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EXTENSIONS OF FUNCTIONS AND SPACES

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ABSTRACT. We investigate, for a given map φ from a topological space X to a topological space Y (denoted by $[X, \varphi, Y]$), those triples $[E, \Phi, Y]$ where E is an extension of X and Φ extends φ to E. A maximal such extension, similar to the Katětov extension of a topological space, is examined.

Introduction. An extension of a topological space X is a pair (E, ψ) where E is a topological space, ψ is a homeomorphism of X into E and $\psi(X)$ is dense in E. Let (E, ψ) be an extension of X and let φ denote a (continuous) map from X into a space Y. A map $\Phi : E \to Y$ is, by definition, an extension of φ if $\varphi(x) = \Phi\psi(x)$ for every $x \in X$. We shall investigate for given $\varphi : X \to Y$ (henceforth denoted by $[X, \varphi, Y]$) simultaneous extensions of both X and φ, that is triples $[(E, \psi), \Phi, Y]$ where (E, ψ) extends X and Φ extends φ. Such a triple $[(E, \psi), \Phi, Y]$ will be called an extension of $[X, \varphi, Y]$. The study of extensions of a space X is the study of extension of $[X, P, \{p\}]$ where $\{p\}$ is the one point space and P is the map from X to $\{p\}$. Since, for arbitrary $[X, \varphi, Y]$ we can at best extend φ to $\overline{\varphi(X)}$, we shall assume throughout this discussion that $\varphi(X)$ *is dense in* Y. We further make the assumption that all spaces are Hausdorff.

1. Extensions.

Definition 1.1. Two extensions $[(E, \psi), \Phi, Y]$, $[(E', \psi'), \Phi', Y]$ of $[X, \varphi, Y]$ are *equivalent* ($=$) if there exists a homeomorphism $\theta : E \to E'$ such that

(i) $\Phi'\theta = \Phi$ (on E) and

(ii) $\theta\psi = \psi'$ (on X).

Let X be a space and $x \in X$. $\mathscr{N}(x)$ will denote the open neighborhood system of x. We use a well-established technique (cf. [2], [4]) of constructing extensions of topological spaces to select a representative from each equivalence class of extensions of $[X, \varphi, Y]$. Let $[(E, \psi), \Phi, Y]$ extend $[X, \varphi, Y]$. Topologize $E' = X \cup \{\psi^{-1}(\mathscr{N}(e)) \mid e \in E \setminus \psi(X)\}$ so that the bijection $x \to \psi(x)$ for $x \in X$, and $\psi^{-1}(\mathscr{N}(e)) \to e$ is a homeomorphism. We denote the extension (E', i) of X, where i is the inclusion map on X, simply by E'. Extend φ to a map Φ' on E' by defining $\Phi'[\psi^{-1}(\mathscr{N}(e))] = \Phi(e)$. Then, $[E', \Phi', Y] = [(E, \psi), \Phi, Y]$. We shall therefore assume, unless otherwise stated, that an extension of $[X, \varphi, Y]$ is a triple $[E, \Phi, Y]$ where E contains X, $E \setminus X$ consists of open filters on X, and $\xi \in E \setminus X$ implies

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that ξ equals the trace of the open neighborhoods of ξ on X.

In the case that Y is regular, we can characterize all extensions of $[X, \varphi, Y]$ in the following way: Let $\mathscr{M} = \{\varphi^{-1}\mathscr{N}(y) \mid y \in Y\}$. Let $E = X \cup \{\xi_\alpha\}_{\alpha \in A}$ where ξ_α is a free open filter in X containing an element of $\mathscr{M}$ for each $\alpha \in A$, and given distinct elements, α and α', of A then there exist $O_\alpha \in \xi_\alpha$ and $O'_\alpha \in \xi_{\alpha'}$ with $O_\alpha \cap O'_\alpha = \varnothing$. Extend φ to a function Φ on E by defining $\Phi(\xi_\alpha) = y$ where $\varphi^{-1}\mathscr{N}(y) \subset \xi_\alpha$, for $\alpha \in A$. There are two "natural topologies" one can impose on E (cf. [2]) for which $[E, \Phi, Y]$ becomes an extension of $[X, \varphi, Y]$. These are the strict and the simple extension topologies, where the *strict topology*, τ_0, is generated by the sets $O \cup \{\xi_\alpha \mid O \in \xi_\alpha\}$ for O open in X, and the *simple topology*, τ_1, is generated by the open sets in X union the sets $\{\xi_\alpha\} \cup O$ where $O \in \xi_\alpha$, for $\alpha \in A$. Hence, for any topology τ on E with $\tau_0 \leq \tau \leq \tau_1$ we have that $[E, \Phi, Y]$ extends $[X, \varphi, Y]$. One can easily show that these extensions represent up to equivalence all possible extensions of $[X, \varphi, Y]$.

