

## WEAK COMPACTNESS IN LOCALLY CONVEX SPACES

BY

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**ABSTRACT.** The notion of weak compactness plays a central role in the theory of locally convex topological vector spaces. However, in the statement of many theorems, completeness of the space, or at least quasi-completeness of the space in the Mackey topology is an important assumption. In this paper we extend the concept of weak compactness in a general way and obtain a number of useful particular cases. If we replace weak compactness by these generalized notions we can drop the completeness assumption from the statement of many theorems; for example, we generalize the classical theorems of Eberlein and Kreĭn. We then consider generalizations of semireflexivity and reflexivity and characterize these properties in terms of our previous ideas as well as in terms of known concepts. In most of the proofs we use techniques of nonstandard analysis.

1. **Notation and definitions.** Suppose  $\langle E, F \rangle$  is a separated pairing of vector spaces (so that we may regard  $F \subset E^\#$ ) and let  $\mathfrak{G}_1$  be a family of subsets of  $F$ . Corresponding to a map  $\phi$  from  $E$  into the set of finite subsets of a set  $S \in \mathfrak{G}_1$  we define a  $\sigma(E, F)$ -neighbourhood of each point  $x \in E$  by

$$U_x(\phi, S) = \{y: |f(y - x)| \leq 1 \text{ for all } f \in \phi(x)\}.$$

The system of  $\sigma(E, F)$ -neighbourhoods  $\{U_x(\phi, S): x \in E\}$  forms a covering of  $E$  which we call the  $(\phi, S)$ -covering of  $E$ . With this notation we introduce the following concept.

1.1. **Definition.** Let  $\langle E, F \rangle$  be a pairing and  $\mathfrak{G}_1$  a family of subsets of  $F$ . Then a subset  $A$  of  $E$  is  $\mathfrak{G}_1$ - $\sigma(E, F)$ -compact if, for each  $S \in \mathfrak{G}_1$  and each map  $\phi$  described above, the  $(\phi, S)$ -covering of  $E$  contains a finite subcover of  $A$ ; that is, there exists a finite subset  $\{x_1, \dots, x_n\}$  of  $E$  such that  $A \subset U_{x_1}(\phi, S) \cup \dots \cup U_{x_n}(\phi, S)$ .

We shall be mostly interested in  $\mathfrak{G}_1$ - $\sigma(E, F)$ -compactness when  $\mathfrak{G}_1$  generates a locally convex topology on  $E$ . If the  $\mathfrak{G}_1$ -topology is the Mackey topology  $\tau(E, F)$  we find it convenient to introduce another definition.

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1.2. **Definition.** Suppose  $\mathfrak{G}_1$  is the family  $\mathfrak{R}$  of all circled, convex,  $\sigma(F, E)$  compact subsets of  $F$ . If a subset  $A$  of  $E$  is  $\mathfrak{R}$ - $\sigma(E, F)$ -compact we say that  $A$  is *nearly  $\sigma(E, F)$ -compact* (or *nearly weakly compact*).

Although our proofs need only be altered slightly for the complex case we restrict our attention to real spaces. We find it easier too to use the term "polar" in the sense of "absolute polar". We denote  $E$  equipped with the  $\mathfrak{G}$ -topology by  $E(\mathfrak{G})$  and emphasise, for example, that if  $E$  is a locally convex topological vector space (hereafter to be abbreviated to LCTVS) then  $E''(\mathfrak{G})$  denotes the bidual  $E''$  equipped with the  $\mathfrak{G}$ -topology whilst  $E(\mathfrak{G})''$  denotes the bidual of  $E(\mathfrak{G})$ . Most of the standard texts on topological vector spaces are suitable references for this paper (see, for example, [6], [10]). We denote the natural embeddings of  $x$  and  $A$  in the bidual by  $\hat{x}$  and  $\hat{A}$  respectively. As in [9] we assume that our spaces together with the real number system are embedded in some full structure  $M$  and we develop our nonstandard theory in an enlargement  ${}^*M$  of  $M$ . We denote nonstandard entities in  ${}^*M$  by boldface type and when there is no confusion we omit the asterisk from standard entities in  ${}^*M$ . Finally we write  $a \simeq b$  whenever  $a, b \in {}^*R$  and  $a - b$  is infinitesimal and denote the standard part of a finite  $a$  by  ${}^o a$ .

