y_1 \cdots x_n y_n$ is an irreducible monomial of length at least two in case $m \neq 1$ or $n \neq 1$. Hence $f(ba) = 0$. In case $n = m = 1$, $f(ba) = \operatorname{tr}(s_1)\operatorname{tr}(t_1 y_1)\operatorname{tr}(x_1) = f(ab)$. Hence, $f(ba) = f(ab)$ in Case 2.

Case 3. y_n is a scalar but s_1 is not. In case $m = n$,

$$f(ab) = \operatorname{tr}(y_n)\operatorname{tr}(x_n s_1)\operatorname{tr}(y_{n-1} t_1)\cdots\operatorname{tr}(y_1 t_{m-1})\operatorname{tr}(x_1 s_m)\operatorname{tr}(t_m); \tag{10}$$

$$f(ab) = \operatorname{tr}(y_n)\operatorname{tr}(x_n s_1)\operatorname{tr}(y_{n-1} t_1)\cdots\operatorname{tr}(x_2 s_m)\operatorname{tr}(y_1 t_m)\operatorname{tr}(x_1) \tag{11}$$

in case $n = m + 1$. In all other cases ab is a sum of irreducible monomials all of length at least two, hence $f(ab) = 0$.

(i) If neither x_1 nor t_m is a scalar, then $ba = s_1 t_1 \cdots s_m t_m$, $x_1 y_1 \cdots x_n y_n$ and $f(ba) = 0$. We also have $f(ab) = 0$ in this case.

(ii) Suppose x_1 is a scalar but t_m is not. In case $m = n$, $f(ab) = 0$, and

$$f(ba) = \operatorname{tr}(x_1)\operatorname{tr}(t_m y_1)\operatorname{tr}(s_m x_2)\cdots\operatorname{tr}(t_1 y_n)\operatorname{tr}(s_1) = 0;$$

$$f(ba) = \operatorname{tr}(x_1)\operatorname{tr}(t_m y_1)\operatorname{tr}(s_m x_2)\cdots\operatorname{tr}(t_1 y_n)\operatorname{tr}(s_1 y_n)\operatorname{tr}(y_n) = f(ab),$$

in case $n = m + 1$. In all other cases ba is a sum of irreducible monomials all of length at least two, hence $f(ba) = 0 = f(ab)$.

(iii) Suppose t_m is a scalar but x_1 is not. In case $m = n$,

$$f(ba) = \operatorname{tr}(t_m)\operatorname{tr}(s_m x_1)\operatorname{tr}(t_{m-1}y_1)\cdots\operatorname{tr}(s_1 x_n)\operatorname{tr}(y_n) = f(ab);$$

$$f(ba) = \operatorname{tr}(t_m)\operatorname{tr}(s_m x_1)\cdots\operatorname{tr}(t_1 y_n)\operatorname{tr}(s_1) = 0,$$

in case $m = n + 1$ as $f(ab) = 0$ also. In all other cases ba is a sum of irreducible monomials all of length at least two. Hence, $f(ba) = 0$ as $f(ab) = 0$ too.

(iv) Suppose both t_m and x_1 are scalars. Then $ba = \operatorname{tr}(x_1)\operatorname{tr}(t_m)s_1 t_1 \cdots s_m y_1 \cdots x_n y_n$ is an irreducible monomial of length at least two in case $n \neq 1$ or $m \neq 1$; hence $f(ba) = 0$, and in case $n = m = 1$, $f(ba) = 0$ since $\operatorname{tr}(s_1) = 0$. By (10), in case $n = m$, $f(ab) = 0$, since $\operatorname{tr}(x_1 s_m) = \operatorname{tr}(x_1)\operatorname{tr}(s_m) = 0$. By (11), in case $n = m + 1$, $f(ab) = 0$, since $\operatorname{tr}(y_1 t_m) = \operatorname{tr}(t_m)\operatorname{tr}(y_1) = 0$, and $f(ab) = 0$ in all other cases. Therefore, $f(ab) = f(ba)$ in Case 3.

Case 4. Both y_n and s_1 are scalars. Then $ab = \operatorname{tr}(s_1)\operatorname{tr}(y_m)x_1 y_1 \cdots x_n t_1 \cdots s_m t_m$. Hence $f(ab) = 0$ if $n \neq 1$ or $m \neq 1$, and

$$f(ab) = \operatorname{tr}(s_1)\operatorname{tr}(y_1)\operatorname{tr}(x_1)\operatorname{tr}(t_1) \qquad \text{if } n = 1 \text{ and } m = 1. \tag{12}$$

(i) If neither x_1 nor t_m is a scalar, then by (1) $f(ba) = 0$ and $f(ab) = 0$ by (12).

(ii) If x_1 is a scalar but t_m is not, then $f(ab) = 0$. In case $m = n$, by (2), $f(ba) = 0$ since $\operatorname{tr}(t_1 y_n) = \operatorname{tr}(t_1)\operatorname{tr}(y_n) = 0$. In case $n = m + 1$, by (3), $f(ba) = 0$ since $\operatorname{tr}(s_1 x_n) = \operatorname{tr}(s_1)\operatorname{tr}(x_n) = 0$. And $f(ba) = 0$ in all other cases as in Case 1 (ii).

(iii) If t_m is a scalar but x_1 is not, then $\operatorname{tr}(ab) = 0$. In case $m = n$, by (4), $f(ba) = 0$ since $\operatorname{tr}(s_1 x_n) = \operatorname{tr}(s_1)\operatorname{tr}(x_n) = 0$. In case $m = n + 1$, by (5), $f(ba) = 0$ since $\operatorname{tr}(t_1 y_n) = \operatorname{tr}(t_1)\operatorname{tr}(y_n) = 0$. And $f(ba) = 0$ in all other cases as in Case 1 (ii).

