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THE ŠILOV BOUNDARY OF $M_0(G)$

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ABSTRACT. Let G be a locally compact abelian group and let $M_0(G)$ be the convolution algebra consisting of those Radon measures on G whose Fourier-Stieltjes transforms vanish at infinity. It is shown that the Šilov boundary of $M_0(G)$ is a proper subset of the maximal ideal space of $M_0(G)$. The measures constructed to prove this theorem are also used to obtain a stronger result for the full measure algebra $M(G)$.

1. **Introduction.** Let G be a nondiscrete locally compact abelian group and $\hat{G}$ its dual group. $M(G)$ is the convolution algebra of bounded complex-valued regular Borel measures on G and $M_0(G)$ is the subalgebra of $M(G)$ (in fact, $M_0(G)$ is an ideal) consisting of those measures whose Fourier-Stieltjes transforms vanish at infinity on $\hat{G}$. We shall prove:

Theorem 1. *The Šilov boundary $\partial M_0(G)$ of $M_0(G)$ is a proper subset of the maximal ideal space of $M_0(G)$.*

The corresponding result for $M(G)$ was proved by Johnson [7]. Our methods are similar to those of that paper, except in one important respect. To see how the proofs differ, let us recall the basic idea of Johnson's proof. This will also serve to introduce some notation.

The maximal ideal space Δ of $M(G)$ can be regarded as a topological subspace of the product $\prod_{\mu \in M(G)} L^\infty(\mu)$ where each factor has the $\sigma(L^\infty(\mu), L^1(\mu))$-topology (see [9]). Thus, associated to each $f \in \Delta$ there is a 'generalized character' $(f_\mu)_{\mu \in M(G)}$. For each $\mu \in M(G)$ let $\Delta(\mu) = \{f_\mu : f \in \Delta\}$. Johnson showed the existence of a measure μ having the following properties:

(a) $\Delta(\mu) = \{a\gamma : |a| \leq 1, \gamma \in \hat{G}\}$;

(b) $C = \mathrm{cl}\{a\gamma : |a| = 1 \text{ or } 0, \gamma \in \hat{G}\} \neq \Delta(\mu)$.

Since the set C contains all f_μ for $f \in \partial M(G)$, we see immediately that $\partial M(G)$ is a proper subset of Δ. It is obviously crucial that there exists in $\Delta(\mu)$ a nonzero constant function $\mathbf{a}$ with absolute value less than 1. In proving this, Johnson makes explicit use of the fact that the Fourier-Stieltjes transform of μ does not vanish at infinity. Of course, the measure which we shall construct to satisfy (a) and (b) must belong to $M_0(G)$. To circumvent the problem which arises, we use a result of Brown and Moran [4], which in turn depends on Taylor's recent work on critical points in measure algebras [11].

In §2 we reduce the proof of Theorem 1 to the construction of suitable measures on a small class of groups. This construction is accomplished in §3.

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Professor C. C. Graham has independently obtained Theorem 1 for infinite products of finite abelian groups. A proof of Theorem 1 has been found independently by Dr. Gavin Brown. In fact, he constructs Hermitian measures of the required type, and so is able to deduce that there is a symmetric maximal ideal of $M_0(G)$ which is not in $\partial M_0(G)$. His paper is entitled *$M_0(G)$ has a symmetric maximal ideal off the Šilov boundary*, and is to appear in the Proceedings of the London Mathematical Society.

2. **The general case.** We shall write Δ° for the maximal ideal space of $M_0(G)$. Since $M_0(G)$ is an L-subalgebra (see [10]) of $M(G)$, Šreider's characterization of the maximal ideal space as the set of 'generalized characters' is available. Moreover, a result of Taylor [10] applies to show that $\partial M_0(G)$ is contained in the closure H of all $f \in \Delta^\circ$ having the property that $|f_\mu|^2 = |f_\mu|$ for all $\mu \in M(G)$. For any subset A of Δ°, we shall denote $\{f_\mu : f \in A\}$ by $A(\mu)$.

Lemma 1. *Suppose that there exists a measure $\mu \in M_0(G)$ having the properties*:
P(i) $\Delta^\circ(\mu) \subset \{a\gamma : |a| \leq 1, \gamma \in \hat{G}\}$;
P(ii) *there exists* $\mathbf{a}$ (*the constant function with value a*) *in* $\Delta^\circ(\mu)(0 < |a| < 1)$.
Then $\partial M_0(G) \neq \Delta^\circ$.