Definition 1.2. A space (X, τ) is semiregular if $\{\overline{O}^o \mid O \in \tau\}$ is a base for τ.

Theorem 1.1. *Let* $[X, \varphi, Y]$ *be given. Then there exists an extension* $[E, \Phi, Y]$ *with* $\Phi(E) = Y$. *In the case that* Y *is semiregular, the extension may be chosen so that* $E \setminus X$ *is homeomorphic to* $Y \setminus \varphi(X)$.

Proof. In the general case, let $E = X \cup \{\varphi^{-1}\mathscr{N}(y) \mid y \in Y \setminus \varphi(X)\}$ with simple topology and define $\Phi(\varphi^{-1}\mathscr{N}(y)) = y$. In the case that Y is semiregular, choose the same set E and function Φ used in the general case and give E the strict topology. Clearly Φ is surjective and a bijection from $E \setminus X$ to $Y \setminus \varphi(X)$. Φ is continuous: Let W be an open neighborhood of $y_0 \in Y \setminus \varphi(X)$ and $V \subset W$ a neighborhood of y_0 with $V = \overline{V^o}$. Let $O = \varphi^{-1}(V) \cup \{\xi \in E \setminus X \mid \varphi^{-1}(V) \in \xi\}$. Clearly $y_0 \in \Phi(O)$. Suppose $\xi \in O$ with $\xi = \varphi^{-1}\mathscr{N}(y)$, $y \notin W$. Since $V = \overline{V^o}$ we have that $\overline{V^o} \cap U \neq \varnothing$ for each $U \in \mathscr{N}(y)$. It then follows, since $\varphi(X)$ is dense in Y, that $\varphi^{-1}(V) \notin \varphi^{-1}\mathscr{N}(y)$, contradicting the fact that $\xi \in O$. Hence, $\Phi(O) \subset W$ and Φ is continuous. Let Φ_R denote the restriction of the map Φ to $E \setminus X$. We complete the proof of the theorem by showing that Φ_R^{-1} is continuous. Let $\varphi^{-1}\mathscr{N}(y_0)$ with basic open neighborhood $U = \{\xi \in E \setminus X \mid \varphi^{-1}(V) \in \xi\}$ in $E \setminus X$ be given, for $V \in \mathscr{N}(y_0)$. Since $V \in \mathscr{N}(y)$ for each $y \in V \cap [Y \setminus \varphi(X)]$ we have $\Phi_R^{-1}(V \cap [Y \setminus \varphi(X)]) \subset U$.

Definition 1.3. (a) A space X is *absolutely closed* if there exists no proper extension of X.

(b) Let $[X, \varphi, Y]$ be given. X is *φ-absolutely closed* if there exists no proper extension of $[X, \varphi, Y]$.

(c) Let $[X, \varphi, Y]$ be given. An open filter, ξ, in X is *φ-convergent* if the filter $\varphi(\xi)$ converges in Y.

An immediate consequence of Theorem 1.1 is that a proper extension exists for a triple $[X, \varphi, Y]$ if φ is not onto. In general, we cannot expect the existence of a proper extension for a triple $[X, \varphi, Y]$; certainly no such extension will exist in the case that X is absolutely closed.

