TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 178, April 1973

LOCAL AND ASYMPTOTIC APPROXIMATIONS OF NONLINEAR OPERATORS BY $(k_1, \cdots, k_N)$-HOMOGENEOUS OPERATORS

BY

R. H. MOORE AND M. Z. NASHED[1]

ABSTRACT. Notions of local and asymptotic approximations of a nonlinear mapping F between normed linear spaces by a sum of N k_i-homogeneous operators are defined and investigated. It is shown that the approximating operators inherit from F properties related to compactness and measures of noncompactness. Nets of equi-approximable operators with collectively compact (or bounded) approximates, which arise in approximate solutions of integral and operator equations, are studied with particular reference to pointwise (or weak convergence) properties. As a by-product, the well-known result that the Fréchet (or asymptotic) derivative of a compact operator is compact is generalized in several directions and to families of operators.

Introduction. In this paper we study local and asymptotic approximations of nonlinear operators by sums of homogeneous operators. The unifying thread is the notion of a locally $(k_1, \cdots, k_N)$-homogeneous or asymptotically $(k_1, \cdots, k_N)$-homogeneous operator defined in §1. The conditions assumed relax the usual conditions of differentiability or linear local (or asymptotic) approximability, and are motivated by consideration of nondifferentiable operators arising for instance in integral equations of the Hammerstein type:

$$x(s) + \int_0^1 k(s, t, x(t))dt = y(s). \tag{A}$$

Analogous approximation concepts are developed for families of operators. The results apply in particular to families of collectively compact operators studied by Anselone [1] and others. Such families occur in numerical analysis: for example, one may consider approximations to equation (A) using numerical quadratures:

Received by the editors April 17, 1972.

AMS (MOS) subject classifications (1970). Primary 47H99, 46G05; Secondary 46B99, 41A65.

Key words and phrases. Local approximations, asymptotic approximations and derivatives, collectively compact operators, nonlinear operators, Fréchet and other differentials, measure of noncompactness, nondifferentiable Hammerstein operators, nonlinear integral equations, approximations by numerical quadratures.

(1) Sponsored by the United States Army under Contract No. DA-31-124-ARO-D-462.

$$\text{(B)} \qquad x(s) + \sum_{j=1}^{n} w_{nj} k(s, t_{nj}, x(t_{nj})) = y(s).$$

Under suitable hypotheses the integral operator $\mathcal{K}$ in (A) is compact, the approximating operators $\mathcal{K}_n$ in (B) are collectively compact, and $\|\mathcal{K}_n x - \mathcal{K}x\| \longrightarrow 0$ for each x, although $\|\mathcal{K}_n - \mathcal{K}\| \not\longrightarrow 0$. Previous applications of the theory for collectively compact operators in this context have assumed differentiability of the operators (see [1], [10]).

In §2 we consider the pointwise limits of nets of operators which are equilocally $(k_1, \cdots, k_N)$-homogeneous and show that local $(k_1, \cdots, k_N)$-homogeneous approximability of an operator F is implied by related properties of a net of operators $\{F_m\}$ converging pointwise to F. The precise formulation of these results is given in Theorems 2.1 and 2.2.

In §3 we establish relations between measures of noncompactness associated with a $(k_1, \cdots, k_N)$-homogeneous operator and its approximating operators, and in §4 we extend these results to families of such operators. It is well-known that the Fréchet derivative and the asymptotic derivative of a completely continuous operator are completely continuous [7, p. 135, p. 207]. Melamed and Perov [9] replaced Fréchet differentiability by a weaker notion of local or asymptotic approximability. Moore [10] extended the result to collectively compact and Fréchet equidifferentiable families of operators. Daneš [3] and Nussbaum [13] showed independently that the Fréchet derivative of an α-set-contraction is an α-set-contraction; thus a property which is more general than compactness of the operator is inherited by the Fréchet derivative of the operator. Some of the corollaries of §§3 and 4 unify and generalize these simple results.

1. Locally and asymptotically $(k_1, \cdots, k_N)$-homogeneous operators and related notions. Throughout this article, X and Y will be real normed linear spaces, and X_1 will denote the open unit ball in X.

Definition 1.1. A map $F: X \longrightarrow Y$ is *locally* $(k_1, \cdots, k_N)$*-homogeneous at* 0 (abbreviated $\mathrm{LH}(k_1, \cdots, k_N)$ at 0) if, in some neighborhood of 0, F can be expressed in the form

(i) $Fx = \Sigma_{i=1}^N A_i x + Bx$

where A_i $(i = 1, \cdots, N)$ and B map X into Y, and

(ii) $A_i(tx) = t^{k_i} A_i x$, $x \in X$, $t \geq 0$,

(iii) $0 < k_1 < k_2 < \cdots < k_N$,

(iv) $\lim_{\|x\| \to 0} \{\|x\|^{-k_n} \|\Sigma_{i=n+1}^N A_i x + Bx\|\} = 0$, $n = 1, \cdots, N$.

