TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 179, May 1973

THE EXISTENCE AND UNIQUENESS OF NONSTATIONARY IDEAL INCOMPRESSIBLE FLOW IN BOUNDED DOMAINS IN R_3

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ABSTRACT. It is shown here that the mixed initial-boundary value problem for the Euler equations for ideal flow in bounded domains of R_3 has a unique solution for a small time interval. The existence of a solution is shown by converting the equations to an equivalent system involving the vorticity and applying Schauder's fixed point theorem to an appropriate mapping.

Introduction. The purpose of this paper is to prove the existence and uniqueness of classical solutions to the mixed initial-boundary value problem for the Euler equations for an ideal incompressible fluid in bounded domains of R_3 for a small time interval. This has been shown by Ebin and Marsden [3] for compact Riemannian manifolds possibly with boundary. In this paper we obtain a new proof by converting the equations to an equivalent system involving the vorticity and solving this by using the Schauder fixed point theorem. The technique is similar to that of Kato [4] where he proved existence of ideal flow in bounded domains of the plane for an arbitrary time interval. The author also used a similar method to obtain an existence result for ideal flow in all R_3 by showing that a solution is the limit of Navier-Stokes flow as the viscosity goes to zero [7]. Existence of a solution for a short time in all R_3 was originally shown by Lichtenstein [5].

The Euler equations for ideal incompressible flow in a domain D in R_3 with boundary bD are

$$\text{(E}'\text{)} \qquad \partial v/\partial t + (v \cdot \operatorname{grad})v = -\operatorname{grad} P + B, \qquad \nabla \cdot v = 0,$$

with constraints $v \cdot n = 0$ on bD and $v(0) = A$, where x is a point in $D \cup bD$; $t \in [0, T]$, $v(x, t)$ is the velocity vector, $P(x, t)$ is the (scalar) pressure, $B(x, t)$ is the external force field vector, $n(x)$ is the outward normal vector to bD and $A(x)$ is the initial velocity vector.

By formally computing the curl of (E′) we get the system

Presented to the Society, January 18, 1972 under the title *The existence and uniqeness of nonstationary ideal flow in bounded domains in* R_3; received by the editors October 12, 1971 and, in revised form, March 20, 1972.

AMS (MOS) subject classifications (1970). Primary 35F25, 35F30, 35Q05, 76C05.

Key words and phrases. Nonstationary ideal flow, Euler equations.

$$
\text{(E)}\quad
\begin{aligned}
&\text{(a)}\qquad \partial w/\partial t + (v\cdot \operatorname{grad})w - (w \cdot \operatorname{grad})v = \nabla \times B \equiv b,\\
&\text{(b)}\qquad w(x,0) = \nabla \times A(x) \equiv a(x),\\
&\text{(c)}\qquad \nabla \times v = w, \qquad \nabla \cdot v = 0,\\
&\text{(d)}\qquad v \cdot n = 0 \qquad \text{on } bD.
\end{aligned}
$$

We solve this system to obtain a solution to (E′) in three steps:

In §I we show that if w is in an appropriate class of functions, then there is a function $v \equiv F_1(w)$ that solves (E), (c) and (d).

In §II we show that for $v = F_1(w)$ there is a solution, denoted $F_2(v)$, to (E), (a) and (b).

In §III we show that the mapping $F_2(F_1(\cdot))$ maps a closed convex compact set of functions in a Banach space continuously into itself provided we restrict the time interval of solution $[0, T]$. Hence we can apply Schauder's fixed point theorem to obtain a function w such that $w = F_2(F_1(w))$ and from this a solution to (E). Finally we show that this gives a classical solution to (E′) with v unique and P unique up to an arbitrary function of t which may be added to P.

The author is indebted to Professor Tosio Kato for his many helpful suggestions and comments.

The following notation is used in this paper:

D is the domain, connected, open and bounded in R_3 with closure $\overline{D}$ and boundary bD of class C^{3+q} for some q, $0 < q < 1$ (see [6, p. 300]). $\overline{D} \times [0, T]$ is denoted $(\overline{D}; T)$, where $T > 0$ is arbitrary.

$C^{q,r}(\overline{D}; T)$ is the Banach space of vector-valued functions from $(\overline{D}; T)$ to R_3 with continuous space derivatives of order $\leq [q]$ (denoted conventionally D_x^α) continuously extendable to $\overline{D}$ and continuous time derivatives of order $\leq [r]$, where $[q]$ and $[r]$ are the largest integers less than or equal to q and r respectively. Space derivatives of order $[q]$ are Hölder-continuous with exponent $q - [q]$ and time derivatives of order $[r]$ are Hölder-continuous with exponent $r - [r]$.