Proof. Under the hypotheses, $H(\mu)$ is contained in the closure in the $\sigma(L^\infty(\mu), L^1(\mu))$-topology of the set

$$A = \{a\gamma : |a| = 1, \gamma \in \hat{G}\} \cup \{\mathbf{0}\}.$$

Because μ belongs to $M_0(G)$, any infinite sequence of characters has the constant function $\mathbf{0}$ as a cluster point. It follows that $H(\mu)$ is contained in A. Thus P(ii) shows that $H(\mu) \neq \Delta^\circ(\mu)$.

We remark that P(i) and P(ii) are together equivalent to

$$\Delta^\circ(\mu) = \{a\gamma : |a| \leq 1, \gamma \in \hat{G}\}.$$

The methods of [7] apply here to obtain P(i), whereas P(ii) requires new techniques. For this reason we prefer to separate the two conditions. We shall be largely concerned with establishing P(ii).

Lemma 2. *Let H be a compact subgroup of G and suppose that there is a measure $\mu \in M_0(G/H)$ satisfying* P(i) *and* P(ii). *Then there is a measure ν in $M_0(G)$ which also satisfies* P(i) *and* P(ii).

Proof. For f in $C_0(G)$, we define

$$\nu(f) = \int \left(\int f(x+y)\, dm_H(y) \right) d\mu(x+H),$$

where m_H is the Haar measure of H, so that the inner integral is constant on cosets of H. This defines a measure $\nu \in M(G)$. It is proved in [8] that ν satisfies P(i) provided that μ does.

Let $\pi : G \to G/H$ be the canonical projection and $\pi^* : M(G) \to M(G/H)$ the induced algebra homomorphism. To prove P(ii), we choose $f \in \Delta^\circ_{G/H}$ such that $f_\mu = \mathbf{a}$ $(0 < |a| < 1)$ and consider the algebra homomorphism

$$\lambda \to \pi^*(\lambda)^\wedge\ (f).$$

(Note that $\pi^*(M_0(G)) \subset M_0(G/H)$ [5, p. 480].) Let g denote this homomorphism and fix σ absolutely continuous with respect to ν ($\sigma \geq 0, \|\sigma\| = 1$). Then

$$\hat{\sigma}(g) = \int g_\nu \, d\sigma = \int f_\mu \, d\pi^*\sigma = a$$

because $\pi^*\sigma$ is absolutely continuous with respect to μ. It follows that $g_\nu = \mathbf{a}$.

Finally, we have to prove that $\nu \in M_0(G)$. (This has been tacitly assumed in the preceding argument.) It is enough to note that unless γ $(\in \hat{G})$ is identically equal to 1 on H, $\hat{\nu}(\gamma) = 0$.

Any infinite compact abelian group has a quotient group isomorphic to $\mathbf{T}$ or $\boldsymbol{\Delta}_p$ (the p-adic integers) or an infinite product of finite cyclic groups. We shall explicitly construct measures on these groups satisfying P(i) and P(ii) in §3. Thus, from the above lemma, we obtain Theorem 1 for all infinite compact abelian groups.

Lemma 3. *Let H and K be locally compact abelian groups and suppose that there exists $\mu \in M_0(H)$ satisfying* P(i) *and* P(ii). *Then the same is true of $H \times K$.*

Proof. It suffices to choose a measure $\nu \in L^1(K)$ and consider the product measure $\mu \times \nu$ on $H \times K$. Clearly, $\mu \times \nu \in M_0(H \times K)$ and it is easily checked that P(i) and P(ii) hold for this measure.

Lemma 4. *Let H be an open subgroup of G with a measure $\mu \in M_0(H)$ satisfying* P(i) *and* P(ii). *Then regarded as a measure on G, $\mu \in M_0(G)$ and satisfies* P(i) *and* P(ii).

Proof. That μ satisfies P(i) and P(ii) is clear. Moreover since $(G/H)^\wedge$ is compact we see immediately that $\mu \in M_0(G)$.

We shall discuss the case when $G = \mathbf{R}$ in §3. Since any locally compact abelian group has an open subgroup of the form $\mathbf{R}^n \times K$ where K is a compact group, the proof of Theorem 1 will be completed when we have proved it for the particular groups studied in the next section.