F is *weakly* locally $(k_1, \cdots, k_N)$-homogeneous at 0 (abbreviated $\mathfrak{W}\mathrm{LH}(k_1, \cdots, k_N)$ at 0) if the limit in (iv) holds weakly as $\|x\| \longrightarrow 0$. A map $F: X \longrightarrow Y$ is $\mathrm{LH}(k_1, \cdots, k_N)$ at $z \in X$ if the map $h \longrightarrow F(z+h) - F(z)$ is

$LH(k_1, \cdots, k_N)$ at 0; $\mathfrak{W}LH(k_1, \cdots, k_N)$ at $z \in X$ is similarly defined.

Definition 1.2. A map $F: X \to Y$ is *asymptotically* $(k_1, \cdots, k_N)$*-homogeneous* (abbreviated $AH(k_1, \cdots, k_N)$ at ∞) if (i)–(iii) hold for $\|x\|$ sufficiently large, and

$(iv)_\infty$ $\lim_{\|x\| \to \infty} \{\|x\|^{-k_n} \|\Sigma_{i=1}^{n-1} A_i x + Bx\|\} = 0$, $n = 1, \cdots, N$.

The notion of $\mathfrak{W}AH(k_1, \cdots, k_N)$ at ∞ is defined using the weak limit. (Of course, here and elsewhere, sums $\Sigma_{i=n+1}^{N}$ or $\Sigma_{i=1}^{n-1}$ are empty if $n = N$ or $n = 1$, respectively, or if $N = 1$.)

The proof of uniqueness of the representations in each of the preceding definitions follows the same pattern as its counterpart for differentials and asymptotic differentials.

Remark 1.1. If the A_i satisfy

(v) $\|A_i x\| \le M_i \|x\|^{k_i}$, $i = 1, \cdots, N$, $x \in X$,

then conditions (iv) follow from the single condition

(vi) $\lim_{\|x\| \to \infty} \|Bx\| \cdot \|x\|^{-k_N} = 0$

because of (iii). Conditions (v) and (vi) are in fact those assumed by Melamed and Perov in [9], in lieu of (iv) in Definition 1.1. Similarly, if (v) and

(vi)′ $\lim_{\|x\| \to \infty} \|Bx\| \cdot \|x\|^{-k_1} = 0$

hold, then conditions $(iv)_\infty$ hold for $AH(k_1, \cdots, k_N)$ at ∞.

It is easy to show that a positively homogeneous operator A of degree $k \ge 0$ (i.e., $A(tx) = t^k Ax$, $t \ge 0$) is bounded (i.e. maps bounded sets into bounded sets) if and only if

(vii) $\|Ax\| \le M\|x\|^k$ for some $M > 0$ and all $x \in X$.

A family $\mathfrak{A}$ of operators A is uniformly bounded if for any bounded set S, $\bigcup_{A \in \mathfrak{A}} A(S)$ is bounded. For a family $\mathfrak{A}$ of operators A all homogeneous of degree k, (vii) holds with a uniform M if and only if the family is uniformly bounded.

Remark 1.2. If F is Fréchet differentiable at z, then it is locally 1-homogeneous at z, but not conversely. But if F is LH(1) at z, then the conditions satisfied are exactly those for F to have a bounded differential in the sense of Suchomolinov (see for instance [11, p. 135]): F is said to have a bounded differential at z if there exists a bounded but not necessarily *linear* operator $B(z; \cdot)$ such that

$$\lim_{\|h\| \to 0} \|h\|^{-1} \cdot \|F(z+h) - F(z) - B(z; h)\| = 0.$$

This implies of course that $B(z; h)$ is homogeneous of degree one in h. Similarly if F is asymptotically Fréchet differentiable, i.e. if there exists a bounded linear operator L such that $\lim_{\|x\| \to \infty} \{\|Fx - Lx\|/\|x\|\} = 0$, then F is AH(1) at ∞, but not conversely.

For later reference we state the following easily proved propositions:

Proposition 1.1. *Let* $\mathfrak{M}$ *be a directed set. Let the operators* A *and* A_m $(m \in \mathfrak{M})$ *be homogeneous of degree* k. *If* $\lim_m \|A_m x - Ax\| = 0$, $x \in X$, *and* $\|A_m x\| \le$

$M\|x\|^k$, $m \in \mathfrak{M}$, $x \in X$, *then* $\|Ax\| \le M\|x\|^k$.

Proposition 1.2. *Let* F *be* LH(k) *at* 0 *with* $F = C + D$, *where* $C(tx) = t^k Cx$, $t \ge 0$, $x \in X$, *and* $\|x\|^{-k}\|Dx\| \to 0$ *as* $\|x\| \to 0$. *If* F *is bounded in some neighborhood of* 0, *then there is* $M > 0$ *such that* $\|Cx\| \le M\|x\|^k$.

Proposition 1.3. *If* F *is* LH$(k_1, \cdots, k_N)$ *at* 0 (*or is* AH$(k_1, \cdots, k_N)$ *at* ∞) *and is bounded, then there are numbers* $M_i > 0$, $i = 1, \cdots, N$, *such that* $\|A_i x\| \le M_i\|x\|^{k_i}$, $i = 1, \cdots, N$, $x \in X$.

Proof. Apply Proposition 1.2 successively to F, $F - A_1, \cdots, F - \sum_{i=1}^{m-1} A_i$.