(iv) Suppose both x_1 and t_m are scalars. Then $f(ab) = 0 = f(ba)$ if $m \neq 1$ or $n \neq 1$; and $f(ba) = \operatorname{tr}(s_1)\operatorname{tr}(t_1)\operatorname{tr}(x_1)\operatorname{tr}(y_1) = f(ab)$ if $n = 1$ and $m = 1$. Therefore, $f(ab) = f(ba)$ in Case 4. This completes the proof for part (a).

Proof of (b). Let $a = \sum_{k=1}^{m} \sum_{k=1}^{n_k} x_{1,b}y_{1,b}\cdots x_{k,b}y_{k,b}$ be a nonzero element in $\bar{A}$, where each $x_{1,b}y_{1,b}\cdots x_{k,b}y_{k,b}$ in the summation is an irreducible monomial. Let E be a maximal linearly independent finite subset containing 1, of the set $\{x_{k,b} \mid b = 1, \cdots, n_k, k = 1, \cdots, m\} \cup \{1\}$. E is a finite linearly independent subset of A_1. Apply the Gram-Schmidt orthogonalization process to E with respect to the inner product $(x|x') = (x\xi_1|x'\xi_1)$ defined on A_1, we obtain an orthonormal set $F = \{a_1, a_2, \cdots, a_n\}$ in A_1, where we specifically design that $a_1 = 1$. Now let $O_1 = \{a_b\}_{b \in N_1}$, where the index set N_1 is a subset of the set of all natural numbers containing $\{1, \cdots, n\}$, be a family of elements in A_1 which is maximal with respect to the following properties: (i) $F \subset O_1$; (ii) $(a_h\xi_1|a_k\xi_1) = 0$ if $h \neq k$, and $(a_{h1}|a_{k1}) = 1$ for $h = k \in N_1$. Let a subset $O_2 = \{b_k\}_{k\in N_2}$ of A_2 be constructed similarly from the set $\{y_{k,h}|\ h = 1, \cdots, n_k, k = 1, \cdots, m\}$. Let $\alpha = \{i_1, \cdots, i_n\}$ be an ordered set of n elements from N_1 such that $i_l \neq 1$, if $l > 1$, and let $\alpha' = \{i_1', \cdots, i_n'\}$ be an ordered set of n elements from N_2 such that $i_l \neq 1$ if $l < n$. Let S be the family of all such couples (α, α') of ordered sets with elements from $N_1 \cup N_2$. For notational simplicity, we shall write α for (α, α') and i_k for i_k'.

Let $e_\alpha = e_{(\alpha,\alpha')} = a_{i_1}b_{i_1}\cdots a_{i_n}b_{i_n}$, $\alpha \in S$. We shall show that $f(e_\alpha^*e_\alpha) = 1$ and $f(e_\alpha^*e_\beta) = 0$, where $e_\beta = a_{j_1}b_{j_1}\cdots a_{j_m}b_{j_m}$, $\alpha \neq \beta$, $\alpha, \beta \in S$.

Case 1. $a_{i_1} = b_{i_n} = 1$. Then $e_\alpha^* = a_{i_n}^*b_{i_{n-1}}^*\cdots a_{i_2}^*b_{i_1}^*$, and

$$\begin{aligned} f(e_\alpha^*e_\alpha) &= f(\overline{a_{i_n}^*b_{i_{n-1}}^*\cdots a_{i_2}^*(b_{i_1}^*b_{i_1})a_{i_2}b_{i_2}\cdots a_{i_n}1}) \\ &= \operatorname{tr}(b_{i_1}^*b_{i_1})f(\overline{a_{i_n}^*b_{i_{n-1}}^*\cdots(a_{i_2}^*a_{i_2})b_{i_2}\cdots a_{i_n}1}) \\ &= \|b_{i_1}\xi_2\|^2\|a_{i_2}\xi_1\|^2\cdots\|a_{i_n}\xi_1\|^2 = 1. \end{aligned}$$

(i) Suppose $a_{j_1} = 1$. If $n = m = 1$, then $b_{j_1} \neq 1$ since $\alpha \neq \beta$. Hence $f(e_\alpha^*e_\beta) = \operatorname{tr}(1\cdot b_{j_1}) = 0$. If $n = m > 1$, then

$$\begin{aligned} f(e_\alpha^*e_\beta) &= f(\overline{a_{i_n}^*b_{i_{n-1}}^*\cdots a_{i_2}^*(b_{i_1}^*b_{j_1})a_{j_2}b_{j_2}\cdots a_{j_m}b_{j_m}}) \\ &= \operatorname{tr}(b_{i_1}^*b_{j_1})\operatorname{tr}(a_{i_2}^*a_{j_2})\cdots\operatorname{tr}(a_{i_n}^*a_{j_m})\operatorname{tr}(b_{j_m}). \end{aligned}$$

Since O_1 and O_2 are orthornormal sets in the inner products defined by traces, either one of $\{\operatorname{tr}(b_{i_k}^*b_{j_k}), \operatorname{tr}(a_{i_k}^*a_{j_k})\}$ is zero or $\operatorname{tr}(b_{j_m}) = 0$. Hence $f(e_\alpha^*e_\beta) = 0$. If $n = m + 1$, then

$$f(e_\alpha^*e_\beta) = \operatorname{tr}(b_{i_1}^*b_{j_1})\operatorname{tr}(a_{i_2}^*a_{j_2})\cdots\operatorname{tr}(a_{i_n}^*a_{j_m})\operatorname{tr}(b_{i_m}^*b_{j_m})\operatorname{tr}(a_{i_n}^*) = 0$$

since $\operatorname{tr}(a_{i_n}) = 0$. In all other cases $e_\alpha^*e_\beta$ is a sum of irreducible monomials all of length at least two, hence $f(e_\alpha^*e_\beta) = 0$.