If $|v|^2 = \sum_{i=1}^3 |v_i|^2$ and

$$
K^{q',r'}(v) = \sup_{\substack{t,t' \in [0,T]; t \neq t' \\ x,x' \in \overline{D}; x \neq x' \\ |x-x'|<1, |t-t'|<1}} \frac{|v(x',t') - v(x,t)|}{|x - x'|^{q'} + |t - t'|^{r'}}
$$

we define a norm for $C^{q,r}(\overline{D}; T)$ as

$$
\|v\|^2_{C^{q,r}} = \sup_{\substack{x \in \overline{D} \\ t \in [0,T]}} \sum_{\substack{|\alpha| \leq [q] \\ \beta \leq [r]}} \left| D_x^\alpha \left(\frac{d}{dt}\right)^\beta v(x,t) \right|^2
+ \sum_{\substack{|\alpha| = [q] \\ \beta = [r]}} \left[K^{q-[q], r-[r]} \left(D_x^\alpha \left(\frac{d}{dt}\right)^\beta v \right) \right]^2 .
$$

When we allow $t \in [0, T]$, $t > 0$, to vary we will note this by using the norm notation $\| \ \|_{C^{q,r}(\overline{D};t)}$ defined in the obvious way. For convenience, $\| \ \|_C = \| \ \|_{C^{0,0}}$. We occasionally use analogously defined $C^q(\overline{D})$ with only space variables ($t \in [0, T]$ is fixed if present).

$v_{,x}$ is the array with ith row $(\partial v_i/\partial x_1, \partial v_i/\partial x_2, \partial v_i/\partial x_3)$, $i = 1, 2, 3$.

$$\|v_{,x}\|_{(\cdot)}^2 = \sum_{i,j=1,3} \left| \frac{\partial v_i}{\partial x_j} \right|_{(\cdot)}^2$$

for various norms $(\cdot)$.

$L_2(D)$ is the Hilbert space of vector-valued functions u over D with inner-product denoted

$$(u^1, u^2) = \sum_{i=1,3} \int_D u_i^1 u_i^2 \, dV.$$

$C_n^q = \{f \in C^q(\overline{D}) \mid f(x) = g(x)n$ for some scalar-valued function $g(x)$ on $bD\}$,

$C_\tau^q = \{f \in C^q(\overline{D}) \mid f \cdot n = 0$ on $bD\}$.

H_n is the finite dimensional subspace of C_n^{1+q} of functions h such that $\nabla \times h = 0$ and $\nabla \cdot h = 0$ $(0 < q < 1)$.

H_τ is the finite dimensional subspace of C_τ^{1+q} of functions h such that $\nabla \times h = 0$ and $\nabla \cdot h = 0$. We let $\{h^i\}$, $i = 1, \ldots, m$, be an L_2-orthonormal basis of H_τ. (See [6, Chapter 7] for a further discussion of these subspaces.)

I.

Theorem I-1. *Let $A \in C^{1+q}(\overline{D})$ and $B \in C^{1+q,0}(\overline{D}; T)$. If $w \in C^{q,q}(\overline{D}; T)$ and $0 = \nabla \cdot w$ (generalized) and, for fixed t, $w \in L_2(D) \ominus H_n$, then there exists a unique $v \in C^{1+q,0}(\overline{D}; T)$ such that*

$$\nabla \cdot v = 0, \qquad \nabla \times v = w, \qquad v \cdot n = 0 \quad \text{on } bD$$

and

$$(d/dt)(v, H) = (-(v \cdot \operatorname{grad})v + B, H), \qquad (v(0), H) = (A, H)$$

for any $H \in H_\tau$.

If we let $v = u + h$ where $u \in L_2(D) \ominus H_\tau$ and $h \in H_\tau$ then there are constants $C_1(r)$, $C_3(r)$ and $C_4(r)$ depending on D and r, $0 < r \leq q$, and a constant C_2 depending on D such that

(i) $$\|u\|_{C^{1+r,0}} \leq C_1(r)\|w\|_{C^{r,0}}$$

and

(ii) $$\|h\|_{C^{1+r,0}} \leq (C_2(\|A_H\|_C + T\|B_H\|_C) + C_3(r)\|w\|_{C^{r,0}}) \exp(C_4(r)\|w\|_{C^{r,0}} T)$$

where A_H and B_H are the projections of A and B respectively onto H_τ.

We denote this unique $v = u + h = F_1(w)$.

Proof. We will have recourse to the following representations of $f \in C^r(\overline{D})$ at several points in this paper.

When interpreted as a statement about vector-valued functions Theorem 7.7.4 of Morrey [6] gives the existence of vector-valued functions p_f and p_f^- for any $f \in C^r(\overline{D})$ such that

$$f = \nabla \times (\nabla \times p_f) - \nabla(\nabla \cdot p_f) + h_f,$$
$$f = \nabla \times (\nabla \times p_f^-) - \nabla(\nabla \cdot p_f^-) + h_f^-$$

with $h_f^- \in H_\tau$, $h_f \in H_n$, $\nabla \times p_f^-$ and $p_f \in C_n^{1+r}$; $\nabla \times p_f$ and $p_f^- \in C_\tau^{1+r}$; $\nabla \cdot p_f$ and $\nabla \cdot p_f^- \in C^{1+r}(\overline{D})$ (as scalar-valued functions) and $\nabla \cdot p_f = 0$ on bD. The three terms of each representation are mutually L_2-orthogonal.

Since $w \in C^{q,q}(\overline{D}; T)$ and $w \in L_2(D) \ominus H_n$, $w = \nabla \times (\nabla \times p_w) - \nabla(\nabla \cdot p_w)$ with these two terms orthogonal. However, since $(w, \nabla q) = 0$ for all smooth scalar-valued functions q with compact support in D, a conventional mollifier and limit argument can be used to establish that $(w, \nabla(\nabla \cdot p_w)) = 0$, for $\nabla \cdot p_w$ is zero on bD. Hence $\nabla(\nabla \cdot p_w) = 0$ and $w = \nabla \times (\nabla \times p_w)$. Letting $u = \nabla \times p_w$ and interpreting it as a two-form, Lemma 7.5.3 of [6] can be used to show that $u \in L_2(D) \ominus H_\tau$, which easily establishes uniqueness as well.