Theorem 1.2. *Let $[X, \varphi, Y]$ be given. X is φ-absolutely closed if and only if every φ-convergent filter in X has nonempty adherent set.*

Proof. Let ξ be a φ-convergent filter with empty adherent set φ-converging to y. Let $E = X \cup \{\xi\}$ with X retaining its topology and with a neighborhood system of the point ξ consisting of all sets of the form $\{\xi\} \cup O$ where $O \in \xi$. Extending φ to Φ on E by defining $\Phi(\xi) = y$ we have that $[E, \Phi, Y]$ extends $[X, \varphi, Y]$. Hence X is not φ-absolutely closed.

Conversely, suppose $[E, \Phi, Y]$ is a proper extension. Let $\xi \in E \setminus X$. Since E is Hausdorff, ξ must be an open filter with no adherent point. Since Φ is continuous, ξ must be φ-convergent.

The following example shows that X may be φ-absolutely closed without being absolutely closed for given $[X, \varphi, Y]$. In the case that Y is compact however, the properties absolutely closed and φ-absolutely closed are equivalent, as is shown in Corollary 1.1 below.

Example 1.1. Let Z denote the open interval $(0,1)$ with the usual topology. Let Y denote the same interval with topology generated by the usual open subsets and the rationals. Let i denote the identity map from Y to Z. Clearly Y is not absolutely closed. It is however i-absolutely closed since any open filter in Y containing $i^{-1}\mathscr{N}(r)$ must have $\{r\}$ as adherent set, for any $r \in (0, 1)$.

Corollary 1.1. *Let $[X, \varphi, Y]$ be given with Y compact. Then X is φ-absolutely closed if and only if X is absolutely closed.*

Proof. We need only prove the necessary part of the theorem. To do this we use the fact that a space X is absolutely closed if (and only if) every maximal open filter in X has nonempty adherent set [1]. Let ξ denote a maximal open filter in X. Suppose ξ is not φ-convergent. Then for each $y \in Y$ there exists an open neighborhood O_y with $\varphi^{-1}(O_y) \notin \xi$. Let $Y = \bigcup_{i=1}^{n} O_{y_i}$. Since ξ is maximal and since $\{\varphi^{-1}(O_{y_i})\}_{i=1}^{n}$ covers X, some $\varphi^{-1}(O_{y_i}) \in \xi$, $1 \leq i \leq n$, a contradiction. Therefore ξ is φ-convergent and, by Theorem 1.2, ξ has nonempty adherent set.

Clearly if $\varphi : X \to Y$ is a homeomorphism then X is φ-absolutely closed. The converse, as shown by Example 1.1, does not in general hold. However, the converse is valid in the case that X is semiregular and φ is injective.

Corollary 1.2. *Let X be semiregular and $\varphi : X \to Y$ be injective. Then X is φ-absolutely closed if and only if φ is a homeomorphism.*

Proof. We need only prove the necessary part of the theorem.

Suppose φ is not a homeomorphism. Then there exists a $y \in Y$ such that $\varphi^{-1}\mathscr{N}(y)$ is not a neighborhood base for any $x \in X$. If φ is not onto then X is not φ-absolutely closed by Theorem 1.1. If φ is onto then let x be such that $\varphi(x) = y$. Choose a neighborhood $O = \overline{O}^o$ of x which contains no element of $\varphi^{-1}\mathscr{N}(y)$. Then, $\varphi^{-1}\mathscr{N}(y) \cup \overline{O}^c$ is a φ-convergent filter with empty adherent set so that X is not φ-absolutely closed.

Definition 1.4. A map $\varphi : X \to Y$ is *perfect* if it is a continuous closed surjection with $\varphi^{-1}(y)$ compact for each $y \in Y$.