(ii) If $a_{j_1} \neq 1$, then $e_\alpha^*e_\beta = a_{i_n}^*b_{i_{n-1}}^*\cdots a_{i_2}^*b_{i_2}^*a_{j_1}b_{j_1}\cdots a_{j_m}b_{j_m}$. Hence $f(e_\alpha^*e_\beta) = 0$.

Case 2. $a_{i_1} = 1$, $b_{i_n} = 1$. Then $e_\alpha^* = 1b_{i_n}^*a_{i_n}^*b_{i_{n-1}}^*\cdots a_{i_2}^*b_{i_1}^*$, and

$$\begin{aligned} f(e_\alpha^*e_\alpha) &= f(1b_{i_n}^*a_{i_n}^*b_{i_{n-1}}^*\cdots a_{i_n}^*(b_{i_1}^*b_{i_1})a_{i_2}b_{i_2}\cdots a_{i_n}b_{i_n}) \\ &= \|b_{i_1}\xi_2\|^2\|a_{i_2}\xi_1\|^2\cdots\|a_{i_n}\xi_1\|^2 = 1. \end{aligned}$$

(i) Suppose $a_{j1} = 1$. If $n = m$,

$$f(e_\alpha^* e_\beta) = \operatorname{tr}(b_{i_1}^* b_{j_1}) \operatorname{tr}(a_{i_2}^* a_{j_2}) \cdots \operatorname{tr}(b_{i_n}^* b_{j_n}) = 0;$$

since $\alpha \neq \beta$, one of the factors must be zero as O_1 and O_2 are orthonormal sets. If $n \neq m$, $e_\alpha^* e_\beta$ is a sum of irreducible monomials all of length at least two. Hence $f(e_\alpha^* e_\beta) = 0$.

Case 3. $b_{i_n} = 1$, $a_{i_j} \neq 1$. Then $e_\alpha^* = a_{i_n}^* b_{i_n}^* \cdots a_{i2}^* b_{i1}^* a_{i1}^* 1$, and

$$f(e_\alpha^* e_\alpha) = \operatorname{tr}(a_{i_1}^* a_{i_1}) \operatorname{tr}(b_{i_1}^* b_{i_1}) \cdots \operatorname{tr}(a_{i_n}^* a_{i_n}) = 1.$$

(i) If $a_{j1} = 1$, then $e_\alpha^* e_\beta = a_{i_n}^* b_{i_{n-1}}^* \cdots a_{i2}^* b_{i1}^* a_{i1}^* b_{j1} a_{j_m} b_{j_m}$, and $f(e_\alpha^* e_\beta) = 0$.

(ii) Suppose $a_{j1} \neq 1$. If $n = m$, then

$$f(e_\alpha^* e_\beta) = \operatorname{tr}(a_{i_1}^* a_{j_1}) \operatorname{tr}(b_{i_1}^* b_{j_1}) \cdots \operatorname{tr}(a_{i_n}^* a_{j_n}) \operatorname{tr}(b_{j_m}) = 0,$$

since $\alpha \neq \beta$, one of the factors above must be zero. If $n = m + 1$, then

$$f(e_\alpha^* e_\beta) = \operatorname{tr}(a_{i_1}^* a_{j_1}) \operatorname{tr}(b_{i_1}^* b_{j_1}) \cdots \operatorname{tr}(a_{i_n}^* a_{j_m}) \operatorname{tr}(a_{i_n}^*) = 0,$$

since $a_{i_n} \neq 1$. In all other cases, $e_\alpha^* e_\beta$ is a sum of irreducible monomials all of length at least two, hence $f(e_\alpha^* e_\beta) = 0$.

Case 4. $a_{i_1} \neq 1$, $b_{i_n} \neq 1$. Then $e_\alpha^* = 1 b_{i_n}^* a_{i_n}^* b_{i_n}^* \cdots a_{i2}^* b_{i1}^* a_{i1}^* 1$, and

$$f(e_\alpha^* e_\alpha) = \operatorname{tr}(a_{i_1}^* a_{i_1}) \operatorname{tr}(b_{i_1}^* b_{i_1}) \cdots \operatorname{tr}(b_{i_n}^* b_{i_n}) = 1.$$

(i) If $a_{j1} = 1$, then $e_\alpha^* e_\beta = 1 b_{i_n}^* \cdots a_{i2}^* b_{i1}^* a_{i1}^* b_{j1} \cdots a_{j_m} b_{j_m}$. Hence $f(e_\alpha^* e_\beta) = 0$.

(ii) Suppose $a_{j1} \neq 1$. If $n = m$, then

$$f(e_\alpha^* e_\beta) = \operatorname{tr}(a_{i_1}^* a_{j_1}) \operatorname{tr}(b_{i_1}^* b_{j_1}) \cdots \operatorname{tr}(b_{i_n}^* b_{j_n}) = 0,$$

since $\alpha \neq \beta$, so one of the factors above must be zero. If $m = n + 1$,

$$f(e_\alpha^* e_\beta) = \operatorname{tr}(a_{i_1}^* a_{j_1}) \cdots \operatorname{tr}(b_{i_n}^* b_{j_n}) f(a_{j_m} b_{j_m}) = 0,$$

since $a_{j_m} \neq 1$. In all other cases, $e_\alpha^* e_\beta$ is a sum of irreducible monomials all of length at least two, hence $f(e_\alpha^* e_\beta) = 0$. We have proven that $f(e_\alpha^* e_\alpha) = 1$ and $f(e_\alpha^* e_\beta) = 0$ if $\alpha \neq \beta$ for all $\alpha, \beta \in S$.

Now, let $a = \Sigma_{\alpha \in S'} c_\alpha e_\alpha$, where c_α's are nonzero complex numbers and S' is a finite subset of S. Hence

$$f(a^* a) = f\Big(\sum_{\alpha,\beta \in S'} \bar{c}_\alpha c_\beta e_\alpha^* e_\beta \Big) = \sum_{\alpha \in S'} |c_\alpha|^2 > 0$$

for any $a \in \bar{A}$.