The operator taking w into u (for fixed t) is clearly a closed linear operator from the subspace Q of $C^r(\overline{D})$, $0 < r \leq q$, defined by

$$Q = \{w \in C^r(\overline{D}) \cap (L_2(\overline{D}) \ominus H_n) \mid 0 = \nabla \cdot w \text{ generalized}\}$$

into $C^{1+r}(\overline{D})$ with associated norms. Hence, by the closed graph theorem, a constant $C_1(r)$ exists satisfying (i) for fixed $t \in [0, T]$. However, the linearity of the operator easily establishes that $u \in C^{1+r,0}(\overline{D}; T)$ and (i) follows.

With u defined from w by the preceding, we let $v = u + \sum_{i=1}^m g^i(t)h^i$ where the scalars $g^i(t)$ are unknown and $\{h^i\}$ is a basis of H_τ. Since $(v, h^i) = g^i(t)$, we can form the ordinary differential equations in $\{g^i\}$:

$$dg^i/dt = (d/dt)(v, h^i) = (-(v \cdot \text{grad})v + B, h^i),$$
$$g^i(0) = (v(0), h^i) = (A, h^i), \qquad i = 1, \ldots, m.$$

An argument similar to that in [4, p. 195] gives, for suitable constants K_1 and K_2 which depend on D,

$$\frac{1}{2}\frac{d}{dt}\sum g^i(t)^2 \le K_1(\|B_H\|_C + \|u\|_C^2)(\sum g^i(t)^2)^{1/2} + K_2\|u\|_C \sum g^i(t)^2.$$

This leads to the estimate

$$(\sum g^i(t)^2)^{1/2} \le (\|A_H\|_{L_2} + tK_1\|B_H\|_C)\exp(tK_2\|u\|_C) + \frac{K_1}{K_2}\|u\|_C(\exp(tK_2\|u\|_C) - 1).$$

Thus $h(t) = \sum g^i(t)h^i$ exists in $[0, T]$. This inequality can be combined with the estimate (i) to yield (ii). Thus $v \equiv u + h \equiv F_1(w)$ exists and is unique and has the required properties.

II. We wish to establish

Theorem II-1. *If $v \in C^{1+q,0}(\overline{D}; T)$, $\nabla \cdot v = 0$, $v \cdot n = 0$ on bD, $a \in C^q(\overline{D})$, $b \in C^{q,0}(\overline{D}; T)$, then we can construct $w \in C^{q,q}(\overline{D}; T)$ uniquely with $w(x, 0) = a(x)$ and, for all $f \in C^1(\overline{D})$,*

$$(d/dt)(w,f) = (w,(v \cdot \mathrm{grad})f) + ((w \cdot \mathrm{grad})v,f) + (b,f).$$

The following theorem from [1, p. 105] is sufficient to establish our estimates:

Theorem A. *Let E be some bounded domain in R_3. If $Y(x,t) \in C^{1,0}(\overline{E}, T)$ and $Z(x,t) \in C(\overline{E}, T)$ and*

$$y(t) \text{ solves } dy/dt = Y(y,t),$$

and

$$z(t) \text{ solves } dz/dt = Z(z,t)$$

then

$$|y(t) - z(t)| \le \{|y(s) - z(s)| + |t - s|\,\|Y - Z\|_C\}\exp(T\|Y_{,x}\|_C).$$

We begin by solving for the streamlines of the "flow" v.

Lemma II-1. *If $v \in C^{1+q,0}(\overline{D};T)$, $\nabla \cdot v = 0$, $v \cdot n = 0$ on bD and $X(x,s;t)$ solves $dX/dt = v(X,t)$ with initial condition $X(x,s;s) = x_2$ then* (i) *$X(x,s;t)$ is unique and continuously differentiable in $x \in D$ and $s, t \in [0,T]$ and*

(a) $$|X(x,s;t) - X(x',s;t)| \leq |x - x'| \exp(T\|v_{,x}\|_C),$$

(b) $$|X(x,s;t) - X(x,s';t')| \leq (|t - t'| + |s - s'|)\|v\|_C \exp(T\|v_{,x}\|_C)$$

and (ii) *for fixed* t *and* s, $X(\cdot, s; t) : x \to X(x,s;t)$ *is a one-to-one measure-preserving map of* $\overline{D}$ *onto* $\overline{D}$ *such that* $\det(X_{,x}) = 1$ *and* $bD \to bD$;

$$X(\cdot, s; s) = \text{Identity}, \quad X(\cdot, s; t) = [X(\cdot, t; s)]^{-1}.$$

Proof. Since $v \in C^{1+q,0}(\overline{D}; T)$, there exists a unique local solution for the streamlines; they can be continued until they reach bD. Hence to show that $\overline{D}$ is mapped into $\overline{D}$ and bD into bD it suffices to show that a solution starting on bD remains there, i.e. on the "sidewall" $bD \times [0, T]$. Since $v \cdot n = 0$ on bD and bD is smooth we can define a system of ordinary differential equations on $bD \times [0, T]$ whose solutions are solutions of the original system and will stay on the "sidewall" by definition and this is sufficient to obtain our result.