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2.  $\mathfrak{G}_1$ - $\sigma(E, F)$ -compactness. Robinson [9, p. 90 and p. 93] introduces the related concepts of monad and near-standardness for points in the enlargement of a topological space. If  $E$  is a LCTVS it is easy to check that a point  $x \in {}^*E$  is weak near-standard (i.e. near-standard in the weak topology) if and only if there is an  $x \in E$  such that  $f(x) \simeq f(x)$  for all  $f \in E'$ . We generalize this property of points of  ${}^*E$  in the following way.

2.1. **Definition.** Let  $\langle E, F \rangle$  be a pairing and let  $\mathfrak{G}_1$  be a family of subsets of  $F$ . We say that a point  $x \in {}^*E$  is  $\mathfrak{G}_1$ - $\sigma(E, F)$ -near-standard if, for each  $S \in \mathfrak{G}_1$ , there is an  $x \in E$  such that  $|f(x - x)| \leq 1$  for all  $f \in S$ . Then we can generalize a result of Robinson [9, p. 93].

2.2. **Theorem.** Let  $\langle E, F \rangle$  be a pairing and let  $\mathfrak{G}_1$  be a family of subsets of  $F$ . A subset  $A$  of  $E$  is  $\mathfrak{G}_1$ - $\sigma(E, F)$ -compact if and only if each point  $x \in {}^*A$  is  $\mathfrak{G}_1$ - $\sigma(E, F)$ -near-standard.

**Proof.** Suppose there exists an  $x \in {}^*E$  which is not  $\mathfrak{G}_1$ - $\sigma(E, F)$ -near-standard. Then there is an  $S \in \mathfrak{G}_1$  such that given  $x \in E$  there is an  $f \in S$  such that  $|f(x - x)| > 1$ . That is to say, there is a map  $\phi$  such that  $x \notin {}^*U_x(\phi, S)$  for

each  $x \in E$ . Now  $A$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -compact, and so there is a finite subset  $\{x_1, \dots, x_n\}$  of  $E$  such that  $A \subset U_{x_1}(\phi, S) \cup \dots \cup U_{x_n}(\phi, S)$ . This equation can be formulated in our formal language which interpreted in  $*M$  yields

$$*A \subset *U_{x_1}(\phi, S) \cup \dots \cup *U_{x_n}(\phi, S).$$

We know that  $x$  does not belong to any of the sets of the right-hand side and consequently it does not belong to  $A$ .

Now on the other hand suppose  $A$  is not  $\mathcal{G}_1$ - $\sigma(E, F)$ -compact. Then there exists an  $S \in \mathcal{G}_1$  together with a map  $\phi$  such that the  $(\phi, S)$ -covering  $\Psi$  of  $E$  has no finite subcover of  $A$ . We define a binary relation  $R(U, y)$  to hold in  $M$  if and only if  $U \in \Psi$  and  $y \in A$  but  $y \notin U$ . By assumption  $R(U, y)$  is concurrent, so that by definition of  $*M$ , there is a point  $x \in *A$  such that  $x \notin *U_x(\phi, S)$  for all  $x \in E$ . This implies that  $x$  is not  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard for  $*\{f_1, \dots, f_n\} = \{*f_1, \dots, *f_n\}$ .

**2.3. Remark.** It is interesting to note that an analysis of the previous proof reveals that when defining  $\mathcal{G}_1$ - $\sigma(E, F)$ -compact sets it suffices to consider only those  $(\phi, S)$ -coverings of  $E$  for which  $\phi$  maps  $E$  into singletons of  $S$ .

As a consequence of Theorem 2.2 we note the following.

**2.4. Corollary.** Suppose  $\mathcal{G}_1$  equals the family of finite subsets  $\mathcal{F}$  of  $F$ . Then a subset  $A$  of  $E$  is  $\mathcal{F}$ - $\sigma(E, F)$ -compact if and only if  $A$  is  $\sigma(E, F)$ -precompact.