Proof of (c). For $e_\alpha \neq 1$, we have $f(e_\alpha) = \overline{f(e_\alpha)} = 0$ and $f(1) = \overline{f(1^*)} = 1$. For a general element, $a = \Sigma\, c_\alpha e_\alpha$ in A. $f(a^*) = f(\Sigma_\alpha\, c_\alpha e_\alpha^*) = \overline{f(a)}$. This completes the proof of the lemma.

Now, we define

$$(a|b) = f(b^* a), \quad \text{for } a, b \in \overline{A}.$$

It is clear from the previous lemmata that this is an inner product on $\overline{A}$ satisfying the following properties:

(i) $(a|b) = (b^*|a^*)$ for $a, b \in \overline{A}$,

(ii) $(ab|d) = (b|a^*d)$ for $a, d, b \in \overline{A}$.

In order to prove that $\overline{A}$ is a Hilbert algebra, it only remains to show that the multiplication $a \mapsto b \cdot a$ is continuous on $\overline{A}$ in the norm topology defined by the inner product for each $b \in \overline{A}$.

Lemma 3. *For each $b \in \overline{A}$, $a \mapsto b \cdot a$ is a bounded linear operator on $\overline{A}$ in norm defined by the inner product.*

Proof. Since any element b in $\overline{A}$ is a finite sum of monomials, and any operator in a von Neumann algebra is a linear combination of four unitary operators in that von Neumann algebra (Dixmier [5, Proposition 3, p. 4]), every element in b is a linear combination of elements of the form $u = \overline{u_1 v_1 \cdots u_r v_r}$, where u_i, v_i are unitary operators in A_1 and A_2 respectively, $i = 1, \cdots, r$. Hence we only need to show that $L_u\colon a \longrightarrow u \cdot a$ is a unitary operator on $\overline{A}$.

Let a be given as in Lemma 2. Let the orthonormal set $\{e_\alpha\}_{\alpha \in S}$ be constructed as in Lemma 2, and let e_α, e_β be given as in Lemma 2. Then

$$(ue_\alpha | ue_\beta) = f(e_\beta^* u^* u^* e_\alpha)$$

$$= f\overline{(1 b_{j_m}^* a_{j_m}^* b_{j_m}^* \cdots a_{j_2}^* b_{j_1}^* a_{j_1}^* v_r^* u_r^* v_{r-1}^* \cdots u_2^* v_1^* (u_1^* u_1) v_1 \cdots u_r v_r a_{i_1} b_{i_1} \cdots a_{i_n} b_{i_n})}$$

$$= f\overline{(1 b_{j_m}^* a_{j_m}^* b_{j_m}^* \cdots a_{j_2}^* b_{j_1}^* (a_{j_1}^* a_{i_1}) b_{j_1} \cdots a_{i_n} b_{i_n})} = (e_\alpha | e_\beta).$$

Hence,

$$(ua | ua) = \Big(\sum_{\alpha \in S'} c_\alpha u e_\alpha \Big| \sum_{\alpha \in S'} c_\alpha u e_\alpha \Big) = \sum_{\alpha, \beta} c_\alpha \overline{c}_\beta (ue_\alpha | ue_\beta) = (a | a).$$

This shows that u is unitary on A, and completes the proof of the lemma.

It can be proved similarly that $R_b\colon a \mapsto a \cdot b$ is a bounded linear operator on $\overline{A}$. Since $\overline{A}$ has an identity, $\overline{A}$ is a Hilbert algebra (Dixmier [5, Chapter 1, §5]). Let H be the completion of $\overline{A}$ under the defined inner product. Let $O_1 = \{a_b\}_{b \in N_1}$ and $O_2 = \{b_k\}_{k \in N_2}$ be maximal orthonormal sets in A_1 and A_2 respectively, and let $\{e_\alpha\}_{\alpha \in S}$ be the orthonormal set in $\overline{A}$ constructed from O_1 and O_2 as in Lemma 2. Then every element x of H can be represented as $\Sigma_{\alpha \in S}\, c_\alpha e_\alpha$, which may have infinite many coefficients c_α being nonzero, such that $\|x\|^2 = \Sigma_{\alpha \in S}\, |c_\alpha|^2 < +\infty$. The left multiplication $L_b\colon a \longrightarrow b \cdot a$ by an element b in A can be extended to a bounded linear operator L_b on H. This is also true for multiplication by elements of A on the right. We call the von Neumann algebra on H generated by all left multiplication operators L_b, $b \in \overline{A}$, $A_1 * A_2$, the *free product* of A_1 and A_2. Let $\hat{A}$ be the subset of H consisting of all elements x with the property that there exists a constant $K(x)$ such that $\|x \cdot a\| \leq K(x)\|a\|$ for all $a \in \overline{A} \subset H$. Hence, for each $x \in A$, $a \mapsto x \cdot a$ can be extended to a bounded linear operator L_x on H. ψ: $x \leftrightarrow L_x$ is a one-to-one correspondence between $\hat{A}$ and $A_1 * A_2$. For let $x = \sum_{\alpha \in S} c_\alpha e_\alpha \in \hat{A}$, and let $x_i = \sum_{\alpha \in F_i} c_\alpha e_\alpha$, where F_i is a finite subset of S, $i = 1, 2, \cdots$, such that $\|x - x_i\| = \sum_{\alpha \in S \setminus F_i} |c_\alpha|^2 \to 0$; then $x_i \in \overline{A}$, $i = 1, 2, \cdots$, $L_{x_i} \to L_x$ strongly, i.e. $L_x \in A_1 * A_2$. On the other hand, it can be shown that $\{L_x | \, x \in \hat{A}\} = \{R_a | \, x \in \overline{A}\}'$ is already a von Neumann algebra, i.e. $A_1 * A_2 \subset \psi(\hat{A})$. By the properties of a Hilbert algebra, we have

(i) $\lambda L_x + \mu L_y = L_{\lambda x + \mu y}$, for $x, y \in \hat{A}$, and complex numbers λ, μ.