Remark 1.3. Condition (iv) in Definition 1.1 is equivalent to the following condition:

(viii) $\lim_{t\to 0} t^{-k_n}\{\sum_{i=n+1}^{N} A_i(th) + B(th)\} = 0$, $n = 1, \cdots, N$, *uniformly with respect to* h in the set $\{h: \|h\| = 1\}$.

Variants of Definition 1.1 can be given by requiring the limit in (viii) to hold uniformly with respect to $h \in S$ for each set S in a given system β of subsets of X. For example if we take for β the system of all finite sets in X, then we get a Gâteaux-type notion of local $(k_1, \cdots, k_N)$-homogeneous approximation. Another interesting choice for β is the system β_c of all sequentially compact subsets of X. If F is Hadamard differentiable at z (see, e.g. [21, p. 124]) then F is LH(1) at z relative to the system β_c but not conversely.

The class of LH- and AH-operators falls in a hierarchy of other notions that are useful in the local and asymptotic approximation of nonlinear operators. We shall say that a mapping $F: X \to Y$ is *k-inner* (*outer*) *bounded*, if there exist positive numbers α, γ, k such that $\|Fx\| \le \gamma\|x\|^k$ if $\|x\| \le \alpha$ (respectively $\|x\| \ge \alpha$). The subclass of 1-outer bounded operators was introduced by Granas [6] under the name of *quasibounded* operators. It is easy to show that F is k-outer bounded if and only if

$$|F|_k := \inf_{0<\rho<\infty} \sup_{\|x\|\ge\rho} \frac{\|F(x)\|}{\|x\|^k} = \lim_{\|x\|\to\infty} \sup \frac{\|Fx\|}{\|x\|^k}$$

is finite. $|F|_1$ is the usual quasinorm of a quasibounded operator. The various

$$\begin{tikzcd}
\mathrm{AH}(k_1, \cdots, k_N) \text{ operator} \arrow[rr, dashed, shift left, "\text{(for a bounded operator)}"] & & \text{Melamed-Perov operator} \arrow[ll, shift left] \arrow[d] \\
& & k\text{-outer bounded operator} \\
& \text{Asymptotically Fréchet differentiable operator} \arrow[uul] \arrow[uur] \arrow[dr] & \\
& & \text{Quasibounded operator} \arrow[uu]
\end{tikzcd}$$

implications among these notions of asymptotic approximations are given in the diagram.

Counterexamples for each of the reverse implications can be easily given.

As an example of an $\mathrm{AH}(k_1, \cdots, k_n)$ operator, consider the Hammerstein operator defined in $\mathfrak{L}_2\,[0, 1]$ by

$$Fx = \int_0^1 K(\cdot\,, t)g(t, x(t))\,dt$$

where we assume the following conditions hold:

(A1) $K(s, t)$ is measurable in s and t with

$$\|K\|^2 := \int_0^1\int_0^1 |K(s, t)|^2\,ds\,dt < \infty.$$

(A2) $g(t, u)$ is continuous in u, measurable in t and satisfies for some $a_i \in \mathfrak{L}_2\,[0, 1]$, $i = 1, \cdots, n$, and $0 < k_1 < k_2 < \cdots < k_n$, the condition

$$\left|g(t, u) - \sum_{i=1}^{n} a_i(t)|u|^{k_i}\right| \leq \sum_{i=1}^{m} g_i(t)|u|^{1-b_i} + g_0(t)$$

for $0 \leq t \leq 1$ and $-\infty < u < \infty$, where $g_0 \in \mathfrak{L}_2$, $g_i \in \mathfrak{L}_{2/b_i}$, $0 < b_i < 1$, $k_1 > 1 - b_i$, $i = 1, \cdots, m$.

(A3) Suppose the functions a_i in (A2) are such that

$$A_i x := \int_0^1 K(\cdot\,, t)a_i(t)|x(t)|^{k_i}\,dt, \qquad 0 < k_i \leq 2,\ i = 1, \cdots, n,$$

map $\mathfrak{L}_2$ into $\mathfrak{L}_2$. (For $k_i > 2$, one must use $\mathfrak{L}_p$ spaces, $p > 2$. For sufficient conditions under which (A3) holds, see Krasnoselskii [7, pp. 27 and 52]).

From (A1)–(A3) it follows easily (as for instance in [12]) that F is $\mathrm{AH}(k_1, \cdots, k_n)$ at ∞.

It is also possible to impose conditions so that F is $\mathrm{AH}(k_1, \cdots, k_n)$ from $\mathfrak{L}_{p_1}$ into $\mathfrak{L}_{p_2}$, or from a Banach space E into $\mathfrak{L}_p$. An interesting situation in integral equations which calls for an asymptotically quadratically null operator Q: $E \to \mathfrak{L}_1$, i.e. $\lim_{\|x\|_E \to \infty} \|Qx\|_{\mathfrak{L}_1}/\|x\|_E^2 = 0$, is given in [17], [18].

2. Pointwise convergence of equilocally (equi-asymptotically) $(k_1, \cdots, k_N)$-homogeneous operators. We denote by $\mathfrak{M}$ a directed set indexing the nets of operators studied in this section.