Theorem 1.3. *Let $[X, \varphi, Y]$ be given with X regular. Then, φ is a perfect map if and only if X is φ-absolutely closed.*

Proof. It is well known that if φ is perfect then every φ-convergent filter has nonempty adherent set, cf. [3, p. 254]. We show, in the case that X is regular, that the converse is also valid. Let X be φ-absolutely closed. By Theorem 1.1, φ is surjective. Suppose there exists a $y \in Y$ such that the fibre $\varphi^{-1}(y)$ is not compact. Then there exists, by regularity, an open cover $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$ of $\varphi^{-1}(y)$ such that the closure of any finite union of elements of $\mathcal{O}$ fails to contain $\varphi^{-1}(y)$. We may assume $\mathcal{O}$ to be closed under finite union. Then, $\varphi^{-1}\mathcal{N}(y) \cup \{\overline{O}_\alpha^c\}_{\alpha \in A}$ is a free filter, since any adherent point of a filter containing $\varphi^{-1}\mathcal{N}(y)$ must be in $\varphi^{-1}(y)$. The filter is therefore free and φ-convergent contradicting the assumption that X is φ-absolutely closed. Finally, suppose φ were not closed. Choose a closed subset C of X and an element y such that $y \in \overline{\varphi(C)} \setminus C$. Since $\varphi^{-1}(y)$ is compact and X is regular, we may choose an open set O with $\varphi^{-1}(y) \subset O$ and $C \cap \overline{O} = \varnothing$. $\varphi^{-1}\mathcal{N}(y) \cup \{\overline{O}^c\}$ is a free φ-convergent filter, contradicting the assumption that X is φ-absolutely closed.

In general, if φ is perfect then no extension exists since no φ-convergent (or indeed φ-adherent) filter is free. In the event that X is not regular however, X may be φ-absolutely closed with φ not closed (see Example 1.1). Also, if X is not regular, X may be φ-absolutely closed with $\varphi^{-1}(y)$ not compact for some $y \in Y$. To see this one need only consider the triple $[X, \varphi, \{y\}]$ where X is absolutely closed but not compact (see Corollary 1.1).

Definition 1.5. (a) A subset S of a space X is *X-absolutely closed* if for any cover $\mathcal{O}$ of S by sets open in X there exists a finite number of elements in $\mathcal{O}$, say $O_1, \ldots, O_n$ with $S \subset \mathrm{Cl}_X \bigcup_{i=1}^n O_i$.

(b) A map $\varphi : X \to Y$ is *H-perfect* if it is a continuous surjection with $\varphi^{-1}(y)$ X-absolutely closed for each $y \in Y$.

One may easily show that $S \subset X$ is X-absolutely closed if and only if S is closed in any extension of X.

Applying part of the proof of the previous theorem one obtains the following theorem.

Theorem 1.4. *Let $[X, \varphi, Y]$ be given (X not necessarily regular). If X is φ-absolutely closed then φ is H-perfect.*

The converse of the above theorem is false. To see this, let $\varphi : X \to Y$ be a continuous surjection with $\varphi^{-1}(y)$ compact for each $y \in Y$. Assume further that X is regular and φ is not closed. Clearly φ is H-perfect. However, by Theorem 1.3, X is not φ-absolutely closed.

Question. Let $[X, \varphi, Y]$ be given. Let φ be H-perfect and closed. Is X φ-absolutely closed?

Theorem 1.5. *Let $[X, \varphi, Y]$ and $[Y, \gamma, Z]$ be given. If X is $(\gamma\varphi)$-absolutely closed then X must be φ-absolutely closed and Y γ-absolutely closed.*

Proof. Suppose X is not φ-absolutely closed. Then there exists $y \in Y$ and a free filter ξ on X which contains $\varphi^{-1}\mathscr{N}(y)$. Since ξ then contains $(\gamma\varphi)^{-1}\mathscr{N}(\gamma(y))$ we have that X is not $(\gamma\varphi)$-absolutely closed.

Suppose Y is not γ-absolutely closed. Then there exists $z \in Z$ and a free filter ξ on Y which contains $\gamma^{-1}\mathscr{N}(z)$. Since $\varphi^{-1}(\xi)$ is a free filter on X containing $(\gamma\varphi)^{-1}\mathscr{N}(z)$ we have that X is not $(\gamma\varphi)$-absolutely closed.