The remainder of (ii) and the differentiability stated in (i) are standard results of ordinary differential equations since $v \in C^{1+q,0}(\overline{D}, T)$ and $\nabla \cdot v = 0$. (See [2, Chapter 1, §7].)

The estimates in (i) are standard or follow from Theorem A; for example

$$\begin{aligned} \|X(x,s;t) - X(x,s';t)| &= |X(x,s;t) - X(X(x,s';s),s;t)| \\ &\leq |X(x,s';s') - X(x,s';s)| \exp(T\|v_{,x}\|_C) \\ &\leq |s - s'| \, \|v\|_C \exp(T\|v_{,x}\|_C). \end{aligned}$$

To obtain a solution to the equations of Theorem II-1 we first solve the system (O) of ordinary differential equations:

(O) $$dy/dt - v_{,x}(X(x,s;t),t)y = b(X(x,s;t),t)$$

with initial condition $y(x,s;0) = a(X(x,s;0))$. We will then establish that $w(x,t) \equiv y(x,t;t)$ is the solution required by Theorem II-1.

Lemma II-2. *Under the conditions assumed for* a, b *and* v *in Theorem* II-1, *there is a unique solution* $y(x,s;t)$ *to* (O) *for* $t \in [0, T]$ *and if*

$$\|y\|_{C(D;T,T)} = \sup_{x \in D; s,t \in [0,T]} |y(x,s;t)|$$

then

(i) $$|y(x,s;t) - y(x,s;t')| \leq |t - t'|(\|v_{,x}\|_C \|y\|_{C(D;T,T)} + \|b\|_C)$$

and

$$\text{(ii)}\qquad \begin{aligned} &|y(x,s;t) - y(x',s';t)| \\ &\quad\le (|x - x'|^q + \|v\|_C^q |s - s'|^q)[\|a\|_{C^q} + t(\|v\|_{C^{1+q,0}} \|y\|_{C(D;T,T)} + \|b\|_{C^q})] \\ &\qquad\cdot \exp(3T\|v_{,x}\|_C), \end{aligned}$$

$$w(x,t) \equiv y(x,t;t) \in C^{q,q}(\overline{D},T).$$

Proof. From [2, p. 74], we know that there is a "fundamental matrix" $F_v(t)$ for linear system (O) and the solution has form

$$\begin{aligned} y(x,s;t) &= F_v(t)F_v(0)^{-1}a(X(x,s;0)) \\ &\quad + F_v(t)\int_0^t F_v^{-1}(\tau)b(X(x,s;\tau),\tau)\,d\tau \end{aligned}$$

and is a continuous function (for fixed v) of the parameters $x \in \overline{D}$ and $s \in [0,T]$. Hence $\|y\|_{C(D;T,T)}$ exists. We use Theorem A and Lemma II-1 to obtain additional smoothness and estimates for $y(x,s;t)$.

$$\begin{aligned} &|y(x,s;t) - y(x',s;t)| \\ &\quad\le \Bigg\{|y(x,s;0) - y(x',s;0)| \\ &\qquad\quad + t\Bigg(\sup_{t\in[0,T]}|v_{,x}(X(x,s;t),t) - v_{,x}(X(x',s;t),t)|\,\|y\|_{C(D,T;T)} \\ &\qquad\qquad\qquad + \sup_{t\in[0,T]}|b(X(x,s;t),t) - b(X(x',s;t),t)|\Bigg)\Bigg\} \\ &\qquad\qquad\qquad\qquad \cdot \exp\Bigg(T\sup_{t\in[0,T]}|v_{,x}(X(x,s;t),t)|\Bigg) \\ &\quad\le \{|X(x,s;0) - X(x',s;0)|^q\|a\|_{C^q} \\ &\qquad + |X(x,s;t) - X(x',s;t)|^q t(\|v\|_{C^{1+q,0}}\|y\|_{C(D;T,T)} + \|b\|_{C^{q,0}})\}\exp(T\|v_{,x}\|_C) \\ &\quad\le |x - x'|^q\exp((q+1)T\|v_{,x}\|_C)\{\|a\|_{C^q} + t\|v\|_{C^{1+q,0}}\|y\|_{C(D;T,T)} + \|b\|_{C^{q,0}}\}. \end{aligned}$$

That

$$|y(x,s;t) - y(x,s';t)| = |y(X(x,s;s),s;t) - y(X(x,s';s),s,t)|$$

can be shown by reference to the definition of X and y; then the Hölder-continuity in x can be used to obtain the result for s. Verifying the Lipschitz-continuity in t is straightforward. These results easily show that $w(x,t) \equiv y(x,t;t) \in C^{q,q}(\overline{D};T)$.

