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FUNDAMENTAL THEORY OF CONTINGENT DIFFERENTIAL EQUATIONS IN BANACH SPACE

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ABSTRACT. For a contingent differential equation that takes values in the closed, convex, nonempty subsets of a Banach space E, we prove an existence theorem and we investigate the extendability of solutions and the closedness and continuity properties of solution funnels. We consider first a space E that is separable and reflexive and then a space E with a separable second dual space. We also consider the special case of a point-valued or ordinary differential equation.

0. **Introduction.** Consider the contingent differential equation

$$Dx \subset F(t,x) \tag{1}$$

where F maps $R \times E$ into the closed, convex, nonempty subsets of E, E a Banach space. A solution to (1) is a function φ mapping some interval I into E such that if $D\varphi(t)$ is the contingent derivative of φ, then $D\varphi(t) \subset F(t,\varphi(t))$ on I. In this paper we prove an existence theorem for the initial value problem associated with (1); we discuss the extendability of solutions and the closedness and continuity properties of solution funnels; and we investigate the initial value problem associated with (1) in the special case where F is point-valued, i.e. when (1) is any ordinary differential equation.

In §1 we state basic definitions, we state the conditions to be placed on F in the hypotheses of the existence theorem, and we give a characterization of solutions.

In particular, if $(t_0,x_0) \in R \times E$ is our initial point and N is a neighborhood of (t_0,x_0), then we assume that for all $(t,x) \in N$, $F(t,x)$ lies in a fixed bounded set and $F(t,x)$ is upper semicontinuous in a certain sense (stated in condition A).

Condition A is interesting in that it extends to Banach spaces Cesari's condition Q [2] and it is similar to Marchaud's concept of regularity [11] and Zaremba's idea of upper semicontinuity [17]. And in the case F is point-valued, condition A reduces to weak continuity.

In §2 we prove our existence theorem in the case of E a reflexive and separable space. In this case and under the above mentioned hypotheses we show that the initial value problem associated with (1) has a solution $\varphi(t)$. Further, the

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Ważewski result [15] holds: The strong derivative of φ, φ', exists and $\varphi'(t) \in F(t, \varphi(t))$ a.e. (strong or weak refers to limits in the strong or weak topology on E).

Also, we discuss the extendability of solutions (using ideas of Corduneanu [5]), we prove that funnels of solutions are closed, and we discuss the continuity properties of solution funnels.

In §3 we investigate the initial value problem for E a general Banach space. Consider the ordinary differential equation.

$$x' = f(t, x) \tag{2}$$

where $f: R \times E \to E$.

It was shown by example in [6] and [16] that if we only assume f is continuous, then the initial value problem for (2) need not necessarily have a solution. (In the special case of $E = E^n$, the continuity of f does, of course, imply the existence of such a solution.)

Subsequently, existence theorems for the initial value problem for (2) were proved where, in addition to the assumption of the continuity of f, it was assumed:

$f = f_1 + f_2$ where f_1 is completely continuous and f_2 satisfies a Lipschitz condition [9];

f is uniformly continuous and its range lies in a compact set [5]; and

f satisfies a Kamke-type condition [13].

In [1] the idea of a weak solution (i.e. a strongly continuous function whose weak derivative satisfies (1)) was used, and it was shown that if f is weakly continuous and bounded and if E is reflexive and separable then there exists a solution to the initial value problem.

In §3 we prove the following: Let E be embedded in its second dual space E^{**}, which is assumed to be separable, and let E^{**} with the weak star topology be denoted by $E^{**}_{w^*}$. If, in a neighborhood of (t_0, x_0), f can be extended so that $f: R \times E^{**} \to E^{**}_{w^*}$ is continuous and f is bounded in the strong norm, then there is a function $\varphi: (t_0 - \delta, t_0 + \delta) \to E^{**}$ with $\varphi(t_0) = x_0$ which is strongly continuous and whose weak* derivative satisfies (2). If, additionally, E is reflexive and its dual space is separable, then φ has a strong derivative which satisfies (2) a.e.

In §3 we also investigate (1) when E is a general Banach space. We show that F can be defined as a mapping from $R \times E^{**}$ into the closed, convex, nonempty subsets of E^{**} in such a way that condition A holds. Then assuming the range of F lies in a fixed bounded set we prove that (1) has a solution $\varphi: R \to E^{**}$. When E is reflexive and E^* separable we are back to the setting of §2.

Some of these results were presented in [4]. An existence theorem for ordinary differential equations under similar conditions is contained in [3].

1. **Definitions and basic theorems.** Let E be a real Banach space with norm $\|\cdot\|$. Denote E, when equipped with the weak topology, by E_w and denote the dual space of E by E^*.

Let W be an open connected set in $R \times E$. Points in W are denoted by P, (t_P, x_P), or just (t, x). For $P, Q \in W$, $\|P - Q\| = \max(|t_P - t_Q|, \|x_P - x_Q\|)$.