In the case that X and Y are regular the above theorem reduces to the well-known result that if $\gamma\varphi$ is a perfect map then both φ and γ must be perfect maps. The following example shows that even if both X is φ-absolutely closed and Y is γ-absolutely closed then X need not be $(\gamma\varphi)$-absolutely closed. In such an example, both X and Y could not be regular since a composite of perfect maps is perfect.

Example 1.2. Let P denote the plane with topology generated by the standard topology of the plane and the set of rational points in the plane. Let X be the subspace $\{(x, 0) | x \in (0, 1)\} \cup \{(x, 1) | x \in (0, 1), x \text{ is irrational}\}$. Let $[Y, i, Z]$ be as in Example 1.1. Let π denote the projection map from X onto Y. As noted in Example 1.1, Y is i-absolutely closed. Since for any $y \in Y$, any open filter in X containing $\pi^{-1}\mathscr{N}(y)$ must contain $(y, 0)$ in its adherent set we have that X is π-absolutely closed. X is not $(i\pi)$-absolutely closed, for adjoining to the filter $(i\pi)^{-1}\mathscr{N}(\frac{1}{2})$ the open subset $\{(x, 1) \mid x > \frac{1}{2}\}$ of X we obtain an $(i\pi)$-convergent filter with no adherent point.

A consequence of the previous theorem is that for a given triple $[X, \varphi, \prod_{\alpha \in A} Y_\alpha]$ if X is $(\pi_{\alpha_0}\varphi)$-absolutely closed for some $\alpha_0 \in A$, then X is φ-absolutely closed. The following example shows that the converse does not hold.

Example 1.3. Let Z denote the set of integers with discrete topology. Let i denote the identity map on $Z \times Z$. By Corollary 1.2, $Z \times Z$ is i-absolutely closed. $Z \times Z$ is not $(\pi_1 i)$-absolutely closed. For let $\mathscr{G}$ denote the set of finite complements of $\{(n_0, m) \mid m \in Z\}$. Then $\mathscr{G}$ is a free open filter in $Z \times Z$ $(\pi_1 i)$-converging to n_0. Similarly $Z \times Z$ may be shown not to be $(\pi_2 i)$-absolutely closed.

Theorem 1.6. *Let $[X, \varphi, Y]$ be given and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for the space Y. Denote by φ_α the restriction of the map φ to the subspace $\varphi^{-1}(U_\alpha)$ of X onto the subspace U_α of Y. Then, X is φ-absolutely closed if and only if $\varphi^{-1}(U_\alpha)$ is φ_α-absolutely closed for each $\alpha \in A$.*

Proof. Suppose $\varphi^{-1}(U_\alpha)$ is not φ_α-absolutely closed. Let ξ be a free open filter on $\varphi^{-1}(U_\alpha)$ which φ_α-converges to y. Clearly $\xi_0 = \xi \cup \varphi^{-1}\mathscr{N}(y)$ φ-converges to y. If x is adherent to ξ_0 in X then $\varphi(x) = y$; therefore $x \in \varphi^{-1}(U_\alpha)$, and x is adherent to ξ in $\varphi^{-1}(U_\alpha)$. It follows that ξ_0 is free. Hence X is not φ-absolutely closed.

Conversely, suppose X is not φ-absolutely closed. Let ξ be a free open filter in

X which φ-converges to $y \in U_\alpha$. $\xi \cap \varphi^{-1}(U_\alpha)$ is then a free open filter on $\varphi^{-1}(U_\alpha)$ which φ_α-converges to y. Hence $\varphi^{-1}(U_\alpha)$ is not φ_α-absolutely closed.

In the case that each X_α is regular, the following theorem is, by Theorem 1.3, the well-known fact that $\prod_{\alpha \in A} \varphi_\alpha$ is a perfect map if and only if each φ_α is a perfect map. In the case that each Y_α is a single point, the following theorem reduces to the well-known fact that a product space is absolutely closed if and only if each factor is absolutely closed [7].