The final result of this section is fundamental to the construction of the measures in these particular cases. All of these measures are infinite convolutions of discrete measures. Using this fact and the results of [4] we obtain a sufficient condition for P(ii) to be satisfied.

Lemma 5. *Let μ be an infinite convolution of discrete measures on the compact abelian group G. Suppose that $\mu \in M_0(G)$ $(\mu > 0, \|\mu\| = 1)$ and satisfies* P(i) *and that μ^n is singular (to Haar measure) for all positive integers n. Then μ also satisfies* P(ii).

Proof. According to Theorem 2 of [4], μ satisfies one of the following two statements:

(a) μ^n is singular to every translate of μ^m unless $n = m$;

(b) there is a finer locally compact group topology τ on G such that $\mu^n \in L^1(G,\tau)$ for some positive integer n.

If (b) holds then evidently $\mu \in M(G,\tau)$. However, $M(G,\tau) \cap M_0(G) = \{0\}$ unless τ is the compact topology (see [5, p. 481]). It follows that if (b) holds, some power of μ belongs to $L^1(G)$, and this contradicts the hypotheses of the lemma. We may assume, therefore, that μ satisfies (a). This being so, the spectrum of μ as a member of $M(G)$ and a fortiori as a member of $M_0(G)$ is the unit disc (see [1, Theorem 2]). (As stated, this theorem requires more conditions on μ. However, the proof only requires that μ satisfies (a).) On the other hand, assuming that P(ii) is not satisfied but P(i) is, the spectrum of μ is

$$\{a\hat{\mu}(\gamma) : |a| = 1, \gamma \in \hat{G}\} \cup \{\mathbf{0}\}.$$

Since $|\hat{\mu}(\gamma)| \geq \frac{1}{2}$ for only finitely many members of $\hat{G}$ the spectrum of μ cannot be the unit disc in this case, and we have obtained the required contradiction.

3. **The construction of measures in** $M_0(G)$. Although the methods of constructing these measures are similar in each of the cases, there are sufficient differences to unduly complicate any attempt at a unified treatment. For this reason we shall present separate discussions of each of the cases, at least until we have established that $\mu \in M_0(G)$ and μ^n is singular to $L^1(G)$ $(n = 1,2,3,\ldots)$.

I. $G = \prod_{n=1}^{\infty} \mathbf{Z}(k_n)$. Let e_n be a generator of $\mathbf{Z}(k_n)$ considered as a subgroup of G, and let, for $0 < \zeta < \frac{1}{2}$,

$$\sigma_n(\zeta) = (k_n^{-1} + \zeta)\delta(0) + \sum_{j=1}^{k_n-1} (k_n^{-1} - \zeta(k_n - 1)^{-1})\delta(je_n).$$

Thus $\sigma_n(\zeta)$ is a discrete measure on $\mathbf{Z}(k_n) \subset G$. Now, for any decreasing sequence $\boldsymbol{\zeta} = (\zeta_n)$ of positive real numbers $(\zeta_n < \frac{1}{2})$, we define

$$\lambda(n,m;\boldsymbol{\zeta}) = \mathop{\ast}_{r=n+1}^{m} \sigma_r(\zeta_r) \qquad (n = 0,1,2,\ldots; m = 1,2,\ldots,\infty)$$

and

$$\lambda(\boldsymbol{\zeta}) = \mathop{\ast}_{r=1}^{\infty} \sigma_r(\zeta_r) = \lambda(0,\infty,\boldsymbol{\zeta}).$$

If $\gamma \in \hat{G}$ and is not identically equal to 1 on $\mathbf{Z}(k_n)$, $\sigma_n(\zeta)^\wedge(\gamma) = \zeta k_n(k_n - 1)^{-1}$. Using this, it is easy to see that $\lambda(\boldsymbol{\zeta}) \in M_0(G)$ if and only if $\boldsymbol{\zeta}$ is a null sequence. Of course, if $\lambda(\boldsymbol{\zeta})$ does not belong to $M_0(G)$, then all of its powers are singular to $L^1(G)$ (see [4]). Let

$$F_n = \left\{(x_n) \in \prod_{n=1}^{\infty} \mathbf{Z}(k_n) : x_1 = x_2 = x_3 = \ldots = x_n = 0\right\}$$

and $\mathcal{J}(n)$ be the collection of all subsets of G which are finite unions of translates of F_n. Note that $\bigcup_{n=1}^{\infty} \mathcal{J}(n)$ is a basis for the closed sets of G.