Definition 2.1. A family $\mathfrak{F} = \{F_m, m \in \mathfrak{M}\}$ of maps from X into Y is *equilocally* $(k_1, \cdots, k_N)$*-homogeneous at* 0 (abbreviated $\mathrm{ELH}(k_1, \cdots, k_N)$ at 0) if, in a neighborhood of 0,

$$F_m x = \sum_{i=1}^{N} A_{m,i} x + B_m x, \qquad m \in \mathfrak{M}, \ x \in X,$$

where the $A_{m,i}$ $(i = 1, \cdots, N)$ and B_m satisfy (ii)–(iv) of Definition 1.1 for $m \in \mathfrak{M}$ with the k_i $(i = 1, \cdots, N)$ independent of m and with the limits in (iv) holding *uniformly* with respect to m.

The family is *equi-asymptotically* $(k_1, \cdots, k_N)$*-homogeneous* (abbreviated $\mathrm{EAH}(k_1, \cdots, k_N)$ at ∞) if $(\mathrm{iv})_\infty$ holds in place of (iv), again uniformly with respect to m. The notions of *weakly equilocally* $(k_1, \cdots, k_N)$*-homogeneous* at 0 (abbreviated $\mathfrak{W}\mathrm{ELH}(k_1, \cdots, k_N)$ at 0) and of weakly *equi-asymptotically* $(k_1, \cdots, k_N)$-homogeneous (abbreviated $\mathfrak{W}\mathrm{EAH}(k_1, \cdots, k_N)$ at ∞) are defined analogously.

Remark 2.1. By the observation preceding Remark 1.2, the operators $A_{m,i}$ will satisfy for a fixed i,

$$\|A_{m,i} x\| \le M_i \|x\|^{k_i}, \qquad x \in X, \ m \in \mathfrak{M}, \tag{2.1}$$

if and only if the family $\{A_{m,i}, m \in \mathfrak{M}\}$ is uniformly bounded. On the other hand, by analogy with Proposition 1.3, if the family $\{F_m, m \in \mathfrak{M}\}$ is uniformly bounded and $\mathrm{ELH}(k_1, \cdots, k_N)$ at 0, then there are numbers $M_i > 0$, $i = 1, \cdots, N$, independent of m, such that (2.1) holds.

We recall that a family $\mathfrak{F}$ of operators on X into Y is called *collectively compact* if for every bounded set $B \subset X$, the set $\bigcup \{F(B): F \in \mathfrak{F}\}$ has compact closure. This notion is useful in the study of approximate solutions of integral and operator equations (cf. [1], [10]).

Theorem 2.1. *Let* $\{F_m, m \in \mathfrak{M}\}$, *where* $\mathfrak{M}$ *is a directed set, be a net of operators on* X *into* Y *which is* $\mathrm{ELH}(k_1, \cdots, k_N)$ *at* 0: $F_m x = \sum_{i=1}^N A_{m,i} x + B_m x$. *Let the family* $\{A_{m,i}, m \in \mathfrak{M}, 0 \le i \le N\}$ *be collectively compact. Suppose the space* Y *is complete (or that* $\bigcup \{F_m(X): m \in \mathfrak{M}\}$ *is complete) and* F *is the pointwise limit of* $\{F_m\}$ *in some open set* S *about* 0,

$$\lim_m \|F_m x - Fx\| = 0, \qquad x \in S. \tag{2.2}$$

Then F *is* $\mathrm{LH}(k_1, \cdots, k_N)$ *at* 0: $Fx = \sum_{i=1}^N A_i x + Bx$ *and moreover*

$$\lim_m \|A_{m,i} x - A_i x\| = 0, \qquad x \in X, \ i = 1, \cdots, N. \tag{2.3}$$

Proof. We proceed inductively. Suppose for $1 \le i \le n-1$ operators A_i homogeneous of degree k_i have been found satisfying (2.3). (We begin with $n = 1$ and no such A_i; then $\sum_{i=1}^{n-1}$ below is empty.) Let $h \in X_1$. Let $\epsilon > 0$ be given. Since

$\{A_{m,n}, m \in \mathfrak{M}\}$ is collectively compact, there is a subnet $A_{m_j,n}$ and a point, denoted by $A_n h$, such that

$$\lim_j \|A_{m_j,n} h - A_n h\| = 0. \tag{2.4}$$

We have

$$\begin{aligned}
&\left\| F(\delta h) - \sum_{i=1}^{n-1} A_i(\delta h) - \delta^{k_n} A_n h \right\| \\
&\quad \le \|F(\delta h) - F_{m_j}(\delta h)\| + \left\| \sum_{i=1}^{n-1} A_{m_j,i}(\delta h) - \sum_{i=1}^{n-1} A_i(\delta h) \right\| \\
&\quad + \|A_{m_j,n}(\delta h) - \delta^{k_n} A_n h\| + \left\| F_{m_j}(\delta h) - \sum_{i=1}^{n} A_{m_j,i}(\delta h) \right\|.
\end{aligned} \tag{2.5}$$

The last term here is equal to $\|\Sigma_{i=n+1}^N A_{m_j,i}(\delta h) + B_{m_j}(\delta h)\|$ which by (iv) of Definition 1.1 can be made less than $\epsilon\delta^{k_n}$ for all m_j by taking δ sufficiently small. Consider any such δ which also satisfies $\delta X_1 \subset S$, where X_1 is the unit ball in X. The limit with respect to j in the other terms of (2.5) is 0. Hence for sufficiently small δ,

$$\left\| \left\{ F(\delta h) - \sum_{i=1}^{n-1} A_i(\delta h) \right\} - \delta^{k_n} A_n h \right\| \le \epsilon \delta^{k_n}.$$

This implies $A_n h$ is the unique limit of all subsets of $\{A_{m,n} h, m \in \mathfrak{M}\}$, and hence is the limit of the net. For any point th, $t \ge 0$, define $A_n(th) = t^{k_n} A_n h$.