Theorem 1.7. *Let* $\{[X_\alpha, \varphi_\alpha, Y_\alpha]\}_{\alpha \in A}$ *be given. Then* $\prod_\alpha X_\alpha$ *is* $\prod_\alpha \varphi_\alpha$*-absolutely closed if and only if each* X_α *is* φ_α*-absolutely closed.*

Proof. Suppose each X_α is φ_α-absolutely closed. Let $y = \{y_\alpha\}_{\alpha \in A} \in \prod_\alpha Y_\alpha$. Let ξ be any open filter in $\prod_\alpha X_\alpha$ which is $\prod_\alpha \varphi_\alpha$-convergent to y. Then, for each $\alpha \in A$, $\pi_\alpha(\xi)$ is a φ_α-convergent open filter in X_α and therefore contains an adherent point, say x_α. $\{x_\alpha\}$ is then an adherent point of ξ.

Conversely, let X_{α_0} not be φ_{α_0}-absolutely closed. Then there exists a free open filter, ξ_{α_0}, with $\varphi_{\alpha_0}(\xi_{\alpha_0})$ converging to $y_{\alpha_0} \in Y_{\alpha_0}$. Let $p = \{p_\alpha\}$ be any point in $\prod_{\alpha \neq \alpha_0} Y_\alpha$ and let ξ denote the open filter in $\prod_\alpha X_\alpha$ generated by the sets

$$\left\{F_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} Y_\alpha \,\middle|\, F_{\alpha_0} \in \xi\right\}$$

$$\cup \bigcup_{\alpha \neq \alpha_0} \left\{\varphi_\alpha^{-1}(U) \times \prod_{\beta \in A; \beta \neq \alpha} Y_\beta \,\middle|\, U \text{ is an open neighborhood of } p_\alpha\right\}.$$

Then $\prod_\alpha \varphi_\alpha(\xi)$ converges to $y_{\alpha_0} \times p$ and is free.

2. **Maximal extensions.**

Definition 2.1. Let $[(E, \psi), \Phi, Y]$ and $[(E', \psi'), \Phi', Y]$ be extensions of $[X, \varphi, Y]$. $[(E, \psi), \Phi, Y]$ is *not less than* ($\geq$) $[(E', \psi'), \Phi', Y]$ if there exists a continuous surjection θ from a subspace S of E containing $\psi(X)$ to E' such that

(i) $\Phi'\theta = \Phi$ (on S) and

(ii) $\theta\psi = \psi'$ (on X).

Considering two equivalent extensions as being equal we have that the relation $\geq$ is a partial order on the set of extensions of $[X, \varphi, Y]$. That the collection is a set follows from our restriction of extensions of X to Hausdorff extensions. That the relation is a preorder is immediate. To see that the relation is a partial order one need only apply the following lemma.

Lemma. *Let X be dense in the spaces E and E'. Let S and S' be subspaces of E and E' containing X. Let θ be a map from S onto E' which leaves X fixed and let θ' be a map from S' onto E which leaves X fixed. Then θ is a homeomorphism between E and E'.*

Proof. We need only show $S = E$ and $S' = E'$. Let $x \in E$. Choose $y \in S'$ such that $\theta'(y) = x$. Choose $s \in S$ with $\theta(s) = y$. The hypotheses of the lemma imply $x = s$.

Question. Is the set of extensions of $[X, \varphi, Y]$ a lattice, and if so, a complete lattice?

Clearly for given $[X, \varphi, Y]$ there is, up to equivalence, a smallest extension, namely $[X, \varphi, Y]$. For a given space X, Katětov [5] has constructed a maximal absolutely closed extension $K[X] = X \cup \mathscr{N}$ where $\mathscr{N}$ denotes the set of maximal free open filters, and the topology on $K[X]$ is the simple topology. We now generalize this construction.

Let $[X, \varphi, Y]$ be given. Let $K(\varphi) = X \cup \mathscr{M}$ with simple topology, where $\mathscr{M}$ denotes the set of maximal φ-convergent open filters with empty adherent set. Extending φ to a map φ^* on $K(\varphi)$ by defining $\varphi^*(\xi) = y$ where $\varphi^{-1}\mathscr{N}(y) \subset \xi$ we have that $[K(\varphi), \varphi^*, Y]$ is an extension of $[X, \varphi, Y]$. The fact that $K(\varphi)$ is Hausdorff follows from the maximality of the elements of $\mathscr{M}$ and from the fact that elements of $\mathscr{M}$ have empty adherent set. We show in the following theorem that $[K(\varphi), \varphi^*, Y]$ is greater than or equal to any extension of $[X, \varphi, Y]$. Since the set of extensions is partially ordered we have that $[K(\varphi), \varphi^*, Y]$ is characterized, up to equivalence, as the greatest extension.