(ii) $L_x L_y = L_{x \cdot y}$, for $x, y \in \hat{A}$, where $x \cdot y$ is the extended multiplication.

(iii) $(L_x)^* = L_{x^*}$, for $x \in \hat{A}$, where $*$ is the $*$ operation of $\overline{A}$ extended to $\hat{A}$.

For simplicity, we identify L_x in $A_1 * A_2$ with x in $\hat{A}$, and write x for L_x. $A_1 * A_2$ is a finite von Neumann algebra with trace $\operatorname{tr}(a) = (a|1)$, for $a \in A_1 * A_2$.

3. **Factors of type II_1.** Let A be a von Neumann algebra on Hilbert space H with a separating cyclic vector ξ. A countable set $O = \{U_\alpha\}_{\alpha \in I}$ of unitary operators in A is called an *ortho-unitary basis* provided (i) $(U_\alpha \xi | U_\beta \xi) = 0$ if $\alpha \neq \beta$; (ii) for all $\alpha \in I$, $U_\alpha^* = c_\tau U_\tau$ for some $U_\tau \in O$ and a complex number c_τ of modulus 1, and for every pair $\alpha, \beta \in I$, $U_\alpha U_\beta = c_\mu U_\mu$ for some $U_\mu \in O$ and a complex number c_μ of modulus 1; (iii) the set of all linear combinations of elements from O is weakly (hence strongly) dense in A. Hence, every element T in A can be expressed as $\sum_{\alpha \in I} c_\alpha U_\alpha$ uniquely, where the series converges unconditionally in strong-operator topology. If $T \in A$ is such that $(T\xi | U_\alpha \xi) = 0$ for all $\alpha \in I$, then $T = 0$. In other words, $\{U_\alpha\}_{\alpha \in I}$ is a maximal orthogonal set in A.

Remark 1. Let G be a discrete group, then the von Neumann algebra $\mathfrak{A}(G)$ on $l^2(G)$ with δ_e as a separating cyclic vector has an ortho-unitary basis $\{U_g\}_{g \in G}$, where U_g is the unitary operator corresponding to the translation by g. Conditions (i) and (ii) are trivially satisfied, and $\mathfrak{A}(G)$ is the von Neumann algebra generated by $\{U_g\}_{g \in G}$.

Remark 2. Let M_n be $n \times n$ matrix algebra, i.e. the factor of type I_n, $n = 1, 2, \cdots$. Then M_n is isomorphic to the von Neumann algebra $M_n = B(H_n) \otimes \mathbf{C}$ on $H_n \otimes H_n$, where H_n is the n-dimensional Hilbert space, $B(H_n)$ is the algebra of all linear operators on H_n, with $\xi = (\xi_1, \cdots, \xi_{n^2})$ as a separating cyclic vector, where $\xi_1 = \xi_{2+n} = \xi_{3+2n} = \cdots = \xi_{k+(k-1)n} = \cdots = \xi_{n^2} = 1$ and all other $\xi_i = 0$. Let $W = a \otimes I$, and $V = b \otimes I$, where $a = (a_{ij})$, $a_{n1} = a_{12} = a_{23} = a_{(n-1)n} = 1$ and all other $a_{ij} = 0$ $(i, j = 1, \cdots, n)$, and $b = (b_{ij})$, $b_{ij} = 0$ $(i \neq j)$, and

$b_{jj} = e^{2\pi i(j-1)/n}$ $(j = 1, \cdots, n)$ are two operators on H_n. Let $U_{kj} = W^k V^j$, $k = 1, \cdots, n$, $j = 1, \cdots, n$. We claim that $\{U_{k,j}\}_{k,j=1}^n$ is an ortho-unitary basis for M_n. Each U_{kj} is clearly a unitary operator. If $k \neq k'$, then obviously we have $(U_{k,j}\xi | U_{k',j'}\xi) = 0$; if $k = k'$, but $j \neq j'$, then

$$(U_{k,j}\xi \mid U_{k',j'}\xi) = 1 + e^{2\pi i(j-j')n-1/n} = 0.$$

We note that $VW = e^{2\pi(n-1)i/n}WV$, $(V^j)^* = V^{n-j}$, $(W^k)^* = W^{n-k}$, $j, k = 1, 2, \cdots, n$. Hence $U^*_{k,j} = (V^j)^*(W^k)^* = V^{n-j}W^{n-k} = e^{2\pi i(n-1)(n-j)(n-k)/n}U_{n-k,n-j}$, $U_{k,j}U_{k',j'} = W^kV^jW^{k'}V^{j'} = e^{2\pi i(n-1)k'j}U_{k+k',j+j'}$, $k, j = 1, 2, \cdots, n$. And as M_n is a n^2-dimensional linear space, $\{U_{k,j}\}_{k,j=1}^n$ spans M_n. Therefore, $\{U_{k,j}\}_{k,j=1}^n$ is an ortho-unitary basis for M_n.

Remark 3. Let A_1 and A_2 be two von Neumann algebras with separating cyclic trace vectors ξ_1 and ξ_2. Let $O_1 = \{a_\beta\}_{\beta \in I_1}$, and $O_2 = \{b_r\}_{r \in I_2}$ be ortho-unitary bases for A_1 and A_2 respectively. Let $O = \{e_\alpha\}_{\alpha \in S}$ be the orthonormal set of operators in $A_1 * A_2$ constructed from O_1 and O_2 as we did in Lemma 3. e_α's are, in fact, unitary operators in $A_1 * A_2$. Conditions (i) and (ii) of an ortho-unitary basis can easily be verified, and the linear combinations of e_α's form a dense set in $A_1 * A_2$. Hence, O is an ortho-unitary basis for $A_1 * A_2$.