Since the above holds for any $h \in X_1$, this defines A_n as a homogeneous operator of degree k_n on X. (Consistent values are obtained if distinct linearly dependent points h_1, h_2 are used to define A_n.) Further, (2.3) holds for $i = n$ by (2.4) and the uniqueness of the limit.

In this manner, the operators A_i, $i = 1, \cdots, N$, are defined. Define $Bx = Fx - \Sigma_{i=1}^N A_i x$. To show (iv) holds, consider n such that $1 \le n \le N$, and let $\epsilon > 0$ be given. We have

$$\begin{aligned}
&\left\| \sum_{i=n+1}^{N} A_i x + Bx \right\| = \left\| Fx - \sum_{i=1}^{n} A_i x \right\| \\
&\quad \le \|Fx - F_m x\| + \left\| F_m x - \sum_{i=1}^{n} A_{m,i} x \right\| + \left\| \sum_{i=1}^{n} (A_{m,i} x - A_i x) \right\|
\end{aligned} \tag{2.6}$$

for all $m \in \mathfrak{M}$. Since by hypothesis for the F_m, (iv) holds uniformly with respect to m: there is $\delta > 0$ such that $\|x\| < \delta$ implies

$$\left\| F_m x - \sum_{i=1}^{n} A_{m,i} x \right\| = \left\| \sum_{i=n+1}^{N} A_{m,i} x + B_m x \right\| \le \frac{\epsilon}{2} \|x\|^{k_n}.$$

Consider any x such that $\|x\| < \delta$. Then on the right side of the inequality (2.6), the first and third terms can each be made less than $(\epsilon/4)\|x\|^{k_n}$, whence

$$\left\| \sum_{i=n+1}^{N} A_i x + Bx \right\| < \epsilon \|x\|^{k_n}$$

so (iv) holds. Thus F is $\mathrm{AH}(k_1, \cdots, k_N)$ at 0.

Remark 2.2. Since the operators $A_{m,i}$ are collectively compact, they are uniformly bounded and hence satisfy (2.1). Thus by (2.3) and Proposition 1.1 we conclude $\|A_i x\| \le M_i \|x\|^{k_i}$, $x \in X$, $i = 1, \cdots, N$. We now give an analog of Theorem 2.1 for weak convergence.

Theorem 2.2. *Let $\{F_m, m \in \mathfrak{M}\}$ be a net of operators on X into Y which are* $\mathfrak{W}$ELH *at* 0. *Let $\{A_{m,n} x: m \in \mathfrak{M}\}$ be bounded for each $x \in X$. Suppose that F is the pointwise limit of $\{F_m\}$ in some open set S about* 0 *and that the space Y is reflexive. Then F is* $\mathfrak{W}\mathrm{LH}(k_1, \cdots, k_N)$ *at* 0 *and for each $x \in X$*

(2.7) $\{A_{m,i} x\}$ *converges weakly to* $A_i x$, $i = 1, \cdots, N$.

Proof. We proceed inductively. Suppose for $1 \le i \le n-1$ operators A_i homogeneous of degree k_i have been found satisfying (2.7). Let $h \in X_1$. Let $\epsilon > 0$ be given. Since $\{A_{m,n} h\}$ is bounded, there is a subnet $\{A_{m_j,n}\}$ and a unique point, denoted by $A_n h$, such that $l(A_{m_j,n}\, h - A_n h) \longrightarrow 0$ for all $l \in Y^*$, the dual space of Y. Consider $l \in Y^*$ such that $\|l\| = 1$. Then for δ such that $\delta h \in S$, we have

$$\left| l\left(F(\delta h) - \sum_{i=1}^{n-1} A_i(\delta h) - \delta^{k_n} A_n h \right) \right|$$

$$\le \|(F - F_{m_j})(\delta h)\| + \left| l\left(\sum_{i=1}^{n-1} (A_{m_j, i} - A_i)(\delta h) \right) \right| \tag{2.8}$$

$$+ |l(A_{m_j, n}(\delta h) - \delta^{k_n} A_n h)| + \left| l\left(F_{m_j}(\delta h) - \sum_{i=1}^{n} A_{m_j, i}(\delta h) \right) \right|$$

for all $m_j \in \mathfrak{M}$. By hypothesis there is $\delta_0 > 0$ such that $0 < \delta < \delta_0$ implies $\delta h \in S$ and for all $m \in \mathfrak{M}$,

$$\left| l\left(F_m(\delta h) - \sum_{i=1}^{n} A_{m,i}(\delta h) \right) \right| < \epsilon \|\delta h\|^{k_n} \le \epsilon \delta^{k_n}.$$