For $A \subset E$, $\overline{\mathrm{co}}\, A$ is the closure of the convex hull of A. And $\mathrm{cf}(E)$ is the collection of all nonempty, convex, closed subsets of E.

Definition 1. A function $f : W \to \mathrm{cf}(E)$ is said to satisfy *condition* A if there exists a countable set $\mathfrak{I} = \{F_n \in E^*\}$ such that, at each $P_0 \in W$,

$$f(P_0) = \bigcap_n Q_n[f(P_0)],$$

where

$$Q_n[f(P_0)] = \overline{\mathrm{co}} \bigcup \{f(P) : |t_P - t_0| < 1/n, |F_i(x_P - x_0)| < 1/n, \quad i = 1, 2, \ldots, n\}.$$

If $\mathfrak{I}$ is dense in E^*, the following hold.

(1) For $E = R^n$ condition A is equivalent to Cesari's condition Q [2]. (This is also called semicontinuity in the sense of Cesari [10].)

(2) For $E = R^n$ and $f(P)$ is point-valued at each P, condition A is equivalent to the continuity of f.

(3) If $f(P)$ is point-valued at each P and if we consider $W \subset R \times E_w$ and $f : W \to E_w$, then condition A is equivalent to the continuity of f. (We show this at the end of the section.)

If $\mathfrak{I}$ is not dense in E^*, but "smaller", then the set of solutions of our contingent equation will be larger.

Condition A yields directly the properties needed in $f(P)$ for an existence theorem and it avoids examining a topology on $\mathrm{cf}(E)$.

Definition 2. A set $A \subset W$ is an *α-set* (Corduneanu [5]) if A is bounded and if the $\inf\{\|P - Q\| : P \in A, Q \in \mathrm{Bdy}(W)\} > 0$.

Definition 3. A mapping $f : W \to \mathrm{cf}(E)$ is said to satisfy *condition* B if for each α-set $A \subset W$ there exists a constant m such that $\|f(P)\| \leq m$ on A.

Definition 4. If $\{x_n\}$ is a sequence in E, $x_n \to x$ *weakly* means that for every F in E^*, $F[x_n] \to F[x]$.

If $\varphi : I \to E$ (I an open interval in R), $\varphi(t)$ is *weakly continuous* means that for every F in E^*, $F[\varphi(t)]$ is a continuous function of t.

When we say that a property holds *nearly everywhere* on I we mean that it holds everywhere except, possibly, at a denumerable number of points.

Definition 5. Let $\Delta_h \varphi(t) = (\varphi(t + h) - \varphi(t))h^{-1}$ and let

$$D\varphi(t) = \{y \in E : \Delta_{h(n)} \varphi(t) \to y \text{ weakly for some sequence } h(n) \to 0+\}.$$

A *contingent differential equation* is any expression of the form

$$Dx \subset f(t, x) \tag{1}$$

where $f : W \to \mathrm{cf}(E)$.

A *solution of* (1) on I is a continuous function $\varphi : I \to E$ such that $\varnothing \neq D\varphi(t) \subset F(t, \varphi(t))$ nearly everywhere on I (i.e. except, possibly, at a denumerable number of points).

Theorem 1. *Let $\varphi : I \to E$ be continuous and assume that $D\varphi(t) \neq \varnothing$ nearly everywhere on I. Then φ is a solution of* (1) *on I if and only if for $t \in I$ and $m > 0$ there exists an $\eta(t,m) > 0$ such that*

$$0 < h < \eta \Rightarrow \Delta_h \varphi(t) \in Q_m[f(t, \varphi(t))].$$

The following is used in the proof of Theorem 1.

Theorem 2. *If A is a closed convex set in E, if $\psi : (a,b) \to E$ is continuous, and if there exist sequences $\{y_n \in A\}$, $\{h(n) \to 0+\}$ such that $[\Delta_{h(n)} \psi(t) - y_n] \to 0$ weakly, nearly everywhere on (a,b), then*

$$\psi(t_2) - \psi(t_1)/(t_2 - t_1) \in A \quad \text{for } t_1, t_2 \in (a,b),\ t_1 \neq t_2.$$

A proof of Theorem 2 when $E = E^n$ may be found in Zaremba [17], but for general Banach spaces we refer to Mlak [12].

Proof of Theorem 1. Let $\varphi(t)$, a solution of (1), and $t \in I$, $m > 0$ be given. Choose $\eta > 0$ such that $|s - t| < \eta$ implies

$$\sup\{|s - t| : |F_i[\varphi(s) - \varphi(t)]|\} < 1/m \quad \text{for } F_i \in \mathcal{F},\ i = 1, 2, \ldots, m.$$

Then $D\varphi(s) \subset F(s, \varphi(s)) \subset Q_m[f(t, \varphi(t))]$ nearly everywhere on $|s - t| < \eta$ implies $\Delta_h \varphi(t) \in Q_m[f(t, \varphi(t))]$ for $0 < h < \eta$.