Theorem 2.1. *Let* $[E, \Phi, Y]$ *be any extension of* $[X, \varphi, Y]$. *Then* $[K(\varphi), \varphi^*, Y] \geq [E, \Phi, Y]$.

Proof. Let $E = X \cup \mathscr{G}$ where $\mathscr{G}$ consists of φ-convergent free filters. For $\xi \in \mathscr{G}$, let $\mathscr{M}(\xi) = \{\alpha \in K(\varphi) \mid \xi \subset \alpha\}$. Define θ on $X \cup \bigcup_{\xi \in \mathscr{G}} \mathscr{M}(\xi)$ by $\theta(x) = x$ for $x \in X$ and $\theta(\alpha) = \xi$ where $\xi \subset \alpha$ for $\alpha \in \bigcup_{\xi \in \mathscr{G}} \mathscr{M}(\xi)$. θ is well defined since, by the Hausdorff property of E, an element of $\bigcup_{\xi \in \mathscr{G}} \mathscr{M}(\xi)$ cannot contain two elements of $\mathscr{G}$. The continuity of θ follows from the fact that $K(\varphi)$ is a simple extension. Clearly θ is a surjection and satisfies properties (i) and (ii) of Definition 2.1.

Let φ be a map on a space X and let E be a subspace of X. The restriction of φ to E will be denoted by φ_E.

Theorem 2.2. *Let* $[X, \varphi, Y]$ *be given and* $X \subset E \subset K(\varphi)$. *Then* $[K(\varphi_E^*), (\varphi_E^*)^*, Y] = [K(\varphi), \varphi^*, Y]$, *where* $[K(\varphi), \varphi^*, Y]$ *is here considered as an extension of* $[E, \varphi_E^*, Y]$. *In particular, we have* $[K(\varphi), \varphi^*, Y] = [K(\varphi^*), \varphi^{**}, Y]$ *so that* $K(\varphi)$ *is* φ^**-absolutely closed.*

Proof. Any extension $[Z, \gamma, Y]$ of $[E, \varphi_E^*, Y]$ is also an extension of $[X, \varphi, Y]$. For the map $\theta : S \to Z$ showing that $[K(\varphi), \varphi^*, Y] \geq [Z, \gamma, Y]$ in the proof of 2.1 one has $E \subset S$, and θ maps E identically. Hence $[K(\varphi), \varphi^*, Y]$ is also the greatest extension of $[E, \varphi_E^*, Y]$.

Remark. Let $[X, \varphi, Y]$ and $[Y, \gamma, Z]$ be given. Since $\xi \in K(\varphi) \setminus X$ implies ξ is a maximal free open filter on X which contains $\varphi^{-1}\mathscr{N}(y)$ for some $y \in Y$, then ξ contains $(\gamma\varphi)^{-1}\mathscr{N}(\gamma(y))$, so that $\xi \in K(\gamma\varphi)$. Hence, $K(\varphi) \subset K(\gamma\varphi)$. The surjection $\varphi^* : K(\varphi) \to Y$ will be denoted by φ_γ when we consider it as a map from $K(\varphi)$ into $K(\gamma)$. Though $K(\varphi)$ is φ^*-absolutely closed, it need not be φ_γ-absolutely closed; indeed, by Theorem 1.1, $K(\varphi)$ is φ_γ-absolutely closed if and only if Y is γ-absolutely closed. By definition, $[K(\gamma\varphi), (\gamma\varphi)^*, Z]$ is the K-absolute closure of $[X, \gamma\varphi, Z]$. By the previous theorem it is also the K-absolute closure of $[K(\varphi), \gamma\varphi^*, Z]$ and of $[K(\varphi), \gamma^*\varphi_\gamma, Z]$. We show it also to be the K-absolute closure of $[K(\varphi_\gamma), \gamma^*\varphi_\gamma^*, Z]$ by showing this last triple to be equivalent to an extension $[E, (\gamma\varphi)_E^*, Z]$ of $[K(\varphi), \gamma\varphi^*, Z]$ for $K(\varphi) \subset E \subset K(\gamma\varphi)$ and then applying the previous theorem. Let $E = K(\varphi) \cup \{\text{trace on } X \text{ of } \xi \text{ for } \xi \in K(\varphi_1) \setminus K(\varphi)\}$. Clearly $E \subset K(\gamma\varphi)$ and one can show that the map θ which leaves the points of $K(\varphi)$ fixed and takes $\xi \in K(\varphi_1) \setminus K(\varphi)$ into the trace on X of ξ satisfies the conditions of Definition 1.1.