We can now begin the construction of μ. Fix a sequence (α_r) of positive real numbers decreasing to zero ($\alpha_r < \frac{1}{2}$). For each positive integer r, let $(\boldsymbol{\alpha}^{(r)})$ be the constant sequence $(\alpha_r, \alpha_r, \alpha_r, \ldots)$. Then $\lambda(\boldsymbol{\alpha}^{(1)})$ is singular so that there is a closed subset J_1 of G such that

$$\lambda(\boldsymbol{\alpha}^{(r)})(J_1) > \tfrac{1}{2}, \qquad m(J_1) < \tfrac{1}{2}.$$

(Here m denotes the Haar measure of G.) Furthermore, J_1 can be chosen to belong to $\mathcal{J}(n(1))$ for some positive integer $n(1)$. Suppose that $n(1), n(2), \ldots, n(k)$ and $J_1, J_2, \ldots, J_k$ have been defined with $J_i \in \mathcal{J}(n(i))$ and $n(1) < n(2) < \ldots < n(k)$. Consider the measure

$$\nu = \lambda(0, n(1); \boldsymbol{\alpha}^{(1)}) * \lambda(n(1), n(2); \boldsymbol{\alpha}^{(2)}) * \ldots * \lambda(n(k-1), n(k), \boldsymbol{\alpha}^{(k)}) * \lambda(n(k), \infty; \boldsymbol{\alpha}^{(k+1)}).$$

This is of the form $\lambda(\boldsymbol{\xi})$ for some sequence $\boldsymbol{\xi}$ which is not a null sequence. It follows that there exists a closed set J_{k+1} such that

$$\nu^j(J_{k+1}) > 1 - (k+2)^{-1} \qquad (j = 1, 2, \ldots, k),$$

and

$$m(J_{k+1}) < (k+2)^{-1}.$$

Further, we may assume that J_{k+1} belongs to $\mathcal{J}(n(k+1))$ for some integer $n(k+1) > n(k)$. In this way we define inductively a sequence $(n(k))$ of positive integers and a sequence (J_k) of closed subsets of G.

Let $n(0) = 0$,

$$\mu = \mathop{*}_{k=1}^{\infty} \lambda(n(k-1), n(k); \boldsymbol{\alpha}^{(k)}),$$
$$\mu_k = \mathop{*}_{p=1}^{k} \lambda(n(p-1), n(p); \boldsymbol{\alpha}^{(p)}), \quad \text{and}$$
$$\nu_k = \mathop{*}_{p=k+1}^{\infty} \lambda(n(p-1), n(p); \boldsymbol{\alpha}^{(p)}),$$

so that $\mu = \mu_k * \nu_k$ $(k = 1, 2, 3, \ldots)$.

By construction, $\mu = \lambda(\boldsymbol{\xi})$ for a null sequence $\boldsymbol{\xi}$ and so belongs to $M_0(G)$. Suppose that μ^r is not singular. Since $m(J_k) \to 0$ as $k \to \infty$, it will follow that $(\mu^r(J_k))_{k=1,2,\ldots}$ is bounded away from 1. However, $\mu^r(J_k) = \mu_k^r * \nu_k^r(J_k)$ and ν_k^r is concentrated on F_k. Thus

$$\mu_k^r * \nu_k^r(J_k) = \int_{F_k} \mu_k^r(J_k - x)\, d\nu_k^r(x) = \mu_k^r(J_k)$$

and in a similar way,

$$\mu^r(J_k) = \mu_k^r(J_k) = \int_{F_k} \mu_k^r(J_k - x)\, d\lambda(n(k), \infty; \boldsymbol{\alpha}^{(k)})^r. \tag{1}$$

The last expression in (1) is just

$$\lambda(0, n(1); \boldsymbol{\alpha}^{(1)})^r * \lambda(n(1), n(2); \boldsymbol{\alpha}^{(2)})^r * \dots * \lambda(n(k-1), \infty; \boldsymbol{\alpha}^{(k)})^r(J_k),$$

which is greater than $1 - (k+1)^{-1}$. This contradicts our earlier statements concerning μ^r. We have shown that $\mu \in M_0(G)$ and μ^r is singular for all positive integers r.