If $\Delta_h \varphi(t) \in Q_m[f(t, \varphi(t))]$ for $t \in I$, $h \in (0, \eta)$, then $D\varphi(t) \subset Q_m[f(t, \varphi(t))]$. But m is arbitrary so

$$D\varphi(t) \subset \bigcap_{n=1}^{\infty} Q_n[f(t, \varphi(t))] = f(t, \varphi(t)).$$

Corollary 1. *In Theorem* 2, *and hence in Theorem* 1, *we may replace the phrase "$\varphi(t)$ is continuous and the stated conditions hold nearly everywhere" by "$\varphi(t)$ is absolutely continuous and the stated conditions hold almost everywhere" and the two theorems are again true.*

Corollary 2. *If $\{\varphi_n(t)\}$ is an equicontinuous sequence of solutions of* (1) *and if $\varphi_n(t) \to \varphi(t)$ weakly on I, then $\varphi(t)$ is a solution of* (1).

Proof. First, for $\epsilon > 0$ there exists a δ such that $|h| < \delta$ implies $\|\varphi_n(t + h) - \varphi_n(t)\| < \epsilon$ for all n and $t \in I$. Then the weak limit lies in the same sphere, i.e., $\|\varphi(t + h) - \varphi(t)\| < \epsilon$, and $\varphi(t)$ is continuous.

Second, in the theorem we may choose $\eta(t,m)$ such that $0 < h < \eta$ implies $\Delta_h \varphi_n(t) \in Q_m[f(t, \varphi(t))]$ for all n sufficiently large.

Third, $Q_m[f(t, \varphi(t))]$ is convex and closed, hence weakly closed, so $\Delta_h \varphi(t) \in Q_m[f(t, \varphi(t))]$.

Addenda. (1) Relation of condition A to continuity: Assume that $f(P)$ is point-valued at each $P \in W$ and consider $W \subset R \times E_w$. We shall show:

(i) f satisfies condition A $\Rightarrow f : W \to E_w$ is continuous;

(ii) $f : W \to E_w$ is continuous, $m = \sup\{\|P\| : P \in W\} < \infty$, and ϑ is dense in $E^* \Rightarrow f$ satisfies condition A.

Since $f(P_0) = \bigcap_{n=1}^{\infty} Q_n[f(P_0)]$ and $Q_1[f(P_0)] \supset Q_2[f(P_0)] \supset \dots$, (i) follows.

Suppose $x_0 \neq f(P_0)$. There exists a $G_0 \in E^*$ such that $G_0(x_0 - f(P_0)) \geq \eta$ and by the continuity of f there is a weak neighborhood

$$A = \bigcap_{i=1}^{M} \{Q : |t_Q - t_0| < \delta, |G_i(x_Q - x_0)| < \delta\},$$

$G_i \in E^*$, of P_0 such that $f(A) \subset \{x : G_0(x - f(P_0)) < \eta\}$.

Choose N such that $N > 2/\delta$ and such that for each $i \in [1, M]$ there is a $j \in [1, N]$ with $\|F_j - G_i\| < \delta/4m$ where $m = \sup\{\|P\| : P \in W\}$.

Then $|F_j(x_Q - x_0)| < 1/N$ ($j \in [1, N]$) implies $|G_i(x_Q - x_0)| \leq |F_j(x_Q - x_0)| + \|F_i - G_i\| \, \|x_O - x_0\| < \delta$ for $i \in [1, M]$ and hence

$$Q_N[f(P_0)] \subset \{x : G_0(x - f(P_0)) < \eta\}.$$

(2) A simple example to emphasize the difference between strong and weak continuity:

Let $E = l^2$ and let e_n be the element of E with one in the nth place and zeros elsewhere. Define g by

$$\begin{aligned} g(t) &= 0, \qquad t \leq 0, \\ &= e_1, \qquad t \geq 1, \\ &= [1/n - 1/(n+1)]^{-1}\{e_n \cdot [t - 1/(n+1)] + e_{n+1} \cdot [1/n - t]\}, \\ &\qquad\qquad t \in [1/(n+1), 1/n]. \end{aligned}$$

Then $g(1/n) = e_n$ and $g(t)$ is continuous in the weak topology. But $g(t)$ is not continuous in the strong topology at $t = 0$.

Now define $f : E \to E$ by $f(x) = g(\langle e_1, x \rangle)$, where $\langle \cdot , \cdot \rangle$ denotes the inner product in E. Thus f is continuous in the weak topology but not in the strong topology.

2. **Existence theorem and fundamental properties of solutions.** In this section we assume, additionally, that E is separable and reflexive.

Theorem 3. *Let* (1) *be given and assume that* $f(t, x)$ *satisfies conditions* A *and* B. *Then for* $(t_0, x_0) \in W$ *there exists an interval* I *containing* t_0 *and a solution* $\varphi(t)$ *of* (1) *on* I *such that* $\varphi(t_0) = x_0$. *Further,* $\varphi'(t)$ (*the strong limit of* $\Delta_h \varphi(t)$ *as* $h \to 0$) *exists and* $\varphi'(t) \in f(t, \varphi(t))$ *almost everywhere on* I.