Proof of Theorem II-1. Let $y(x,t;t)$ be the solution of (O) of Lemma II-2. Then if $w(x,t) = y(x,t;t)$, $w(x,0) = y(x,0;0) = a(X(x,0;0)) = a(x)$ and

$$\left.\frac{dy(x,t;\tau)}{d\tau}\right|_{\tau=t} = v_{,x}(X(x,t;t),t)y(x,t;t) + b(X(x,t;t),t)$$
$$= v_{,x}(x,t)w(x,t) + b(x,t) = (w\cdot\mathrm{grad})v + b.$$

$$\int_D (y(x,t+\Delta t;t) - y(x,t;t))f(x)\,dx$$
$$= \int_D (y(X(x,t+\Delta t;t),t;t) - y(X(x,t;t),t;t))f(x)\,dx$$
$$= \int y(x,t;t)f(X(x,t;t+\Delta t)) - f(X(x,t;t))\,dx$$

by the measure-preserving and inverse properties of $X(\cdot,s;t)$ (see Lemma II-1 (ii)). Hence we can multiply by $(\Delta t)^{-1}$ and take the limit to obtain

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_D (y(x,t+\Delta t;t) - y(x,t;t))f(x)\,dx = (w,(v\cdot\mathrm{grad})f).$$

This establishes that $w(x,t)$ is the weak solution required.

III. In this section we apply Schauder's fixed point theorem using the two mappings F_1 and F_2 defined in §§I and II applied to the set

$$S = \{w \in C^{q,q}(\overline{D};T) \mid \|w\|_C \le K_3;\ k^{q,q}(w) \le K_4;\ \nabla\cdot x = 0;$$
$$w \in L_2(D) \ominus H_n \text{ (for fixed } t \in [0,T])\}$$

defined for suitable constants K_3, K_4 and $T > 0$ (see introduction for notation).

Lemma III-1. *The set S is a compact convex set in the space $C(\overline{D};T)$.*

Proof. Convexity is immediate. Compactness in $C(\overline{D};T)$ follows from the equicontinuity implied by the requirement $K^{q,q}(w) \le K_4$ and the Arzela-Ascoli theorem and straightforward verification of the closure requirements.

Lemma III-2. *Let $C_1 = C_1(q)$, $C_3 = C_3(q)$, $C_4 = C_4(q)$ and*

$$C_5 = (C_1 + C_3 e)^{-1} C_2 e(\|A_H\|_C + \|B_H\|_C),$$
$$K_3 = (3/2)\|A\|_{C^q} + 1/2,$$
$$K_4 = (K_3(2e(C_1 + C_3 e)^q + 1) + C_5)^{1/(1-q)} - (K_3 + C_5),$$
$$T^{1-q} \le (1/3)\{(K_3 + K_4)(C_1 + C_3 e + (C_4)/3)$$
$$+ C_2 e(\|A_H\|_C + \|B_H\|_C) + \|b\|_{C^{q,0}}\}^{-1},$$

and assume $T \leq 1$. Then $w \in F_2(F_1(S))$ implies $\|w\|_C \leq K_3$ and $K^{q,q}(w) \leq K_4$.

Proof. If $w \in S$ and $v = F_1(w)$, then Theorem I-1 and the restriction on T gives

$$\begin{aligned}\|v\|_{C^{1+q,0}} &\leq (C_1 + C_3 e)(K_3 + K_4) + C_2 e(\|A_H\|_C + T\|B_H\|_C)\\ &\leq (3T^{1-q})^{-1} \leq (3T)^{-1}.\end{aligned}$$

Let $y(x, s; t)$ be the solution of system (O) of Lemma II-2 with $v = F_1(w)$. Then

$$\begin{aligned}&y(x, s; t) - a(X(x, s; 0))\\ &\qquad = \int_0^t v_{,x}(X(x, s; \tau), \tau) y(x, s; \tau) + b(X(x, s; \tau), \tau)\, d\tau,\end{aligned}$$

so

$$|y(x, s; t)| \leq \|a\|_C + t\left\{\|v\|_{C^{1+r,0}} \sup_{\tau\in[0,t]} |y(x, s; \tau)| + \|b\|_C\right\}.$$

Since $y(x, t; t) = w'(x, t) \equiv F_2(F_1(w))$ we get

$$\begin{aligned}&\|w'\|_{C(\bar{D};t)} \leq \|a\|_C + \tfrac{1}{3}(t/T)\|w\|_{C(\bar{D};t)} + t\|b\|_C \quad \text{or}\\ &\|w'\|_{C(\bar{D};T)} \leq (3/2)\|a\|_{C^q} + \tfrac{1}{2} = K_3.\end{aligned}$$

Similarly

$$\begin{aligned}|y(x, s; t) - y(x, s; t')| &\leq |t - t'|^q T^{1-q}((K_3)/(3T^{1-q}) + \|b\|_C)\\ &\leq |t - t'|^q (K_3 + 1)/3 \leq |t - t'|^q (K_4)/2.\end{aligned}$$

Lemma II-2 gives

$$|y(x, s; t) - y(x', s'; t)| \leq (|x - x'|^q + \|v\|_{C^q}|s - s'|^q)eK_3.$$

Hence $|w'(x, t) - w'(x', t)| \leq |x - x'|^q(eK_3) \leq |x - x'|^q K_4$. We also have

$$\begin{aligned}&|y(x, s; t) - y(x, s'; t)|\\ &\qquad \leq |s - s'|^q eK_3(C_2 e(\|A_H\|_C + T\|B_H\|_C) + (K_3 + K_4)(C_1 + C_3 e))^q.\end{aligned}$$

To show that this final inequality yields the desired result, we note that the right-hand side has form $|s - s'|^q K_5(K_6 + K_4)^q$; we require that this be less than $\frac{1}{2}K_4$. Since $0 < q < 1$, the existence of such K_4 follows from considering the equivalent problem of finding K_4 such that

$$2K_5 \leq (K_6 + K_4)^{1-q} - K_6(K_6 + K_4)^{-q},$$

which is clearly obtainable for sufficiently large K_4 and will remain true for larger K_4. Verification that our estimate is adequate is straightforward.