II. $G = \boldsymbol{\Delta}_p$. We regard G as the space of all sequences (x_n) of integers with $0 \le x_n \le p - 1$ [6, §10]. Let e_n be the sequence with 1 in the nth place and 0's elsewhere. Then e_1 is a topological generator for G and $p^{n-1}e_1 = e_n$. Let

$$\sigma_n(\zeta) = (p^{-1} + \zeta)\delta(0) + \sum_{j=1}^{p-1} (p^{-1} - \zeta(p-1)^{-1})\delta(je_n),$$

for $0 < \zeta < \frac{1}{2}$. As before, if $(\zeta_n) = \boldsymbol{\zeta}$ is a decreasing sequence of such numbers, we define

$$\lambda(n, m; \boldsymbol{\zeta}) = \mathop{\ast}_{k=n+1}^{m} \sigma_k(\zeta_k) \qquad (n = 0, 1, 2, \dots; m = 1, 2, \dots, \infty)$$

and $\lambda(\boldsymbol{\zeta}) = \lambda(0, \infty; \boldsymbol{\zeta})$.

If γ is a continuous character G defined by $\gamma(e_1) = \exp 2\pi i \ell p^{-n}$ where $0 < \ell \le p - 1$, then

$$\sigma_q(\zeta)^\wedge(\gamma) = 1 \qquad (q > n),$$
$$\sigma_n(\zeta)^\wedge(\gamma) = \zeta(1 + (p-1)^{-1})$$

and

$$|\sigma_q(\zeta)^\wedge(\gamma) - 1| \le \pi p^{q-n-1}(p-1)\ell \qquad (q \searrow n).$$

Thus, if $\zeta_r \to 0$ and $n \ge N$,

$$|\lambda(\boldsymbol{\zeta})^\wedge(\gamma)| \le |\sigma_n(\zeta)^\wedge(\gamma)| \le \zeta_N(1 + p^{-1})$$

so that $\lambda(\boldsymbol{\zeta})$ belongs to $M_0(G)$. Conversely if $\zeta_n \ge \epsilon$ for all n and $\gamma_n(e_1) = \exp 2\pi i p^{-n}$,

$$|\lambda(\boldsymbol{\zeta})^\wedge(\gamma_n)| \ge \left(\prod_{q=1}^{n-1} (1 - \pi p^{-q-1}(p-1))\right)\zeta_n(1 + p^{-1})$$

and the right-hand side is bounded away from zero as $n \to \infty$.

Let

$$F_n = \{(x_n) : x_1 = x_2 = \ldots = x_n = 0\} = \mathrm{cl}(gp(e_{n+1}))$$

and $\mathcal{J}(n)$ be the collection of all finite unions of translates of F_n. Precisely the same induction procedure now applies to give a measure $\mu \in M_0(G)$, $\mu = \lambda(\xi)$ where ξ is a null sequence and μ^n is singular to Haar measure ($n = 1, 2, 3, \ldots$). The proof given in case I only requires F_n to be a subgroup on which ν_n and $\lambda(n, \infty; \zeta)$ are concentrated.

III. $G = \mathbf{T} = \mathbf{R}/\mathbf{Z}$. Here let $e_n = 2^{-n}$ and

$$\sigma_n(\zeta) = (\tfrac{1}{2} + \zeta)\delta(0) + (\tfrac{1}{2} - \zeta)\delta(e_n).$$

Let $\lambda(n, m; \zeta)$ and $\lambda(\zeta)$ be defined as in I. It is already known that $\lambda(\zeta)$ belongs to $M_0(G)$ if and only if ζ is a null sequence (see [3]). Let us define $F_n = [0, 2^{-n})$ and $\mathcal{J}(n)$ as the set of all finite unions of translates of F_n by members of the subgroup generated by e_n. Although the sets in $\mathcal{J}(n)$ are not closed, it is clearly possible to carry out the induction to obtain a sequence $(n(k))$ of positive integers and a sequence (J_k) of subsets of G having the same properties with regard to the measures $\lambda(n, m; \zeta)$ as in I. In trying to show that the powers of the constructed measure μ are singular to Haar measure, a more serious difficulty is encountered. This is that F_n is no longer a subgroup of G, so that the argument which justifies the singularity of μ^r in the preceding cases fails to work here. To deal with this problem, we first have to choose J_k with smaller Haar measure. Thus, for a fixed decreasing null sequence (α_r) we define inductively a sequence $(n(k))$ of positive integers, and a sequence (J_k) with $J_k \in \mathcal{J}(n_k)$ such that