Proof. Choose an α-set $A \subset W$ such that $P_0 = (t_0, x_0)$ is in the interior of A and let m be the constant given by condition B. Choose $a > t_0$ such that

$$R = \{(t,x) : |t - t_0| \leq a - t_0, \|x - x_0\| \leq m(a - t_0)\} \subset A.$$

With no loss of generality assume $t_0 \leq t \leq a \leq t_0 + 1$.

Form a partition Δ_n of $[t_0, a] : t_0 < t_1 < \ldots < t_n = a$ with $(t_i - t_{i-1}) \leq 1/n$ $(i = 1, \ldots, n)$.

Define the polygonal line

$$\varphi_n(t_0) = x_0,$$

$$\varphi_n(t) = \varphi_n(t_{i-1}) + (t - t_{i-1})v_{i-1} \qquad \text{if } t_{i-1} < t \leq t_i \ (i = 1, \ldots, n)$$

where $v_{i-1} \in f(t_{i-1}, \varphi_n(t_{i-1}))$ and $|v_{i-1}| \leq m$.

Since $\{\varphi_n(t)\}$ is a uniformly bounded, equicontinuous family, a subsequence $\{\varphi_k(t)\}$ converges weakly, uniformly on $[t_0, a]$, to a function $\varphi(t)$ on $[t_0, a]$. Then $\varphi(t) \in E$ for each t, $\varphi(t_0) = x_0$, and as in the proof of Theorem 1, Corollary 2, $\varphi(t)$ is continuous.

Let $m > 0$, $t_1 \in (t_0, a)$ be given. We claim there exists $N(m, t_1)$, $\eta(m, t_1) > 0$ such that $\varphi_n'(t) \in Q_m[f(t_1, \varphi(t_1))]$ for $|t - t_1| < \eta$, $n \geq N$ ($'$ is here the right derivative in the strong sense). It will then follow from Theorem 2 that

$$(\varphi_n(t_3) - \varphi_n(t_2))(t_3 - t_2)^{-1} \in Q_m[f(t_1, \varphi(t_1))]$$

for $n \geq N$, $|t_i - t_1| < \eta$ $(i = 2, 3)$, hence by the convexity of Q_m that

$$\Delta_h \varphi(t_1) \in Q_m[f(t_1, \varphi(t_1))],$$

and thus, by Theorem 1, that $\varphi(t)$ is a solution of (1) on $[t_0, a]$.

To prove the claim assume $1/m < \min(t_1 - t_0, a - t_1)$ and choose $\eta \in (0, 1/2m)$ such that $|t - t_1| \leq 2\eta$ implies $|F_i[\varphi_n(t) - \varphi_n(t_1)]| < 1/2m$ for every n and $i = 1, \ldots, m$.

Choose $N > 2/\eta$ such that $n \geq N$ implies $|F_i[\varphi_n(t_1) - \varphi(t_1)]| < 1/2m$, $i = 1, \ldots, M$. Then $|t - t_1| \leq \eta$, $n \geq N$, implies $\varphi_n'(t) = v_j \in f(t_j, \varphi_n(t_j))$ for t_j a point of the subdivision Δ_n and further $|t - t_j| < \eta/2$. Then $|t_1 - t_j| < 2\eta$ so

$$|F_i[\varphi_n(t_1) - \varphi_n(t_j)]| < 1/2m \qquad (i = 1, \ldots, m),$$

$$\max\{|t_j - t_1|, |F_i[\varphi(t_j) - \varphi(t_1)]| \ (i = 1, \ldots, m)\} < 1/m,$$

and $\varphi_n'(t) \in Q_m[f(t_1, \varphi(t_1))]$.

To see that $\varphi'(t)$ exists a.e. and hence that $\varphi'(t) \in D\varphi(t) \subset f(t, \varphi(t))$ a.e. on I we employ a theorem of Pettis [14] which states: For a reflexive space E, a function of bounded variation $\psi : I \to E$ is strongly differentiable a.e. and its derivative is integrable in the sense of Bochner.

From the definition of $\varphi_n(t)$, $\|\varphi_n(t_2) - \varphi_n(t_1)\| \leq m|t_2 - t_1|$ $(t_2, t_1 \in I)$. Hence, $\|\varphi(t_2) - \varphi(t_1)\| \leq m|t_2 - t_1|$ and φ is of bounded variation.

Remark. In Theorem 3 we may weaken condition B as follows: For each α-set $A \subset W$ there exists a constant $m > 0$ such that $f(P) \cap \{x : \|x\| \leq m\} \neq \varnothing$ on A.

We now state some fundamental properties of solutions of (1). We sketch only a few proofs since they involve standard techniques from the theory of ordinary differential equations.