Lemma III-3. *The map $F_2(F_1(\ \ ))$ is continuous in the $C(\overline{D}; T)$ topology from S into $C(\overline{D}; T)$.*

Proof. Suppose $w_i \in S$, $i = 0, 1, 2, \ldots$, and $\|w_i - w_0\|_C \to 0$ as $i \to \infty$. Then, using the notation of Theorem I-1, $F_1(w_i) = u_i + h_i \equiv v_i$.

First we wish to establish that $\{v_{i,x}\}$ is an equicontinuous set. To show that $\{h_{i,x}\}$ is equicontinuous, since $h_i \in H_\tau$ (finite dimensional) for fixed t, it suffices to show that $\{h_{i,x}\}$ is equicontinuous in t; but it is immediate that they are uniformly Lipschitz-continuous from the bounds of Lemma III-2 and the defining ordinary differential equations for h_i of Theorem I-1. Equicontinuity of $\{u_{i,x}\}$ in $x \in \overline{D}$ follows from inequality (i) of Theorem I-1. To show equicontinuity in $t \in [0, T]$, we note that since $w_i \in C^{q,q}(\overline{D}; T)$, $w_i \in C^{q-r,q}(\overline{D}; T)$ for any r, $0 < r < q$, and

$$\begin{aligned}
&|u_{i,x}(x,t) - u_{i,x}(x,s)| \\
&\quad \leq C_1(q-r)\Bigg(\|w_i(\cdot,t) - w_i(\cdot,s)\|_{C(\overline{D})} \\
&\qquad\qquad + \sup_{\substack{x,x' \in \overline{D} \\ |x-x'|<1 \\ x \neq x'}} \frac{|w_i(x,t) - w_i(x,s) - w_i(x',t) + w_i(x',s)|^{(q-r)/q + r/q}}{|x-x'|^{q-r}}\Bigg) \\
&\quad \leq C_1(q-r)\left(K_4|t-s|^q + \frac{(2K_4|x-x'|^q)^{(q-r)/q}(2K_4|t-s|^q)^{r/q}}{|x-x'|^{q-r}}\right) \\
&\quad \leq C_1(q-r)3K_4|t-s|^r.
\end{aligned}$$

We wish to show that $\|v_i - v_0\|_{C^{1,0}} \to 0$ as $i \to \infty$. Suppose not. Then, using the equicontinuity just established and the Arzela-Ascoli theorem, there is a subsequence $\{v_j\}$ and $v' \in C^{1,0}(\overline{D}; T)$ and $\varepsilon > 0$ such that $\|v_j - v_0\|_{C^{1,0}} \geq \varepsilon$ and $\|v_j - v'\|_{C^{1,0}} \to 0$ as $j \to \infty$. However, the criteria for the existence of a unique v_0 satisfying the constraints of Theorem I-1 are maintained under $C^{1,0}(\overline{D}; T)$ convergence (defining $h' \equiv \sum (v', h^i)h^i$; etc.), hence $v' = v_0$ and we have a contradiction.

Let $X_i(x, s; t)$ be the streamline associated with v_i and

$$w_i' = F_2(F_1(w_i) \equiv y_i(x, t; t)$$

where $y_i(x, s; t)$ is the solution to system (O) defined by v_i. Then Theorem A gives

$$|X_i(x,s;t) - X_0(x,s;t)| \leq |t-s|\, \|v_i - v_0\|_C e^{1/3}$$

and, using this,

$$\begin{aligned}
&|y_i(x,s;t) - y_0(x,s;t)| \\
&\quad \le \Big\{ |a(X_i(x,s;0)) - a(X_0(x,s;0))| \\
&\qquad + t \sup_{\tau\in[0,t]} (K_3 |v_{i,x}(X_i(x,s;\tau),\tau) - v_{0,x}(X_0(x,s;\tau),\tau)| \\
&\qquad\qquad + |b(X_i(x,s;\tau),\tau) - b(X_0(x,s;\tau),\tau)|) \Big\} \exp(T\|v_{0,x}\|_C) \\
&\quad \le e\Big\{ (\|a\|_{C^q} + TK^{q,0}(b))(T\|v_i - v_0\|_C e^{1/3})^q \\
&\qquad + TK_3\Big(\sup_{t\in[0,T]} (|v_{i,x}(X_i(x,s;t),t) - v_{i,x}(X_0(x,s;t),t)| \\
&\qquad\qquad + |v_{i,x}(X_0(x,s;t),t) - v_{0,x}(X_0(x,s;t),t)|)\Big)\Big\} \\
&\quad \le e\{\|a\|_{C^q} + TK^{q,0}(b) + (K_3/3)(T\|v_i - v_0\|_C e^{1/3})^q + TK_3\|v_i - v_0\|_{C^{1,0}}\}
\end{aligned}$$

from the restriction on T (see Lemma III-2). Since $\|v_i - v_0\|_{C^{1,0}} \to 0$ as $i \to \infty$, this establishes the continuity.