Pick and fix any such δ. Passing to the limit with respect to m_j in (2.8) we obtain, for any $l \in Y^*$ such that $\|l\| = 1$,

$$\left| l\left(F(\delta h) - \sum_{i=0}^{n-1} A_i(\delta h) - \delta^{k_n} A_n h \right) \right| \le \epsilon \delta^{k_n}$$

whenever $\delta < \delta_0$. Hence, as a consequence of the Hahn-Banach theorem,

$$\left\| F(\delta h) - \sum_{i=1}^{n-1} A_i(\delta h) - \delta^{k_n} A_n h \right\| \leq \epsilon \delta^{k_n} \quad \text{for } \delta < \delta_0.$$

This proves that $A_n h$ does not depend on the choice of the subnet, so $A_n h$ is the unique weak limit of the net $\{A_{m,n} h\}$. The rest of the proof is the same as in Theorem 2.1, replacing norms in (2.6) by continuous linear functionals and using weak limits in the arguments following (2.6).

3. Measures of noncompactness related to $\mathrm{LH}(k_1, \cdots, k_N)$ operators. Let S be a bounded set in X. The measure of *noncompactness* of S, denoted $\kappa(S)$, is the infimum of all numbers $\epsilon > 0$ such that S can be covered by a finite number of ϵ-balls in X. The following known properties are needed for the proofs of the theorems below:

(a) $\kappa(S) = 0 \iff S$ is precompact (i.e. totally bounded).

(b) $\kappa(S \cup T) = \max(\kappa(S), \kappa(T))$.

(c) $S \subset T \implies \kappa(S) \leq \kappa(T)$.

(d) $\kappa(S(T, \epsilon)) \leq \kappa(T) + \epsilon$, where $S(T, \epsilon) = \{x \in X: \text{distance } (x, T) \leq \epsilon\}$.

(e) $\kappa(S + T) \leq \kappa(S) + \kappa(T)$.

(f) $\kappa(\lambda S) = |\lambda| \kappa(S)$, λ real.

(g) $\kappa(S) = \kappa(\overline{S})$.

(h) $\kappa(X_1) = 0$ if X is finite dimensional; $\kappa(X_1) \doteq 1$ if X is infinite dimensional.

Here and in the following X_1 denotes the open unit ball in X.

The definition of measure of noncompactness used here seems to be due to Goldenstein and Markus [5]. Closely related notions of measures of noncompactness have been introduced by Kuratowski [8], Darbo [4], Sadovskii [16] and others (see [2], [14], [15]).

Theorem 3.1. *If $F: X \longrightarrow Y$ is $\mathrm{LH}(k)$ at z,*

(3.1) $$F(z + h) = F(z) + A(z)h + B(z, h), \qquad \|h\| \leq \epsilon_0,$$

then there exists

(3.2) $$\lim_{\epsilon \to 0^+} \frac{\kappa(F(z + \epsilon X_1))}{\epsilon^k} = \kappa(A(z)X_1).$$

Proof. The proof of Theorem 3 of Daneš [3] or Lemma 4 of Nussbaum [13] can be adapted to this setting since the additivity of A assumed there is not really essential.

Corollary 3.1. *If F is $\mathrm{LH}(k_1, \cdots, k_N)$ at z, then there exists*

$$\lim_{\epsilon \to 0^+} \kappa\left(F(z + \epsilon X_1) - \sum_{i=1}^{j-1} A_i(z)(\epsilon X_1)\right)\epsilon^{-k_j} = \kappa(A_j(z)X_1) \quad \textit{for } j = 1, 2, \cdots, N.$$

Indication of Proof. Apply the theorem to the operators $F(z+h) - \sum_{i=1}^{j-1} A_i(z)h$ and use (iv).

A mapping $F\colon X \longrightarrow Y$ is said to satisfy a *Hölder condition of order k in the sense of measure of noncompactness with constant* $\alpha \geq 0$, briefly called α-mcH^k mapping, if for any bounded subset S of X, $\kappa(F(S)) \leq \alpha[\kappa(S)]^k$.

Corollary 3.2. *Let Ω be an open subset of X and let the mapping $F\colon \Omega \longrightarrow Y$ be* LH(k) *at $z \in \Omega$. If F is an α-mcH^k mapping, then so is the operator $A(z)$ defined in* (3.1) *and* $\kappa(A(z)X_1) \leq \alpha$.

Proof. This follows easily from (3.2) and the properties above.

When F is an α-set-contraction ([3], [14], [15]) which is Fréchet differentiable at z, Corollary 3.2 reduces to a result of Daneš [3, Theorem 3 and Corollary] and Nussbaum [13, Lemma 4]. From this result also follows the well-known fact that the Fréchet derivative of a compact operator is compact (since T is compact if and only if $\kappa(T(X_1)) = 0$). Note, however, that Corollary 3.2 yields the stronger result that the bounded derivative (which is not necessarily a linear operator; see Remark 1.2) of a compact operator is compact.

Other corollaries can be given which are specializations to a single operator of the corollaries to Theorem 4.1 below.