$$\lambda(0, n(1); \alpha^{(1)})^r * \lambda(n(1), n(2); \alpha^r * \ldots * \lambda(n(k-2), n(k-1); \alpha^{(k-1)})^r$$

$$* \lambda(n(k-1), \infty; \alpha^{(k)})^r(J_k) > 1 - k^{-1} \qquad (1 \leq r \leq k)$$

and $m(J_k) < k^{-1}(2k+1)^{-1}$. Now let $I_k = \bigcup_{j=-k}^{k} (J_k + je_{n(k)})$, so that $I_k \in \mathcal{J}_{n(k)}$ and

$$m(I_k) \leq \sum_{j=-k}^{k} m(J_k + je_{n(k)}) < k^{-1}.$$

We define μ, μ_k, ν_k as in I. Note that ν_k is concentrated on $F_{n(k)}$ so that ν_k^r is concentrated on $\bigcup_{j=0}^{r} (F_{n(k)} + je_{n(k)})$. In fact, there exist positive real numbers $\beta_0^{(r)}, \beta_1^{(r)}, \ldots, \beta_r^{(r)}$ ($\beta_j^{(r)} = \nu_k^r(F_{n(k)} + je_{n(k)})$) such that $\sum_{j=0}^{r} \beta_j^{(r)} = 1$, and for all $J \in \mathcal{J}(n(k))$

$$\nu_k^r(J) = \left(\sum_{j=0}^{r} \beta_j^{(r)} \delta(je_{n(k)})\right)(J). \tag{2}$$

Similarly, $\lambda(n(k), \infty; \alpha^{(k)})$ is concentrated on $F_{n(k)}$; so that we can find positive real numbers $\eta_0^{(r)}, \eta_1^{(r)}, \ldots, \eta_r^{(r)}$ such that $\sum_{j=0}^{r} \eta_j^{(r)} = 1$ and for all $j \in \mathcal{J}(n(k))$,

$$\lambda(n(k), \infty; \alpha^{(k)})^r(J) = \left(\sum_{j=0}^{r} \eta_j^{(r)} \delta(je_{n(k)})\right)(J). \tag{3}$$

In this case, $\eta_j^{(r)} = \lambda(n(k), \infty; \alpha^{(k)})^r(F_{n(k)} + je_{n(k)})$.

If $0 \leq j, \ell \leq r \leq k$, $J_k - je_{n(k)} \subset I_k - \ell e_{n(k)}$ so that

$$\mu_k^r(I_k - \ell e_{n(k)}) \geq \delta(je_{n(k)}) * \mu_k^r(J_k). \tag{4}$$

From (3), we obtain

$$\left(\sum_{j=0}^{r} \eta_j^{(r)} \delta(je_{n(k)}) * \mu_k^r\right)(J_k) = (\mu_k * \lambda(n(k), \infty; \alpha^{(k)}))^r(J_k) \geq 1 - k^{-1}$$

which, combined with (4), gives

$$\mu_k^r(I_k - \ell e_{n(k)}) \geq 1 - k^{-1}, \tag{5}$$

for $0 \leq \ell \leq r \leq k$. However, from (2) it follows that if $J \in \mathcal{J}(n(k))$,

$$\mu^r(J) = \left(\sum_{j=0}^{r} \beta_j^{(r)} \delta(je_{n(k)}) * \mu_k^r\right)(J)$$

so that, with (5), we have

$$\mu^r(I_k) = \sum_{j=0}^{r} \beta_j^{(r)} \mu_k^r(I_k - je_{n(k)}) \geq 1 - k^{-1}.$$

Since $m(I_k) \leq k^{-1}$ it is clear that all powers of μ are singular.

The property P(i) remains to be proved for the measures μ constructed above. We shall, in large part, rely on Johnson's arguments here. Before proving P(i), it will be convenient to note that if we let, in II, $k_n = p$, and, in III, $k_n = 2$, then in all cases

$$\sigma_n(\zeta) = (k_n^{-1} + \zeta)\delta(0) + \sum_{j=1}^{k_n-1} (k_n^{-1} - \zeta(k_n - 1)^{-1})\delta(je_n).$$

The proof of the following lemma can be abstracted from [7].