Definition 6. Let $\varphi(t)$ be a solution of (1) passing through (t_0, x_0). Let $(\alpha_\varphi, \omega_\varphi)$ be the domain of φ and $\Gamma_\varphi^+ = \{(t, \varphi(t)) : t_0 \leq t < \omega_\varphi\}$. Then $\psi(t)$ is a *right extension* of $\varphi(t)$, and $\varphi(t)$ is *extendable to the right* if (i) ψ is a solution of (1) passing through (t_0, x_0); and (ii) $\omega_\varphi < \omega_\psi$, $\varphi(t) = \psi(t)$ on $[t_0, \omega_\varphi)$. If φ is not extendable to the right, then φ is *fully extended to the right* .

Theorem 4. *The solution* $\varphi(t)$ *of* (1) *is extendable to the right if and only if* Γ_φ^+ *is an* α*-set. Further, each solution of* (1) *which is not fully extended to the right has a right extension which is fully extended to the right.*

Proof. See Corduneanu [5].

We may similarly discuss left extensions and extensions of solutions.

Definition 7. For $P \in W$, let $\Phi(P)$ be the family of all solutions of (1) passing through P. If all members of $\Phi(P)$ are defined on $[\gamma, \delta]$, then $Z(P; \gamma, \delta)$ or simply $Z(P) = \{(t, \varphi(t)) : \gamma \leq t \leq \delta, \varphi \in \Phi(P)\}$.

For $A \subset W$, assume all members of $\Phi(P)$ are defined on $[\gamma, \delta]$ for each $P \in A$. Then $Z(A) = \bigcup \{Z(P) : P \in A\}$.

Theorem 5. *If* $A \subset W$ *is closed and bounded and if all solutions of* (1) *through any point of* A *are defined on* $[\gamma, \delta]$, *then* $Z(A)$ *is an* α*-set and is closed.*

Proof. We sketch the proof. First assume A is a point P. Using weak compactness we can extend the proof to sets.

If $\{\varphi_n(t, P)\}$ is a sequence of solutions of (1) passing through P whose graphs lie in an α-set B for $t \in I \subset [\gamma, \delta]$, then $\{\varphi_n(t, P)\}$ is a uniformly bounded, equicontinuous family. Hence by Corollary 2 of Theorem 1 some subsequence of $\{\varphi_n(t, P)\}$ converges weakly to a solution of (1) on I.

Also, such an α-set B always exists, viz., the usual small rectangle with center at P.

Now if $Z(P)$ is not an α-set, then there exists a sequence $Q_n = (t_n, \varphi_n(t_n, P)) \in Z(P)$ such that $t_n \to t_0 \in [\gamma, \delta]$ and either $\|\varphi_n(t_n, P)\| \to \infty$ (as a limit) or $Q_n \to \mathrm{Bdy}(W)$ as $t_n \to t_0$. By standard methods we may find a subsequence $\varphi_k(t, P) \to \varphi(t, P)$ weakly where $\varphi(t, P)$ is a solution of (1) through P which cannot be defined at $t = t_0$. This gives a contradiction.

If $Z(P)$ is an α-set, then $\|f(t, x)\| \leq m$ on $Z(P)$ and $\Phi(P)$, the family of solutions of (1) passing through P, is equicontinuous. The closedness of $Z(P)$ again follows from Theorem 1, Corollary 2.

Theorem 6. *Let $\{A_n\}$ be a sequence of closed bounded sets in W with $A_1 \supset A_2 \supset \ldots$ and assume that all solutions of* (1) *passing through any point of A_1 are defined on* $[\gamma, \delta]$. *Then* $\bigcap_{n=1}^{\infty} Z(A_n) = Z(\bigcap_{n=1}^{\infty} A_n)$.

Proof. Suppose $Q \in \bigcap_{n=1}^{\infty} Z(A_n)$. Then $Q = (t_Q, \varphi_n(t_Q, P_n))$ where φ_n is a solution of (1) and $P_n \in A_n$. Now $\{\varphi_n(t, P_n)\}$ is an equicontinuous family since $Z(A_1)$ is an α-set. So some subsequence of $\{\varphi_n(t, P_n)\}$ converges weakly, uniformly on $[\gamma, \delta]$, to $\varphi(t_0, P_0)$. Then $P_0 \in \bigcap_{n=1}^{\infty} A_n$ and $Q = (t_Q, \varphi(t_Q, P_0))$.

The other half of the proof is immediate.