4. Measures of noncompactness related to equilocally or equi-asymptotically homogeneous families of operators and collectively compact operators. We begin with a generalization of Theorem 3.1 to families of operators.

Theorem 4.1. *Let $\{F_m,\ m \in \mathfrak{M}\}$ be a family of maps from a neighborhood of $z \in X$ into Y. If the family is* ELH(k) *at z:*

$$F_m(z+h) = F_m(z) + A_m(z)h + B_m(z,h), \qquad \|h\| < \epsilon_0,$$

and

$$\kappa\Big(\bigcup_{m \in \mathfrak{M}} F_m(z)\Big) = 0, \tag{4.1}$$

then there exists

$$\lim_{\epsilon \to 0^+} \kappa\Big(\bigcup_{m \in \mathfrak{M}} F_m(z + \epsilon X_1)\Big)\Big/\epsilon^k = \kappa\Big(\bigcup_{m \in \mathfrak{M}} A_m(z)X_1\Big). \tag{4.2}$$

Remark. Condition (4.1) holds in particular if $\mathfrak{M}$ is a directed set, and the generalized sequence $\{F_m(z),\ m \in \mathfrak{M}\}$ converges to some point or is a Cauchy sequence (in case Y is not complete).

Indication of Proof of Theorem 4.1. For $0 < \epsilon \leq \epsilon_0$ let

$$\delta(\epsilon) = \sup\{\|B_m(z,h)\|/\|h\|^k\colon h \in X,\ \|h\| < \epsilon,\ m \in \mathfrak{M}\}.$$

Then $\delta(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ since the condition (iv)

$$\lim_{\|h\|\to 0} \frac{\|B_m(z, h)\|}{\|h\|^k} = 0$$

holds *uniformly* with respect to m. Also,

$$\bigcup_m \frac{F_m(z + \epsilon X_1)}{\epsilon^k} \subset \bigcup_m \frac{F_m(z)}{\epsilon^k} + \bigcup_m A_m(z)X_1 + \delta(\epsilon)X_1$$

and

$$\bigcup_m A_m(z)X_1 \subset \bigcup_m \frac{F_m(z)}{\epsilon^k} - \bigcup_m \frac{F_m(z + \epsilon X_1)}{\epsilon^k} + \delta(\epsilon)X_1$$

since for each $m \in \mathfrak{M}$, the corresponding inclusion relations hold. Thus, by (4.1) and the properties of κ,

$$\left|\kappa\left(\bigcup_m F_m(z + \epsilon X_1)\right)\Big/\epsilon^k - \kappa\left(\bigcup_m A_m(z)X_1\right)\right| \leq \delta(\epsilon),$$

which proves the theorem.

Corollary 4.1. *If in addition to the hypotheses of Theorem* 4.1 *there is* $\alpha \geq 0$ *such that, for any bounded set* $S \subset X$, $\kappa(\bigcup_m F_m(S)) \leq \alpha[\kappa(S)]^k$, *then* $\kappa(\bigcup_m A_m(z)X_1) \leq \alpha$.

Proof. Since $\kappa(z + \epsilon X_1) = \epsilon$ we have

$$\kappa\left(\bigcup_m F_m(z + \epsilon X_1)\right) \leq \alpha[\kappa(z + \epsilon X_1)]^k = \alpha\epsilon^k$$

so Theorem (4.2) yields the result.

Corollary 4.2. *If in addition to the hypotheses of Theorem* 4.1 *the family* $\{F_m, m \in \mathfrak{M}\}$ *is collectively compact, then the family* $\{A_m(z), m \in \mathfrak{M}\}$ *is also collectively compact.*

Proof. In this case $\alpha = 0$.

Corollary 4.3. *If the family* $\{F_m, m \in \mathfrak{M}\}$ *is collectively compact and Fréchet equidifferentiable at* z, *then the derivatives* $\{F_m'(z), m \in \mathfrak{M}\}$ *form a collectively compact family.*

Proof. The equidifferentiability assures that the limit in (iv) will be uniform with respect to m. Also (4.1) clearly holds. Hence Corollary 4.2 applies.

Corollary 4.4. *Let* $\{F_m, m \in \mathfrak{M}\}$ *be a family of operators from a neighborhood of* $z \in X$ *into* Y. *If the family is collectively compact, and* ELH$(k_1, \cdots, k_N)$ *at* z, *then* $\{A_{m,i}, m \in \mathfrak{M}, i = 1, \cdots, N\}$ *and* $\{B_m, m \in \mathfrak{M}\}$ *are collectively compact.*

Indication of Proof. Apply Corollary 4.2 successively for $j = 1, \cdots, N$ to the families $\{F_m(z+h) - \sum_{i=1}^{j-1} A_i(z)h,\ m \in \mathfrak{M}\}$.

Corollary 4.4 reduces for the case of a single *bounded* operator (see Remark 1.1) to the theorem of Melamed and Perov in [9]. Corollary 4.4 also generalizes a result of Moore [10, Theorem 2].

We conclude by stating an analog of Theorem 4.1 for EAH(k) mappings.