Theorem 1. *Let A_1 and A_2 be two von Neumann algebras with separating cyclic trace vectors ξ_1 and ξ_2 on Hilbert space H_1 and H_2 respectively, where the linear dimension of A_1 is at least two and that of A_2 is at least three. Suppose that A_1 and A_2 have ortho-unitary bases O_1 and O_2 respectively. Then the free product $A_1 * A_2$ of A_1 and A_2 is a factor of type* II_1.

Proof. Let $O = \{e_\alpha\}_{\alpha \in S}$ be the ortho-unitary basis of $A_1 * A_2$ constructed from O_1 and O_2. Let $T = \Sigma_{\alpha \in S}\, c_\alpha e_\alpha$ be an arbitrary element in the center of $A_1 * A_2$, where the c_α's are complex coefficients. Suppose that T is not a scalar. By rearrangements, we can assume that $r = |c_{\alpha_1}| \neq 0$ and $e_{\alpha_1} \neq 1$. Let $e_{\alpha_1} = a_{\beta_1}b_{\beta_1} \cdots a_{\beta_n}b_{\beta_n}$, where $n \geq 2$ if $a_{\beta_1} = 1$ and $b_{\beta_n} = 1$. Let $p = \max\{p(e_\alpha) \mid |c_\alpha| \geq r\}$.

Case 1. $a_{\beta_1} \neq 1$ and $b_{\beta_n} \neq 1$. Let a_{β_0} be a nonidentity element of O_1 and let b_{β_0} be a nonidentity element of O_2 orthogonal to b_{β_n}. Let $U = a_{\beta_0}b_{\beta_0}$. Let $a^*_{\beta_0} = c'_{\beta_0}a_{r_0}$, $b^*_{\beta_0} = c''_{\beta_0}b_{r_0}$, where $|c'_{\beta_0}| = 1$. $|c''_{\beta_0}| = 1$, $a_{r_0} \in O_1$, $b_{r_0} \in O_2$. Then $U^* = c_{\beta_0}1b_{r_0}a_{r_0}1$, where $c_{\beta_0} = c'_{\beta_0}c''_{\beta_0}$. Let $b_{\beta_n}b_{r_0} = c_{r_0}b_{r_1}$, where $b_{r_1} \neq 1$ since b_{r_0} is orthogonal to $b^{-1}_{\beta_n}$. We have

$$U^kT(U^*)^k = c_{r_0}(c_{\beta_0})^kc_{\alpha_1}(a_{\beta_0}b_{\beta_0})^ka_{\beta_1}b_{\beta_1} \cdots a_{\beta_n}b_{r_1}(a_{r_0}b_{r_0})^{k-1}a_{r_0}1$$

$$+ \sum_{\alpha \in S, \alpha \neq \alpha_1} c_\alpha U^k e_\alpha (U^*)^k.$$

The first term has a coefficient with absolute value r, and the length of the irreducible monomial is $2k+n$. And the second term is orthogonal to the first term because the trace is unitary invariant. But $U^kT(U^*)^k = T$ for all $k = 1, 2, \cdots$, and when $k > 1/a(p-n)$, we arrived at a contradiction by the definition of p. Hence $c_{\alpha_1} = 0$.

Case 2. $a_{\beta_1} = 1$ and $b_{\beta_n} = 1$. Let $a_{\beta_0} \neq 1$, $a_{\beta_0} \in O_1$, and b_{β_0} be a nonidentity of O_2 orthogonal to $b^*_{\beta_1}$. Let $U = a_{\beta_0} b_{\beta_0}$. Then $U^* = 1b^*_{\beta_0} a^*_{\beta_0} 1 = c_{\beta_0} 1 b_{r_0} a_{r_0} 1$, where $|c_{\beta_0}| = 1$, and $b_{r_0} \in O_2$, $a_{r_0} \in O_1$. Let $b_{\beta_0} b_{\beta_1} = c_{r_1} b_{r_1}$, where $|c_{r_1}| = 1$ and $b_{r_1} \neq 1$, $b_{r_1} \in O_2$. We have

$$U^kT(U^*)^k = c_{r_1} c_{\alpha_1} c^k_{\beta_0} (a_{\beta_0} b_{\beta_0})^{k-1} a_{\beta_0} b_{r_1} a_{\beta_2} b_{\beta_2} \cdots a_{\beta_n} (b_{r_0} a_{r_0})^{k-1} a_{r_0} 1$$

$$+ \sum_{\alpha \in S, \alpha \neq \alpha_1} c_\alpha U^k e_\alpha (U^*)^k, \qquad k = 1, 2, \cdots.$$

The first term has a coefficient with absolute value r, and the length of the irreducible monomial is $2k+n-1$; the second term is orthogonal to the first term. But $U^kT(U^*)^k = T$, $k = 1, 2, \cdots$, and when $k > 1/2(p-n+1)$, we have a contradiction by the definition of p. Hence $c_{\alpha_1} = 0$.

Case 3. $a_{\beta_1} = 1$ and $b_{\beta_n} \neq 1$. Let $a_{\beta_0} \neq 1$, $b_{\beta_0} \neq 1$, $a_{\beta_0} \in O_1$, $b_{\beta_0} \in O_2$. Let $a^*_{\beta_0} = c'_{r_0} a_{r_0}$, $b^*_{\beta_0} = c''_{r_0} b_{r_0}$, where $|c'_{r_0}| = |c''_{r_0}| = 1$, $a_{r_0} \in O_1$, $b_{r_0} \in O_2$. Let $U = 1b_{\beta_0} a_{\beta_0} 1$; hence $U^* = c_{r_0} a_{r_0} b_{r_0}$, where $c_{r_0} = c'_{r_0} c''_{r_0}$. We have

$$U^kT(U^*)^k = c_{\alpha_1} c_{r_0} 1 b_{\beta_0} (a_{\beta_0} b_{\beta_0})^{k-1} a_{\beta_0} b_{\beta_1} a_{\beta_2} b_{\beta_2} \cdots a_{\beta_n} b_{\beta_n} (a_{r_0} b_{r_0})^k$$

$$+ \sum_{\alpha \in S, \alpha \neq \alpha_1} c_\alpha U^k e_\alpha (U^*)^k, \qquad k = 1, 2, \cdots.$$

The first term has a coefficient with absolute value r and the length of the irreducible monomial is $2k+n$; the second term is orthogonal to the first term. As $U^kT(U^*)^k = T$, $k = 1, 2, \cdots$, when $k > 1/2(p-n)$, we have a contradiction by the definition of p. Hence $c_{\alpha_1} = 0$.