Specific continuity properties of solutions follow from Theorem 6. For example:

Corollary 1. *Let $\|P_n - P_0\| \to 0$ as $n \to \infty$ ($P_n, P_0 \in W$) and let $\varphi_n(t)$ be a fully extended solution of* (1) *through P_n with (α_n, ω_n) as its domain of definition, $n = 0, 1, \ldots$. Then*

$$\limsup \alpha_n \leq \alpha_0 < \omega_0 \leq \liminf \omega_n.$$

Corollary 2. *For $P \in W$ let $Z_t(P) = Z(P) \cap \{(s, x) : s = t, x \in E\}$. Let $\|P_n - P_0\| \to 0$ as $n \to \infty$ ($P_n, P_0 \in W$) and assume $Z_t(P_0) \neq \varnothing$. Then given $\epsilon > 0$, $F \in E^*$ with $F(x) \leq 0$ for all $x \in Z_t(P_0)$, there exists an N such that $n \geq N$ implies $F(y) \leq \epsilon$ for all $y \in Z_t(P_n)$.*

3. **Differential equations in a nonreflexive Banach space.** In this section we again consider the initial value problem. We drop the requirement that E be reflexive but we do require that the second dual space of E be separable. Hence the dual space of E and E itself are separable. First we make some remarks on ordinary differential equations.

Consider the initial value problem

$$x' = f(t, x), \qquad x(0) = 0, \tag{2}$$

where $f : R \times E \to E$ is point-valued.

As the example of Dieudonné [6] shows, even if f is uniformly continuous a solution of (1) may not exist in E. But one does exist in E^{**}, the second dual space of E, and in fact this is also true in general.

Let E be embedded in E^{**}. On E^{**} we shall use both the norm $\|\cdot\|$ topology and the weak* topology, i.e. the topology induced by the functionals in E^* considered as a subset of E^{***}. We shall denote E^{**}, when equipped with the weak* topology, by E^{**}_w.

Let $I = [-1, 1]$, $W = \{(t, x) \in R \times E^{**} : |t| \leq 1, \|x\| \leq 1\}$, and $\mathcal{B} = \{x(\cdot) : I \to E^{**} : x(t) \text{ is continuous}\}$, $\|x(\cdot)\| = \sup_I \|x(t)\|$. Since E^{**} is separable, $\mathcal{B}$ is separable.

For $\epsilon > 0$, $F \in E^*$, $x(\cdot) \in \mathcal{B}$, let $N_{\epsilon,F}[x(\cdot)] = \{y(\cdot) \in \mathcal{B} : |F[y(t) - x(t)]| < \epsilon \text{ on } I\}$. Using these sets and all finite intersections of these sets we have a base for a weak topology on $\mathcal{B}$ and we denote $\mathcal{B}$ with this topology by $\mathcal{B}_w$. Since $\mathcal{B}$ is separable, $\mathcal{B}_w$ is separable and since E^* is separable, $\mathcal{B}_w$ satisfies the second axiom of countability. Hence, in $\mathcal{B}_w$, sequential compactness will imply compactness.

Let

$$\mathcal{E} = \{x(\cdot) \in \mathcal{B} : x(0) = 0, \|x(\cdot)\| \leq 1 \text{ and } \|x(t) - x(s)\| \leq |t - s|\}.$$

If $\{x_n(\cdot)\}$ is a sequence in $\mathcal{E}$, then $\{x_n(\cdot)\}$ is uniformly bounded. And since the unit sphere in E^{**} is weak* compact we may apply Ascoli's theorem to obtain a subsequence $\{x_k(\cdot)\}$ and an $x_0(\cdot)$ such that $F[x_k(t) - x_0(t)] \to 0$ for every $F \in E^*$ uniformly in t.

Now $x_k(t) \to x_0(t)$ in E^{**}_w implies $\|x_0(t)\| \leq \underline{\lim}_n \|x_n(t)\|$. Hence $\|x_0(\cdot)\| \leq 1$, $\|x_0(t) - x_0(s)\| \leq |t - s|$, and $x_0(0) = 0$ so $x_0(\cdot) \in \mathcal{E}$ and $\mathcal{E}$ is a compact set in $\mathcal{B}_w$.

Assume that $f : W \cap (R \times E) \to E$ can be extended to a function $f : W \to E^{**}$ such that:

(i) $\|f(t,x)\|$ is bounded on W. We denote the bound by $\|f\|$ and for simplicity we assume $\|f\| \leq 1$.

(ii) $f : W_w \to E^{**}_w$ is continuous and it is uniformly continuous in x. By W_w we mean W with the $R \times E_w$ topology and by uniform continuity in x we mean that given $\epsilon > 0$, $F \in E^*$, there exists a weak* neighborhood $M(x)$ such that, for all $\|x\| \leq 1$, $y \in M(x)$ implies $|F[f(t,y) - f(t,x)]| < \epsilon$ for all $t \in I$.