**Proof.** Suppose  $A$  is  $\mathcal{F}$ - $\sigma(E, F)$ -compact. Then given an  $x \in *A$  and  $S \in \mathcal{F}$  there exists an  $x \in E$  such that  $|f(x - x)| \leq 1$  for all  $f \in S$ . But  $*S = *\{f_1, \dots, f_n\} = \{*f_1, \dots, *f_n\}$  so that  $|f(x - x)| \leq 1$  for all  $f \in *S$ . This means that  $x$  is pre-near-standard in the  $\sigma(E, F)$ -topology and hence (see [8, p. 77])  $A$  is  $\sigma(E, F)$ -precompact. The converse is also similarly established.

For the remainder of this section we assume that  $\mathcal{G}_1$  is a family of weakly bounded subsets of  $F$  which cover  $F$  and such that the  $\mathcal{G}_1$ -topology on  $E$  is compatible with the duality  $\langle E, F \rangle$ . Furthermore, we assume that the polars of its sets form a basis of 0-neighbourhoods for the  $\mathcal{G}_1$ -topology. This requires of course that  $\mathcal{G}_1$  satisfies the two conditions:

- (I) If  $S_1, S_2 \in \mathcal{G}_1$  then there is an  $S_3 \in \mathcal{G}_1$  such that  $S_1 \cup S_2 \subset S_3$ ,
- (II) If  $\lambda$  is a real number and  $S \in \mathcal{G}_1$  there is an  $S_1 \in \mathcal{G}_1$  such that  $\lambda S \subset S_1$ .

**2.5. Definition.** Let  $E$  be a TVS. A point  $x \in *E$  is *bounded* if there is a bounded set  $B$  of  $E$  such that  $x \in *B$ .

**2.6. Lemma.** Suppose  $x$  is a bounded point in  $*E(\mathcal{G}_1)$ . Then  $x$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard if and only if  $\hat{x}$  is weak near-standard in  $F'(\mathcal{G}_1)$ .

**Proof.** We show the necessity of the condition first. We define  $x'' \in F'$  (the dual of  $F$  when equipped with the strong topology) by  $x''(f) = \mathcal{Q}[f(x)]$  for all  $f \in F$ . Let  $x''' \in F'(\mathcal{G}_1)'$ . The restriction of  $x'''$  to  $E$  may be assumed to be an element of  $F$  which we denote by  $g$ . If  $x$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard then given a set  $S \in \mathcal{G}_1$  there is a point  $x(S) \in E$  such that  $|f(x - x(S))| \leq 1$  for all  $f \in S$ . This implies that  $x'' - x(S) \in S^\circ$ , the polar of  $S$ . Consequently  $\{x(S)\}$  may be considered to be a net convergent to  $x''$  in the  $\mathcal{G}_1$ -topology. In particular,  $g(x(S)) \rightarrow x''(g)$  and  $x'''(x(S)) \rightarrow x'''(x'')$  so that  $x'''(x'') = x''(g)$ . Therefore,  $x'''(x'') = x''(g) \simeq g(x) = x'''(\hat{x})$  and thus  $\hat{x}$  is weak near-standard (see the comment at the beginning of this section).

Now let us suppose  $\hat{x}$  is weak near-standard. Then there exists an  $x'' \in F'$  such that  $x'''(x'') \simeq x'''(\hat{x})$  for all  $x''' \in F'(X_1)'$ .

It follows (see [9, p. 91]) that  $x''$  belongs to the weak closure of  $E$  in  $F'(\mathcal{G}_1)$  and thus to the closure of  $E$  in  $F'(\mathcal{G}_1)$ . This noted it is then an easy consequence that  $x$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard.