Case 4. $a_{\beta_1} \neq 1$ and $b_{\beta_n} = 1$. Let a_{β_0} and b_{β_0} be nonidentity elements of O_1 and O_2 respectively. Let $a^*_{\beta_0} = c'_{r_0} a_{r_0}$, $b^*_{\beta_0} = c''_{r_0} b_{r_0}$, where $|c'_{r_0}| = |c''_{r_0}| = 1$, $a_{r_0} \in O_1$, $b_{r_0} \in O_2$. Let $U = a_{\beta_0} b_{\beta_0}$; hence $U^* = c_{r_0} 1 b_{r_0} a_{r_0} 1$. We have

$$U^kT(U^*)^k = c_{\alpha_1} c^k_{r_0} (a_{\beta_0} b_{\beta_0})^k a_{\beta_1} b_{\beta_1} \cdots a_{\beta_n} b_{r_0} (a_{r_0} b_{r_0})^{k-1} a_{r_0} 1$$

$$+ \sum_{\alpha \in S, \alpha \neq \alpha_1} c_\alpha U^k e_\alpha (U^*)^k, \qquad k = 1, 2, 3, \cdots.$$

By the same reason as in the previous cases, when we let $k \geq 1/2(p-n)$, the equality $U^kT(U^*)^k = T$ together with the definition of p implies a contradiction. Hence $c_{\alpha_1} = 0$.

This proves that if T is in the center then T is a scalar. Hence $A_1 * A_2$ is a factor. Since $A_1 * A_2$ is clearly an infinite-dimensional vector space, $A_1 * A_2$ is a factor of type II_1 since $A_1 * A_2$ has a trace vector. This completes the proof.

Let $O = \{e_\alpha\}_{\alpha \in S}$ be the ortho-unitary bases for the von Neumann algebra $A_1 * A_2$, constructed from O_1 of A_1 and O_2 of A_2. We define a multiplication on S by $\alpha \cdot \beta = r$, if $e_\alpha \cdot e_\beta = c_r e_r$, where c_r is a complex number of modulus 1. This multiplication is uniquely defined. For if $e_\alpha \cdot e_\beta = c_\mu e_\mu$ with $e_\mu \neq e_r$, then $1 = (e_\alpha \cdot e_\beta | e_\alpha \cdot e_\beta) = c_r c_\mu (e_r | e_\mu) = 0$ a contradiction. If $e_\alpha e_\beta = c_r e_r$, and $(e_r e_\mu) = c_\lambda e_\lambda$, where $|c_r| = |c_\lambda| = 1$, then $(\alpha\ \beta) r = \lambda$. Let $e_\beta e_r = c_\mu e_{\mu'}$, and $e_\alpha e_{\mu'} = c_{\lambda'} e_{\lambda'}$, where $|c_\mu| = |c_\lambda| = 1$. Then $\alpha(\beta\ r) = \lambda'$. But $c_r c_\lambda e_\lambda = c_{\lambda'} c_\mu e_{\lambda'}$, by the same argument as before, we have $e_{\lambda'} = e_\lambda$ or $\lambda = \lambda'$. If $e_\iota = 1$, then ι is clearly the identity of S. Also, we have $\alpha^{-1} = \beta$ if $e_\alpha^* = c_\beta e_\beta$, where $|c_\beta| = 1$. Hence S is a group. Let E' be the subset of O consisting of all e_α ending in $a_{\beta_0} 1$, where $a_{\beta_0} \in O_1$, $a_{\beta_0} \neq 1$. Let $F = \{\beta \colon e_\beta \in E\}$. Let $e_{r_0} = a_{\beta_0} 1$, $e_{r_1} = 1 b_{\beta_1}$, $e_{r_2} = 1 b_{\beta_2}$, where $b_{\beta_1}, b_{\beta_2} \in O_2$, $b_{\beta_1} \neq 1$, $b_{\beta_2} \neq 1$. We have (i) $F \cup r_0 F r_0^{-1} = S \backslash \{\iota\}$. (ii) F, $r_1 F r_1^{-1}$ and $r_2 F r_2^{-1}$ are three disjoint subsets of $S \backslash \{\iota\}$. The following is a slight variation of Lemma 10 in Pukanszky [13]. The proof, which we omit, is exactly the same after replacing $f(g_i a g_i^{-1})$ by $c_a f(g_i a g_i^{-1})$, where c_a is a complex number of modulus 1.

Lemma 4. *Let S be a group with identity ι. Suppose there exists a subset $F \subset S \backslash \{\iota\}$ and three elements in S such that* (i) $F \cup g_1 F g_1^{-1} = S \backslash \{\iota\}$, (ii) *the sets F, $g_2 F g_2^{-1}$, $g_2 F g_3^{-1}$ are pairwise subsets of $S \backslash \{\iota\}$. Let f be a complex-valued function on S such that $\Sigma_{\alpha \in S} |f(a)|^2 < +\infty$,*

$$\left(\sum_{\alpha \in S} |c_a f(g_i a g_i^{-1}) - f(g)|^2 \right)^{1/2} < \epsilon \qquad (i = 1, 2, 3).$$

Then $(\Sigma_{\alpha \in S \backslash \{\iota\}} |f(a)|^2)^{1/2} < 14\epsilon$.