Lemma III-4. *If* $w \in F_2(F_1(S))$, *then* $\nabla \cdot w = 0$ *(in the generalized sense).*

Proof. If we assume $v \in C^{3,0}(\overline{D};T)$, $b \in C^{2,0}(\overline{D};T)$ and $a \in C^2(\overline{D})$ then, by standard results concerning differentiable dependence on parameters of ordinary differential equations [2, p. 31] applied to both the streamlines and the system (O), $y(x,t;t) \equiv w(x,t) \in C^{2,1}(\overline{D};T)$ and so the weak solution of Theorem II-1 of (E) (a), (b) is classical. Hence we can take the divergence of system (E) (a), (b) to obtain $(\partial/\partial t)(\nabla\cdot w) + v\cdot \text{grad}\,(\nabla\cdot w) = 0$ and $\nabla\cdot w(x,0) = \nabla\cdot(\nabla\times A) = 0$ since several terms cancel and $\nabla\cdot v = 0 = \nabla\cdot b$. Now, to show that $\nabla\cdot w(x,t) = 0$ it is sufficient to show that $\nabla\cdot w(X(x,s;t),t) = 0$ since $X(x,s;t)$ is a 1-1 map of $\overline{D}$ onto $\overline{D}$ for fixed s and t.

$$0 = \nabla\cdot w(X(x,s;0),0)$$

and

$$\begin{aligned}
\frac{d}{dt}\nabla\cdot w(X(x,s;t),t) &= \frac{\partial}{\partial t}(\nabla\cdot w(X(x,s;t),t)) + \left(\frac{\partial}{\partial x_i}(\nabla\cdot w)\right)\cdot v_i \\
&= \frac{\partial}{\partial t}(\nabla\cdot w) + v\cdot \text{grad}\,(\nabla\cdot w) = 0
\end{aligned}$$

was just established. Hence $\nabla\cdot w = 0$.

For our purpose it is sufficient to show that $(w, \nabla g) = 0$ for any smooth scalar-valued function g with support in some compact $D' \subset D$. Using Friedrich's mollifier we can construct sequences $\{a^i\}$, $\{b^i\}$ and $\{V^i\}$ of the required smoothness that preserve the relations $\nabla \cdot a^i = 0$; $\nabla \cdot b^i = 0$ and $\nabla \cdot v^i = 0$ and converge uniformly together with the necessary derivatives to a, b and v respectively on any compact set $D'' \subset D$. Although the associated streamlines $X^i(x, s; t)$ may lead out of the original compact set D', the continuity arguments of Lemma III-3 and the result that $X(x, s; t)$ takes bD into bD of Lemma II-1(ii) show that by taking v^i sufficiently close to v in a chosen set D'' compactly contained in D and containing D' in its interior, streamlines starting in D' will stay in D'' for s and $t \in [0, T]$. Hence we can define solutions w^i of (E)(a), (b) using a^i, b^i and v^i and the previous reasoning will yield $\nabla \cdot w^i = 0$ in D'. The continuity arguments of Lemma III-3 show the dependence of w^i on v^i and the representation of solutions $y^i(x, s; t)$ to (O) with v^i of Lemma II-2 show the dependence on a^i and b^i; together these allow us to assert that w^i approaches w uniformly in $D' \times [0, T]$ and hence $(w, \nabla g) = 0$.

Lemma III-5. *$w \in F_1(F_2(S))$ implies that $w \in L_2(D) \ominus H_n$ (for fixed $t \in [0, T]$).*

Proof. First, note that if $h \in H_n$ and hence $h = g(x)n$ on bD where $g(x)$ is a scalar $(\nabla \times F, h) = (F, \nabla \times h) = 0$ for any $F \in C'(\overline{D})$.

Now, if w is the weak solution of (E)(a), (b) of Theorem II-1

$$(d/dt)(w, h) = (\nabla \times B, h) + (w, (v \cdot \operatorname{grad})h) + ((w \cdot \operatorname{grad})v, h)$$

and

$$(w(0), h) = (\nabla \times A, h) = 0,$$

$$((w \cdot \operatorname{grad})v, h) = (w, \nabla(v \cdot h)) - (v, (w \cdot \operatorname{grad})h).$$

But, on bD, $v \cdot h = g(x)v \cdot n = 0$. Hence, using Lemma III-4 and the limit argument employed in Theorem I-1, $(w, \nabla(v \cdot h)) = 0$. Then, since $(\nabla \times B, h) = 0$

$$\frac{d}{dt}(w, h) = \int_D \left(\sum_{i,j} w_i v_j \frac{\partial h_i}{\partial x_j} - v_j w_i \frac{\partial h_j}{\partial x_i} \right) dx = 0$$

since $\nabla \times h = 0$ implies $\partial h_i / \partial x_j = \partial h_j / \partial x_i$. Hence $(w, h) = 0$ for all $t \in [0, T]$.