**2.7. Theorem.** *Let  $\langle E, F \rangle$  be a pairing and let  $\mathcal{G}_1$  be a family of subsets of  $F$  satisfying the above conditions. Then a subset  $A$  of  $E$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -compact if and only if  $\hat{A}$  is relatively weakly compact as a subset of  $F'(\mathcal{G}_1)$ .*

**Proof.** Suppose that  $A$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -compact. As  $\mathcal{G}_1$  covers  $F$  it follows that  $A$  is  $\sigma(E, F)$ -bounded and hence bounded. By Theorem 2.2 each point  $x \in {}^*A$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard and thus, by Lemma 2.6, each  $\hat{x}$  is weak near-standard in  $F'(\mathcal{G}_1)$ . But TVS's are regular and the necessity of the condition is then implied by a result of Luxemburg [8, p. 65]. Conversely, suppose  $A$  has the stated property. Again each point  $\hat{x} \in {}^*\hat{A}$  is weak near-standard in  $F'(\mathcal{G}_1)$  and it follows in turn that  $x$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard. Thus Theorem 2.2 implies the result.

Suppose  $\mathcal{G}_2$  is another family of subsets of  $F$  satisfying the previous conditions on  $\mathcal{G}_1$  and which also generates the  $\mathcal{G}_1$ -topology. Then we have the following immediate but noteworthy result.

**2.8. Corollary.** *The subset  $A$  of  $E$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -compact if and only if it is  $\mathcal{G}_2$ - $\sigma(E, F)$ -compact.*

**2.9. Lemma.** *Let  $A$  be a bounded subset of  $E(\mathcal{G}_1)$ . Then a point  $x \in {}^*A$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard if and only if given a set  $S \in \mathcal{G}_1$  there exists an  $x$  belonging to the convex hull of  $A$  such that  $|f(x - x)| \leq 1$  for all  $f \in S$ .*

**Proof.** It is immediate that the condition is sufficient. Therefore we suppose that  $x$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -near-standard. By Lemma 2.6 there exists an

$x'' \in F'(\mathcal{G}_1)$  such that  $x'''(\hat{x}) \simeq x'''(x'')$  for all  $x''' \in F'(\mathcal{G}_1)'$ . This implies  $x''$  belongs to the weak closure of  $A$  in  $F'(\mathcal{G}_1)$  and thus to the closure of its convex hull. Hence given  $S \in \mathcal{G}_1$  there is a point  $x$  belonging to the convex hull of  $A$  such that  $|f(x) - x''(f)| < 1$  for all  $f \in S$ . Therefore  $|f(x - x)| \leq 1$  for all  $f \in S$ .

**2.10. Theorem.** *Let  $A$  be a subset of  $E$ . If  $A$  is  $\mathcal{G}_1$ - $\sigma(E, F)$ -compact and the closed convex hull of  $A$  in  $E(\mathcal{G}_1)$  is complete, then  $A$  is relatively  $\sigma(E, F)$ -compact.*

**Proof.** By Lemma 2.9 given a point  $x \in {}^*A$  and a set  $S \in \mathcal{G}_1$  there is a point  $x(S)$  belonging to the convex hull of  $A$  such that  $|f(x(S) - x)| \leq 1$  for all  $f \in S$ . Thus  $\{x(S)\}$  is a Cauchy net in the convex hull of  $A$  (where the sets  $\{S\}$  are ordered by containment). By the completeness assumption  $\{x(S)\}$  has a limit  $x$  which must be the standard part of  $x$  in the  $\sigma(E, F)$ -topology. By [8, p. 65]  $A$  is relatively  $\sigma(E, F)$ -compact.

We comment that it is a simple consequence of Lemma 2.9 and the proof of Theorem 2.2 that, when defining  $\mathcal{G}_1$ - $\sigma(E, F)$ -compactness we may require that the finite subset  $\{x_1, \dots, x_n\}$  of Definition 1.1, be chosen in the convex hull of  $A$ .