Theorem 2. *Let A_1 and A_2 be two von Neumann algebras with separating cyclic trace vectors. Suppose A_1 and A_2 have ortho-unitary bases O_1 and O_2 respectively, and that the linear dimension of A_1 is at least two, that of A_2 at least three. Then, the free product $A_1 * A_2$ is a factor of type* II_1 *without property* Γ (*Murray and von Neumann* [11, *Definition* 6.1]).

Proof. It is proved in Theorem 1 that $A_1 * A_2$ is a factor of type II_1. Suppose $A_1 * A_2$ has property Γ. Let $1/14 > \epsilon > 0$, and let $T_i = e_{r_i}$, $i = 0, 1, 2$, be given, where e_{r_i} ($\epsilon O = \{e_\alpha\}_{\alpha \in S}$, the ortho-unitary basis of $A_1 * A_2$ constructed from O_1 and O_2), $i = 0, 1, 2$, are as described in the discussion preceded by Lemma 4. Let $U = \Sigma_{r \in S} f(r) e_r$ be a unitary operator in $A_1 * A_2$ such that

(i) $\mathrm{tr}(U) = 0$, or $f(\iota) = 0$, where $e_\iota = 1$,

(ii) $\|U^*T_iU - T_i\|_2 = \|T_iUT_i^* - U\|_2 < \epsilon$, $i = 0, 1, 2$, where $\|\cdot\|_2$ is the trace norm of $A_1 * A_2$.

Let $e_{r_i}^* = c'_{r_i} e_{r_i^{-1}}$, $e_{r_i^{-1}} e_r = c'_r e_{r_i^{-1} r}$, $e_{r_i^{-1} r} \cdot e_{r_i} = c''_r e_{r_i^{-1} r r_i}$, $i = 0, 1, 2$, $r \in S$, S is given the group structure as described in the paragraph immediately before Lemma 4, c'_{r_i}, c'_r, c''_r are all complex numbers of modulus 1. We have $\Sigma_{r \in S \setminus \{\iota\}} |f(r)|^2 = 1$, and by (ii)

$$\left\| \sum_{r \in S} (c_r f(r) e_{r_i^{-1} r r_i} - f(r) e_r) \right\|_2 = \left\{ \sum_{r \in S} |c_r f(r_i r r_i^{-1}) - f(r)|^2 \right\}^{1/2} < \epsilon,$$

$i = 0, 1, 2$. Let F be the subset of S described in the paragraph immediately before Lemma 4. Then, by Lemma 4, we have $\Sigma_{r \in S \setminus \{\iota\}} |f(r)|^2 < 14\epsilon < 1$, a contradiction. This completes the proof of the theorem.

Remark 4. Let G_1 and G_2 be two discrete groups, and let $G_1 * G_2$ be the free product of G_1 and G_2. Then $\mathfrak{A}(G_1 * G_2) = \mathfrak{A}(G_1) * \mathfrak{A}(G_2)$. In fact, let U_g be the unitary operator in $\mathfrak{A}(G)$ corresponding to the translation by g for $g \in G_i$, $i = 1, 2$. Then $O_i = \{U_g\}_{g \in G_i}$ is an ortho-unitary basis for $\mathfrak{A}(G_i)$, $i = 1, 2$. The ortho-unitary basis $O = \{e_\alpha\}_{\alpha \in S}$ for $\mathfrak{A}(G_1) * \mathfrak{A}(G_2)$ constructed from O_1 and O_2 is just $\{U_g\}_{g \in G_1 * G_2}$ because e_α's are in one-to-one correspondence with the reduced sequences among all words in the free product $G_1 * G_2$ (Magnus, Karrass and Solitar [7, Theorem 4.1]). Hence $\mathfrak{A}(G_1) * \mathfrak{A}(G_2) = \mathfrak{A}(G_1 * G_2)$. For example, $\phi_2 = Z * Z$, the free product of two integer groups. Hence, $\mathfrak{A}(\Phi_2) = \mathfrak{A}(Z) * \mathfrak{A}(Z)$, the free product of two infinite-dimensional abelian von Neumann algebras.

In the case of $\mathfrak{A}(\Phi_2) = \mathfrak{A}(Z) * \mathfrak{A}(Z)$, there are three known types of outer automorphisms: (i) Interchange the two isomorphic free factors, i.e. ϕ: $a \leftrightarrow b$, where a and b are the two generators of Φ_2. This is first pointed out by Kadison [5, p. 308, Example 15]. (ii) Map a into ab, b into b, and extend this map into an automorphism of $\mathfrak{A}(\Phi_2)$. (iii) Any outer automorphism of $\mathfrak{A}(Z)$ can be extended to an outer automorphism of $\mathfrak{A}(Z) * \mathfrak{A}(Z)$. This is pointed out by Behncke [1]. Now, let M_2 be the 2×2 matrix algebra with a separating cyclic trace vector on a four-dimensional Hilbert space, and let M_3 be the 3×3 matrix algebra with a separating cyclic trace vector on a nice-dimensional Hilbert space. Let $M_2 * M_3$ be the free product of M_2 and M_3. $M_2 * M_2$ is a factor of type II_1 without property Γ as we remarked before, M_2 and M_3 have ortho-unitary bases. We can see that the three known types of outer automorphisms do not occur in the case of $M_2 * M_3$ for (i) M_2 and M_3 are nonisomorphic, so they cannot be interchanged by an isomorphism; (ii) if a is a generator for M_2, b a generator for M_3, ab would generate an infinite-dimensional algebra, hence cannot be isomorphic to M_2; (iii) all automorphisms of

M_2 and M_3 are inner. We cannot determine all outer automorphisms of $M_2 * M_3$ at present. It may turn out that $M_2 * M_3$ is a factor of type II_1 without any outer automorphism. $M_2 * M_3$ is also a possible candidate for a factor of type II_1 non-isomorphic to any factor of type II_1 constructed from group construction.

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