Lemma 6. *Let* $\mu = \ast_{n=1}^{\infty} \sigma_n(\zeta_n)$ *and let* H *be the subgroup generated by* $\{e_n : n = 1, 2, \ldots\}$. *Suppose that all characters of* H *which satisfy*

$$\int |b_n \theta(\zeta_n) - b_{n-1}| d\sigma_n(\zeta_n) \to 0 \tag{6}$$

for some sequence (b_n) *of complex numbers of absolute value* 1 *can be extended to continuous characters of* G. *Then* μ *has property* P(i).

Of course, in our situation (ζ_n) is a null sequence; using this, (6) becomes

$$k_n^{-1} \sum_{j=0}^{k_n-1} |\theta(e_n)^j - c_n| \to 0,$$

where $c_n = b_n^{-1} b_{n-1}$.

If (k_n) is a bounded sequence $c_n \to 1$ and hence so does $\theta(e_n)$. This is sufficient to prove (when (k_n) is bounded) that θ can be extended to a continuous character of G (see [7]). If (k_n) is unbounded, then G is an infinite product of finite cyclic groups and so $\theta(e_n)$ is a k_nth root of unity. If $\theta(e_n)$ does not equal 1, then at least half of its powers differ from c_n by $\frac{1}{2}$, so that

$$k_n^{-1} \sum_{j=0}^{k_n-1} |\theta(e_n)^j - c_n| \geq \tfrac{1}{4}.$$

It follows that for sufficiently large n, $\theta(e_n) = 1$ and so θ extends to a continuous character. In view of Lemma 6, μ has property P(i) and so by Lemma 5 and the construction of μ, it is a measure of the required type. This completes the proof of Theorem 1 for compact abelian groups.

It remains to deal with the case of the real line. Let μ be the measure constructed in III, but now regarded as belonging to $M(\mathbf{R})$. It is well known that μ must belong to $M_0(\mathbf{R})$ [2, p. 388]. Let $\pi : \mathbf{R} \to \mathbf{T}$ be the natural epimorphism and $\pi^* : M(\mathbf{R}) \to M(\mathbf{T})$ the induced algebra homomorphism. Then $\pi^*(\mu)$ is the measure of III. The argument given in Lemma 2 now applies to show that μ has property P(ii). That μ also satisfies P(i) follows by an application of Lemma 6.

4. The Šilov boundary of $M(G)$. It is not difficult to adapt the above arguments to give an alternative proof of Johnson's theorem on the Šilov boundary of $M(G)$. In fact, we can obtain a stronger statement which is not obtainable using the measures considered in [7]. In order to describe this we shall need to examine some of the properties of the maximal ideal space Δ of $M(G)$. There is a natural semigroup structure on Δ which derives from the usual multiplication of members of $L^\infty(\mu)$. Also if f belongs to Δ, then $|f|$ defined by $|f|_\mu = |f_\mu|$ $(\mu \in M(G))$ belongs to Δ, as does $\bar{f}$ $((\bar{f})_\mu = \bar{f}_\mu$ $(\mu \in M(G))$. Further, if $f \in \Delta$, we can write $f = |f|h$ where $h \in \Delta$ and $|h|^2 = |h|$ (see [10]). Taylor has also noted that $|f|^z \in \Delta$ for $\mathrm{Re}(z) > 0$. For any subset S of Δ, we write $T(S)$ for the smallest closed subsemigroup of Δ which contains S and is such that if $f \in T(S)$, then $\bar{f}$, h (as defined above) and $|f|^z$ ($\mathrm{Re}\, z > 0$) also belong to $T(S)$.

Theorem 2. *$T(\partial M(G))$ is a proper subset of Δ.*

Proof. Since $M_0(G)$ is an L-ideal every element of $\Delta^\circ(\mu)$ is also in $\Delta(\mu)$. Furthermore, the arguments of the preceding sections show that

$$\Delta(\mu) \subset \{a\gamma : |a| \leq 1, \gamma \in \hat{G}\}.$$

Thus $\Delta(\mu) = \Delta^\circ(\mu) \cup \{\mathbf{0}\}$. However,

$$\partial M(G)(\mu) \subset \{a\gamma : |a| = 1, \gamma \in \hat{G}\} \cup \{\mathbf{0}\}.$$

This last set is invariant under the various operations described above so that

$$T(\partial M(G))(\mu) \subset \{a\gamma : |a| = 1, \gamma \in \hat{G}\} \cup \{\mathbf{0}\}$$

and the result is proved.

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