We have established the following commutative diagram, where i denotes the inclusion map. It is complete in the sense that the K-absolute closure of any triple in the diagram is represented in the diagram.

$$
\begin{tikzcd}
K(\gamma\phi) \arrow[rrrddd, "(\gamma\phi)^*"] & & & \\
E \arrow[u, "i"] \arrow[r, "\theta"] & K(\phi_\gamma) \arrow[rd, "\phi_\gamma^*"] & & \\
K(\phi) \arrow[u, "i"] \arrow[ru, "i"] \arrow[rr, "\phi_\gamma"] \arrow[rrd, "\phi^*"] & & K(\gamma) \arrow[rd, "\gamma^*"] & \\
X \arrow[u, "i"] \arrow[rr, "\phi"] & & Y \arrow[u, "i"] \arrow[r, "\gamma"] & Z
\end{tikzcd}
$$

Let $\{[X_\alpha, \varphi_\alpha, Y_\alpha]\}_{\alpha \in A}$ be given and consider $[\prod_\alpha X_\alpha, \prod_\alpha \varphi_\alpha, \prod_\alpha Y_\alpha]$. If each X_α is φ_α-absolutely closed then, by Theorem 1.7, we have $K(\prod_\alpha \varphi_\alpha) = \prod_\alpha K(\varphi_\alpha) = \prod_\alpha X_\alpha$. In general, equality does not hold. For example, in the case that each Y_α is a single point then $K(\varphi_\alpha) = K(X_\alpha) =$ the Katětov absolute closure of X_α and $K(\prod_\alpha \varphi_\alpha) = K(\prod_\alpha X_\alpha) =$ the Katětov absolute closure of $\prod_{\alpha \in A} X_\alpha$. The following theorem gives necessary and sufficient conditions for $\prod_\alpha K(\varphi_\alpha) = K(\prod_\alpha \varphi_\alpha)$ in this special case.

Theorem 2.3 (Liu [6]). *Let X_α be nonempty spaces for $\alpha \in A$. Then $K(\prod_\alpha X_\alpha) = \prod_\alpha K(X_\alpha)$ iff at least one of the following two conditions is satisfied.*

(a) *X_α is absolutely closed for each $\alpha \in A$.*

(b) *There exists* X_{α_0} *which is not absolutely closed.* X_α *is finite for all* $\alpha \neq \alpha_0$. *Moreover, all but finitely many* X_α*' s have only one point.*

Using arguments similar to those appearing in [6] one may generalize the above theorem to the following result .

Theorem 2.4. *Let* X_α *be nonempty spaces for* $\alpha \in A$. *Then* $K(\prod_\alpha \varphi_\alpha) = \prod_\alpha K(\varphi_\alpha)$ *iff at least one of the following two conditions are satisfied .*

(a) X_α *is* φ_α*-absolutely closed for each* $\alpha \in A$.

(b) *There exists* X_{α_0} *which is not* φ_{α_0}*-absolutely closed.* X_α *is finite for all* $\alpha \neq \alpha_0$. *Moreover, all but finitely many* X_α*'s have only one point.*

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