Before we leave this section we wish to make some final observations. Suppose we introduce a locally convex topology on  $E$  and consider a family  $\mathcal{G}$  of bounded subsets of  $E'$  which we assume cover  $E'$ . Let  $F = E(\mathcal{G})'$  and let  $\mathcal{G}_1 = \mathcal{G}$ , the family of equicontinuous subsets of  $F$ . In case the  $\mathcal{G}$ -topology is compatible with the duality  $\langle E, E' \rangle$  we have as a consequence of Theorem 2.7 a similar result where  $F'(\mathcal{G}_1)$  is replaced by  $E''(\mathcal{G})$ . Indeed, a similar substitution exists if  $\mathcal{G}$  is a family of strongly bounded sets (so that the  $\mathcal{G}$ -topology on  $E''$  is a linear topology). We state this result as a theorem but as we do not use this fact directly later we do not generalize our earlier proof.

**2.11. Theorem.** *Suppose  $E$  is a LCTVS and that  $\mathcal{G}$  is a family of strongly bounded subsets of  $E'$ . Then  $A$  is  $\mathcal{G}$ - $\sigma(E, E(\mathcal{G})')$ -compact if and only if  $\hat{A}$  is relatively weakly compact in  $E''(\mathcal{G})$ .*

**3. Eberlein's theorem.** The main purpose of this section is to give a non-standard proof of Theorem 3.2. From this result we derive Eberlein's theorem. We suppose again that  $E$  is a LCTVS and, as in the previous section,  $\mathcal{G}$  denotes a family of  $\sigma(E', E)$ -bounded subsets of  $E'$  unless the contrary is stated. We use  $H$  to denote  $E(\mathcal{G})$  and if  $C$  is a subset of  $E$  then  $C^\perp$  denotes  $\{f \in E' : f(x) = 0 \text{ whenever } x \in C\}$ .

**3.1. Lemma.** *Let  $E$  be a LCTVS and suppose that  $x$  is a bounded point in  ${}^*E(\mathcal{G})$ . Then  $x$  is  $\mathcal{G}$ - $\sigma(E, E(\mathcal{G})')$ -near-standard if and only if, for each  $S \in \mathcal{G}$ , there is a finite subset  $C(S)$  of  $E$  such that for each  $f \in C(S)^\perp \cap S$ ,  $f(x) \leq 1$ .*

**Proof.** The necessity of the condition is clear. It therefore only remains to prove the sufficiency of the condition. As  $x$  is bounded in  $H$  we can define, as in Lemma 2.6,  $x'' \in H''$  by  $x''(f) = {}^\circ[f(x)]$  for all  $f \in H'$ . We claim that  $x'' \in \bar{H}$ , the closure of  $H$  in  $H''(\bar{\mathcal{G}})$ , and establish this claim by contradiction. If  $x'' \notin \bar{H}$  the separation theorem [10, p. 65] implies there is a continuous functional  $x''' \in H''(\bar{\mathcal{G}})'$  and real number  $c$  such that  $x'''(\bar{H}) < c - 1 < c < x'''(x'')$ , in other words, such that  $x'''(\bar{H}) = 0$  and  $x'''(x'') > 1$ . As  $x''' \in H''(\bar{\mathcal{G}})'$ ,  $x'''$  is bounded on a 0-neighbourhood on  $H''(\bar{\mathcal{G}})$  [10, p. 74] and hence we may assume  $x'''$  is bounded by less than unity on a polar  $S^\circ$  in  $H''$  of a set  $S \in \bar{\mathcal{G}}$ . It follows by Helly's theorem (see [5, Problem 16H, p. 151]) that there is an  $f \in S$  such that  $x'''(x'') = x''(f)$  and such that  $f(x) = 0$  for every  $x \in C(S)$ . But then  $f(x) > 1$  and  $f \in C(S)^\perp \cap S$ . This is a contradiction and so  $x'' \in \bar{H}$ . But then  $x$  is  $\bar{\mathcal{G}}$ - $\sigma(E, E(\bar{\mathcal{G}})')$ -near-standard. For suppose  $S \in \bar{\mathcal{G}}$ , then there is an  $x$  belonging to  $H$  such that  $|x''(f) - f(x)| < 1$  for all  $f \in S$ . But then  $|f(x - x'')| \leq 1$  for all  $f \in S$ .