Theorem 4.2. *Let* $\{F_m, m \in \mathfrak{M}\}$ *be a family of maps from* X *into* Y. *If the family is* EAH(k) *at* ∞, $F_m(x) = A_m(x) + B_m x$, for $\|x\|$ *greater than some* α, *then there exists*

$$\lim_{t\to\infty} \kappa\Big(\bigcup_{m\in\mathfrak{M}} F_m(tX_1)\Big)\Big/ t^k = \kappa\Big(\bigcup_{m\in\mathfrak{M}} A_m(X_1)\Big).$$

Indication of Proof. For $t > \alpha$, let

$$\delta(t) = \sup\{\|B_m x\|/\|x\|^k : x \in X,\ \|x\| \geq t,\ m \in \mathfrak{M}\}.$$

Then $\delta(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Also for each $m \in \mathfrak{M}$, if $S_1 = \{x: \|x\| = 1\}$, then $F_m(tS_1) \subset A_m(tS_1) + B_m(tS_1)$ and $A_m(tS_1) \subset F_m(tS_1) + B_m(tS_1)$. Hence

$$t^{-k}F_m(tS_1) \subset A_m(S_1) + \delta(t)X_1 \quad \text{and} \quad A_m(S_1) \subset t^{-k}F_m(tS_1) + \delta(t)X_1.$$

The remainder of the proof is as in Theorem 4.1.

Corollaries analogous to those for Theorem 4.1 can be given.

REFERENCES

1. P. M. Anselone, *Collectively compact operator approximations*, Prentice-Hall, Englewood Cliffs, N. J., 1971.

2. J. Daneš, *Generalized concentrative mappings and their fixed points*, Comment. Math. Univ. Carolinae 11 (1970), 115–135. MR 41 #7668.

3. ———, *Some fixed point theorems in metric and Banach spaces*, Comment. Math. Univ. Carolinae 12 (1971), 37–51.

4. G. Darbo, *Punti uniti in trasformazioni a codiminio noncompatto*, Rend. Sem. Mat. Univ. Padova 24 (1955), 84–92. MR 16, 1140.

5. L. S. Gol′dešteĭn and A. S. Markus, *On the measure of non-compactness of bounded sets and of linear operators*, Studies in Algebra and Math. Anal., Izdat. "Karta Moldovenjaske", Kishinev, 1965, pp. 45–54. (Russian) MR 35 #789.

6. A. Granas, *Über eine Klasse nichtlinearer Abbildungen in Banachschen Raumen*, Bull. Acad. Polon. Sci. Cl. III 5 (1957), 867–871. (Russian) MR 19, 968.

7. M. A. Krasnosel′skiĭ, *Topological methods in the theory of nonlinear integral equations*, GITTL, Moscow, 1956; English transl., Macmillan, New York, 1964. MR 20 #3464; MR 28 #2414.

8. K. Kuratowski, *Topologie*. Vol. 1, PWN, Warsaw, 1958; English transl., Academic Press, New York; PWN, Warsaw, 1966. MR 19, 873; MR 36 #839.

9. V. B. Melamed and A. I. Perov, *A generalization of a theorem of M. A. Krasnosel′skiĭ on the complete continuity of the Fréchet derivative of a completely continuous operator*, Sibirsk. Mat. Ž. 4 (1963), 702–704. (Russian) MR 28 #476.

10. R. H. Moore, *Differentiability and convergence for compact nonlinear operators*, J. Math. Anal. Appl. 16 (1966), 65–72. MR 33 #4736.

11. M. Z. Nashed, *Differentiability and related properties of nonlinear operators: Some aspects of the role of differentials in nonlinear functional analysis*, Nonlinear Functional Analysis and Applications, L. B. Rall (editor), Academic Press, New York, 1971, pp. 103–309.

12. M. Z. Nashed and J. S. W. Wong, *Some variants of a fixed point theorem of Krasnoselskii and applications to nonlinear integral equations*, J. Math. Mech. 18 (1969), 767–777. MR 38 #6416.

13. R. D. Nussbaum, *Estimates for the number of solutions of operator equations*, Applicable Anal. 1 (1971), 183–200.

14. ———, *The fixed point index and asymptotic fixed point theorems for k-set-contractions*, Bull. Amer. Math. Soc. 75 (1969), 490–495. MR 39 #7589.

15. W. V. Petryshyn, *Structure of the fixed points sets of k-set-contractions*, Arch. Rational Mech. Anal. 40 (1970/71), 312–328. MR 42 #8358.

16. B. N. Sadovskiĭ, *On measures of noncompactness and concentrative operators*, Problemy Mat. Anal. Slož. Sistem. 2 (1968), 89–119.

17. P. P. Zabreĭko and A. I. Povolockiĭ, *Eigenvectors for the Hammerstein operators*, Dokl. Akad. Nauk SSSR 183 (1968), 758–761 = Soviet Math. Dokl. 9 (1968), 1439–1442. MR 39 #833.

18. ———, *Existence and uniqueness theorems for solutions of the Hammerstein equations*, Dokl. Akad. Nauk SSSR 176 (1967), 759–762 = Soviet Math. Dokl. 8 (1967), 1178–1182. MR 36 #4293.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MILWAUKEE, WISCONSIN 53201 (Current address of R. H. Moore)

MATHEMATICS RESEARCH CENTER, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

Current address (M. Z. Nashed): School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332