Theorem III-1. *If $A \in C^{1+q}(\overline{D})$, $\nabla \cdot A = 0$, $A \cdot n = 0$ on bD, $B \in C^{1+q,0}(\overline{D}; T)$, then there exists a unique $v \in C^{1+q,r}(\overline{D}; T)$, $0 < r < q < 1$, such that $\partial v/\partial t$ exists in $C^{q,0}(\overline{D}; T)$ and P exists with $\nabla P \in C^{q,0}(\overline{D}; T)$, unique up to a function of t that may be added to P such that*

$$\text{(E}'\text{)} \qquad \begin{aligned} &\partial v/\partial t + (v \cdot \operatorname{grad})v = -\operatorname{grad} P + B, \\ &\nabla \cdot v = 0, \quad v \cdot n = 0 \quad \text{on } bD, \quad \text{and} \quad v(x, 0) = A(x) \end{aligned}$$

for $t \in [0, T]$ with T satisfying the constraints of Lemma III-2.

Proof. With A and B construct the mappings $F_1(\ )$ and $F_2(\ )$ of Theorems I-1 and II-1. The previous lemmas establish that the requirements for applying Schauder's fixed point theorem to the mapping $F_2(F_1(\cdot))$ from S (defined in the beginning of this section) into itself are met and hence there is a fixed point $w \in S$. Let $v = F_1(w)$; then v and $\nabla \times v = w$ give the weak solution to system (E) of Theorem II-1. To establish the differentiability in t of v, we note that for any $f \in C^{1+q}(\overline{D})$ we can use the representation result of Morrey [6] mentioned in Theorem I-1 to obtain $f = \nabla \times (\nabla \times p_f^-) - \nabla(\nabla \cdot p_f^-) + h_f^-$. Then

$$\begin{aligned}(v,f) &= (v, \nabla \times (\nabla \times p_f^-)) - (v, \nabla(\nabla \cdot p_f^-)) + (v, h_f^-)\\ &= (\nabla \times v, \nabla \times p_f^-) + (v, h_f^-) = (w, \nabla \times p_f^-) + (v, h_f^-).\end{aligned}$$

The t-derivatives of the two terms on the right-hand side exist by Theorems I-1 and II-1 and we get, letting $\nabla \times p_f^- = Q$,

$$(d/dt)(v,f) = (d/dt)(w,Q) + (-(v \cdot \mathrm{grad})v + B, h_f^-);$$

$$\begin{aligned}(d/dt)(w,Q) &= (\nabla \times v, (v \cdot \mathrm{grad})Q) + (((\nabla \times v) \cdot \mathrm{grad})v, Q) + (\nabla \times B, Q)\\ &= (\nabla \times v, (v \cdot \mathrm{grad})Q) - (v, ((\nabla \times v) \cdot \mathrm{grad})Q)\\ &\quad + (\nabla \times v, \nabla(v \cdot Q)) + (B, \nabla \times Q)\\ &= (-(v \cdot \mathrm{grad})v, \nabla \times Q) + (\nabla(v \cdot v), \nabla \times Q) + (B, \nabla \times Q)\\ &= (-(v \cdot \mathrm{grad})v + B, \nabla \times Q)\end{aligned}$$

since $v \cdot Q = 0$ on bD and $\nabla \times Q \cdot n = 0$ on bD. Now $R \equiv -(v \cdot \mathrm{grad})v + B \in C^{q,0}(\overline{D}; T.)$ We can use the previous representation theorem to obtain $R = \nabla \times (\nabla \times p_R^-) - \nabla(\nabla \cdot p_R^-) + h_R^-$. Then

$$(d/dt)(v,f) = (R, \nabla \times (\nabla \times p_f^-) + h_f^-) = (\nabla \times (\nabla \times p_R^-) + h_R^-, f).$$

Hence dv/dt exists in $C^{q,0}(\overline{D}, T)$. $dv/dt = \nabla \times (\nabla \times p_R^-) + h_R^-$ and furthermore $dv/dt + (v \cdot \mathrm{grad})v - B = \nabla(\nabla \cdot p_R^-)$. We let $P = -\nabla \cdot p_R^-$ and our solution is obtained. Also,

$$\begin{aligned}(v(0),f) &= (w(0), \nabla \times p_f^-) + (A, h_f^-)\\ &= (\nabla \times A, \nabla \times p_f^-) + (A, h_f^-)\\ &= (A, \nabla \times (\nabla \times p_f^-) - \nabla(\nabla \cdot p_f^-) + h_f^-)\\ &= (A,f)\end{aligned}$$

since $(A, \nabla(\nabla \cdot p_f^-)) = 0$ ($\nabla \cdot A = 0$ and $A \cdot n = 0$ on bD). This establishes that the initial value is achieved. To establish uniqueness: suppose v_i, P_i are solutions, $i = 1, 2$. Then, letting $\bar{v} = v_1 - v_2$, $\overline{P} = P_1 - P_2$,

$$d\bar{v}/dt + (\bar{v} \cdot \operatorname{grad})v_1 + (v_2 \cdot \operatorname{grad})\bar{v} = -\nabla \bar{P}.$$

Taking the L_2 inner-product with $\bar{v}$ and noting that $((v_2 \cdot \operatorname{grad})\bar{v}, \bar{v}) = 0 = (-\nabla \bar{P}, \bar{v})$, we get

$$(d/dt)\tfrac{1}{2}\|\bar{v}\|_{L_2}^2 = (-(\bar{v} \cdot \operatorname{grad})v_1, \bar{v}) \leq K\|v_{1,x}\|_C \|\bar{v}\|^2$$

with $\bar{v}(0) = 0$. Such a relationship can hold only if $\bar{v} = 0$. The qualified uniqueness of P follows from the uniqueness of v.

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