**3.2. Theorem.** Let  $E$  be a LCTVS and let  $A$  be a subset of  $E$ . If  $\hat{A}$  is relatively countably weakly compact in  $H''(\bar{\mathcal{G}})$  then the  $\sigma(E, E(\bar{\mathcal{G}})')$ -closure of  $A$  is  $\bar{\mathcal{G}}$ - $\sigma(E, E(\bar{\mathcal{G}})')$ -compact.

**Proof.** Let  $\bar{A}$  denote the closure of  $A$  in the  $\sigma(E, E(\bar{\mathcal{G}})')$ -topology. Then  $\bar{A}$  is bounded in  $E(\bar{\mathcal{G}})$  so that, if  $x \in \bar{A}$ ,  $x$  is a bounded point in  ${}^*E(\bar{\mathcal{G}})$ . Let us suppose  $\bar{A}$  is not  $\bar{\mathcal{G}}$ - $\sigma(E, E(\bar{\mathcal{G}})')$ -compact. Then by Theorem 2.2 and Lemma 3.1 for some  $x \in \bar{A}$  there is a set  $S \in \bar{\mathcal{G}}$  such that for each finite set  $C$  of  $E$  there is an  $f \in C^\perp \cap S$  such that  $f(x) > 1$ .

We construct three sequences  $\{x_n\} \subset \bar{A}$ ,  $\{y_n\} \subset A$  and  $\{f_n\} \subset S$  in the following manner. We choose  $x_0 = y_0$  arbitrarily in  $A$ , then  $f_0 \in S$  such that  $f_0(y_0) = 0$  and  $f_0(x) > 1$ . Now the statement

$$\exists x(x \in \bar{A} \wedge f_0(x) > 1)$$

holds in  ${}^*M$  (for  $x$  satisfies both conditions), and so it is true in  $M$ . Hence there is an  $x_1 \in \bar{A}$  such that  $f_0(x_1) > 1$ . As  $x_1 \in \bar{A}$  there exists  $y_1 \in A$  such that  $|f_0(x_1 - y_1)| < \frac{1}{2}$ . Suppose now that we have chosen  $y_k, x_k$  for  $k = 0, 1, \dots, n-1$ , and  $f_j$  for  $j = 0, 1, \dots, n-2$  satisfying

$$f_j(y_i) = 0 \text{ and } f_j(x) > 1, \quad i = 0, 1, \dots, j,$$

$$f_j(y_k) > 1, \quad 0 \leq j < k \leq n-1,$$

$$|f_j(y_i - x_i)| < \frac{1}{2}, \quad j = 0, 1, \dots, i-1.$$

Then we choose  $f_{n-1} \in S$  such that  $f_{n-1}(y_i) = 0$  and  $f_{n-1}(x) > 1$ ,  $i = 0, 1, \dots, n-1$ . The abbreviated statement

$$\exists x((x \in \bar{A}) \wedge (f_i(x) > 1, \quad i = 0, 1, \dots, n-1))$$

is true in  ${}^*M$  (for again  $x$  satisfies these conditions) and so it is true in  $M$ . This means we can choose  $x_n \in \bar{A}$  such that  $f_i(x_n) > 1$ ,  $0 \leq i < n$ , and in turn  $y_n \in A$  such that  $|f_k(x_n - y_n)| < \frac{1}{2}$ ,  $0 \leq j < n$ . Therefore we can choose sequences  $\{y_n\}$  in  $A$ ,  $\{f_n\}$  in  $S$  satisfying  $f_n(y_i) = 0$  if  $n \geq i$ ,  $f_n(y_i) > \frac{1}{2}$  if  $n < i$ . As  $A$  is relatively countably weakly compact in  $H''(\bar{\mathcal{G}})$ ,  $\{y_n\}$  has a weak limit point  $x''$  in  $H''(\bar{\mathcal{G}})$ . Subsequently,  $f_k(y_i) - x''(f_k) \geq \frac{1}{2}$ ,  $0 \leq i \leq k$ . Now, because  $S^\circ$  is a 0-neighbourhood in  $H''(\bar{\mathcal{G}})$ , the Banach-Alaoglu theorem implies  $S^\circ$  is  $\sigma(H''(\bar{\mathcal{G}})', H'')$ -compact. Consequently  $\{f_k\}$  has a limit point  $x'''$  in the  $\sigma(H''(\bar{\mathcal{G}})', H'')$ -topology. But then  $x'''(y_i) - x'''(x'') \geq \frac{1}{2}$ ,  $i = 1, 2, \dots$ , contradicting the assumption that  $x''$  is a weak limit point of  $\{y_n\}$  in  $H''(\bar{\mathcal{G}})$ .