Define $T : E^* \times I \times \mathcal{E} \to R$ by

$$T(F, t, x(\cdot)) = \int_0^t F[f(s, x(s))]\, ds.$$

Then (i) T is linear in F and $|T(F, t, x(\cdot))| \leq \|F\|\, |t|\, \|f\|$;

(ii) $|T(F, t_2, x(\cdot)) - T(F, t_1, x(\cdot))| \leq \|F\|\, |t_2 - t_1|\, \|f\|$;

(iii) for $\epsilon > 0$, $F \in E^*$, $x_1(\cdot) \in \mathcal{E}$, there is a neighborhood $N_{\epsilon,F}[x_1(\cdot)]$ such that $x_2(\cdot) \in N_{\epsilon,F}[x_1(\cdot)]$ implies $|T(F, t, x_2(\cdot)) - T(F, t, x_1(\cdot))| < \epsilon |t|$.

Fix $x(\cdot) \in \mathcal{E}$ and $t \in I$ and let $\Phi(t, x(\cdot)) \in E^{**}$ be the bounded linear functional defined by T. This, in fact, is the Dunford third integral [7]. Then

(iv) $\|\Phi(t, x(\cdot))\| \leq |t|\, \|f\|$;

(v) $\|\Phi(t_2, x(\cdot)) - \Phi(t_1, x(\cdot))\| \leq |t_2 - t_1|\, \|f\|$.

Let $y(t) = \Phi(t, x(\cdot))$. Then

(vi) $\|y(t)\| \leq 1$ for $t \in I$ and $y(0) = 0$;

(vii) $\|y(t_2) - y(t_1)\| \leq |t_2 - t_1|$;

(viii) for $\epsilon > 0$, $F \in E^*$, $x_1(\cdot) \in \mathcal{E}$ there is a neighborhood $N_{\epsilon,F}[x_1(\cdot)]$ such that $x_2(\cdot) \in N_{\epsilon,F}[x_1(\cdot)]$ implies $|F[y_2(\cdot) - y_1(\cdot)]| < \epsilon$.

We thus have a mapping $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ where $\mathcal{T}$ is continuous in the $\mathcal{B}_w$ topology and we can use the Schauder-Tychonov fixed point theorem [8, p. 405] to infer the existence of an $x(t) \in \mathcal{E}$ (i.e. $x(0) = 0$, $\|x(t)\| \leq 1$, and $\|x(t) - x(s)\| \leq |t - s|$) such that

$$x(t) = \int_0^t F[f(s, x(s))]\, ds \quad \text{for all } F \in E^*.$$

Further, if $F \in E^*$ and $\epsilon > 0$ then there exists a δ such that $|h| < \delta$ implies

$$\left| F\left[\frac{x(t+h) - x(t)}{h} - f(t, x(t)) \right] \right| = \left| \frac{1}{h} \int_t^{t+h} F[f(s, x(s)) - f(t, x(t))]\, ds \right| < \epsilon,$$

i.e., the weak derivative of $x(t)$, $\dot{x}(t)$, exists and $\dot{x}(t) = f(t, x(t))$.

We have thus proved

Theorem 7. *Let* (2) *be given and assume that in a neighborhood W of the origin f can be extended to a function mapping $R \times E^{**}$ into E^{**} in such a way that*

(i) $\|f(t, x)\|$ *is bounded on W; and*

(ii) $f(t, x)$ *is weakly continuous in (t, x) and is uniformly weakly continuous in x.*

Then there exists a function $x(t)$, with values in E^{**}, such that $x(0) = 0$, $\|x(t)\|$ is Lipschitz continuous, and the weak derivative of $x(t)$, $\dot{x}(t)$, exists and $\dot{x}(t) = f(t, x(t))$.

We can also give an existence theorem for the contingent differential equation (1) in the case where E is not reflexive, but E^{**} is separable.

We first observe that if $f : R \times E \to \mathrm{cf}(E)$ satisfies condition A, then f can be extended to a function $\hat{f} : R \times E^{**} \to \mathrm{cf}(E^{**})$ where $\hat{f}$ satisfies condition A (with respect to $\mathcal{G}$ which is considered as a subset of E^{***}).

For $P \in E^{**}$ let

$$\hat{f}(P) = \bigcap_{n=1}^{\infty} \overline{\mathrm{co}} \bigcup \{f(Q) : |t_Q - t_P| < 1/n, |F_i[x_Q - x_P]| < 1/n, \quad i = 1, \ldots, n\}$$

where $Q \in E \subset E^{**}$ and $F_1, F_2, \ldots \in \mathcal{G}$.

Assume $f : R \times E \to \mathrm{cf}(E)$ satisfies condition A. Then

(i) $\hat{f} : R \times E^{**} \to \mathrm{cf}(E^{**})$ and if $P \in E$, $\hat{f}(P) = f(P)$.

(ii) $\hat{f}$ satisfies condition A.

To see (ii), we note that (a)

$$\begin{aligned} \hat{f}(P_0) &= \bigcap_{n=1}^{\infty} \overline{\mathrm{co}} \bigcup \{f(P) : |t_P - t_{P_0}| < 1/n, |F_i[x_P - x_{P_0}]| < 1/n, \quad i = 1, \ldots, n\} \\ &\subset \bigcap_{n=1}^{\infty} \overline{\mathrm{co}} \bigcup \{\hat{f}(P) : |t_P - t_{P_0}| < 1/n, |F_i[x_P - x_{P_0}]| < 1/n, \quad i = 1, \ldots, n\}; \end{aligned}$$

(b) for n fixed, we have for $m \geq n$

$$R_n[\hat{f}(P_0)] = \bigcup\{\hat{f}(P) : |t_P - t_{P_0}| < 1/n, |F_i[x_P - x_{P_0}]| < 1/n, \quad i = 1,2,\dots,n\}$$
$$\subset \bigcup\{\overline{\mathrm{co}} \bigcup\{f(Q) : |t_Q - t_P| < 1/m, |F_i[x_Q - x_P]| < 1/m, i = 1,\dots,m\} : \quad |t_P - t_{P_0}| < 1/n, |F_i[x_P - x_{P_0}]| < 1/n, i = 1,2,\dots,n\}$$
$$\subset \overline{\mathrm{co}} \bigcup\{f(Q) : |t_Q - t_{P_0}| < 1/m + 1/n, |F_i[x_Q - x_{P_0}]| \quad < 1/m + 1/n, i = 1,2,\dots,n\}.$$

Since this is true for every $m \geq n$, we have

$$Q_n[\hat{f}(P_0)] = \overline{\mathrm{co}}\ R_n[\hat{f}(P_0)] \subset \overline{\mathrm{co}} \bigcup\{f(Q) : |t_Q - t_{P_0}| \leq 1/n, |F_i[x_Q - x_{P_0}]| \leq 1/n, \quad i = 1,2,\dots,n\}.$$

Now if we take the intersection from $n = 1$ to $n = \infty$ on the left and then on the right, we obtain

$$\bigcap_{n=1}^{\infty} Q_n[\hat{f}(P_0)] \subset \hat{f}(P_0),$$

and we have proved (ii).

By using the method in Theorem 3 we have the following:

Theorem 8. *If f satisfies condition* A *and if $\hat{f}$ is bounded on every α-set, then the equation $Dx \subset \hat{f}(t,x)$ has solution for the initial value problem.*

Remark. For the continuity properties of solutions we need to know more about the set ϑ, considered as a subset of E^{***}, and separability of E^*. This can then be discussed as in §2.

Bibliography

1. Ju. G. Borisovič, *A weak topology and periodic solutions of differential equations*, Dokl. Akad. Nauk SSSR **136** (1961), 1269–1272 = Soviet Math. Dokl. **2** (1961), 176–179. MR **22** #8179.

2. L. Cesari, *Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints*. I, II, Trans. Amer. Math. Soc. **124** (1966), 369–430. MR **34** #3392; #3393.

3. S. N. Chow and J. D. Schuur, *An existence theorem for ordinary differential equations in Banach spaces*, Bull. Amer. Math. Soc. **77** (1971), 1018–1020.

4.———, *Contingent differential equations in Banach spaces*, Notices Amer. Math. Soc. **19** (1972), A-137. Abstract 691-34-41.

5. C. Corduneanu, *Equazioni differenziali negli spazi di Banach, teoremi di esistenza e di prolungabilità*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 226–230. MR **20** #3312.

6. J. Dieudonné, *Deux exemples singuliers d'équations différentielles*, Acta Sci. Math. Szeged **12** (1950), Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, pars B, 38–40. MR **11**, 729.

7. N. Dunford, *Integration of abstract functions*, Bull. Amer. Math. Soc. **42** (1937), 24; See also, B.

J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 277–304.

8. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964. MR **30** #1270.

9. M. A. Krasnosel′ skiĭ and S. G. Kreĭn, *Nonlocal existence theorems and uniqueness theorems for systems of ordinary differential equations*, Dokl. Akad. Nauk SSSR **102** (1955), 13–16. (Russian) MR **17**, 151.

10. A. Lasota and C. Olech, *On Cesari's semicontinuity condition for set valued mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **16** (1968), 711–716. MR **39** #138.

11. A Marchaud, *Sur les champs de demi-cônes et équations différentielles du premier ordre*, Bull. Soc. Math. France **62** (1934), 1–38.

12. W. Mlak, *Note on the mean value theorem*, Ann. Polon. Math. **3** (1956), 29–31. MR **18**, 724.

13. C. Olech, *On the existence and uniqueness of solutions of an ordinary differential equation in the case of Banach space*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 667–673. MR **26** #5247.

14. B. J. Pettis, *A note on regular Banach spaces*, Bull. Amer. Math. Soc. **44** (1938), 420–428.

15. T. Ważewski, *Sur une condition équivalente à l'équation au contingent*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 865–867. MR **24** #A3328.

16. J. A. Yorke, *A continuous differential equation in Hilbert space without existence*, Funkcial. Ekvac. **13** (1970), 19–21. MR **41** #8792.

17. S. Zaremba, *Sur les équations au paratingent*, Bull. Sci. Math. (2) **60** (1936), 139–160.

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