If  $\mathcal{G}$  is a family of strongly bounded subsets of  $E'$  we may replace  $H''(\bar{\mathcal{G}})$  by  $E''(\mathcal{G})$  (cf. Theorems 2.7 and 2.11). As the proof is a straightforward development of the above proof we omit it.

**3.3. Theorem.** *Suppose  $\mathcal{G}$  is a family of strongly bounded subsets of  $E'$ . If  $A$  is relatively countably weakly compact in  $E''(\mathcal{G})$ , then the  $\sigma(E, E(\mathcal{G})')$ -closure of  $A$  is  $\bar{\mathcal{G}}\text{-}\sigma(E, E(\mathcal{G})')$ -compact.*

**3.4. Corollary (Eberlein's theorem).** *Let  $E$  be a LCTVS and let  $A$  be a subset of  $E$ . If  $A$  is relatively countably weakly compact then  $A$  is nearly weakly compact. Furthermore, if the closed convex hull of  $A$  is complete in the Mackey topology, then  $A$  is relatively weakly compact.*

**Proof.** By Theorem 3.2  $A$  is nearly weakly compact. The end remark is a consequence of Theorem 2.10.

**3.5. Remarks.** Eberlein's theorem in its general context was first established by Grothendieck [3]. We comment, too, that Theorem 2.7 ensures that the converse of Theorem 3.2 holds. It would be interesting to obtain the natural generalization of Kreĭn's theorem by nonstandard methods. We give the result here as a corollary to Kreĭn's theorem and Theorem 2.7.

**3.6. Corollary.** *Let  $E$  be a LCTVS and  $A$  be a subset of  $E$ . If  $A$  is  $\bar{\mathcal{G}}\text{-}\sigma(E, E(\mathcal{G})')$ -compact then its convex hull is also  $\bar{\mathcal{G}}\text{-}\sigma(E, E(\mathcal{G})')$ -compact.*

**Proof.** Suppose  $B$  is the convex hull of  $A$ . Since every Cauchy net from  $B$  in  $H''(\bar{\mathcal{G}})$  has a limit point the closure of  $B$  is complete. As  $A$  is  $\bar{\mathcal{G}}\text{-}\sigma(E, E(\mathcal{G})')$ -compact  $A$  is relatively weakly compact by Theorem 2.7. Therefore Kreĭn's theorem ensures that  $B$  is relatively weakly compact in  $H''(\bar{\mathcal{G}})$  and consequently  $B$  is  $\bar{\mathcal{G}}\text{-}\sigma(E, E(\mathcal{G})')$ -compact using Theorem 2.7 once more.

Finally we note the following result which is contained in the proof of Theorem 3.2 (cf. Condition 9 [4]).

**3.7. Theorem.** *Let  $E$  be a LCTVS and let  $A$  be a subset of  $E$ . Then  $A$  is not  $\widetilde{\mathfrak{G}}\text{-}\sigma(E, E(\mathfrak{G})')$ -compact if and only if there exists a sequence  $\{y_n\}$  in  $A$  and a sequence  $\{f_n\}$  in a set  $S \in \widetilde{\mathfrak{G}}$  such that  $f_n(y_i) = 0$  if  $n \geq i$ ,  $f_n(y_i) > r$  if  $n < i$ , for some positive  $r$ .*

**4. Generalizations of semireflexive spaces.** We intend now to consider a class of generalizations of semireflexive spaces. We find again that nonstandard techniques are helpful in the investigation of these properties. Before we consider these generalizations though we prove a nonstandard variant of Helly's theorem [12, p. 103].

**4.1. Lemma.** *Let  $\langle E, F \rangle$  be a pairing, let  $\phi \in F$ , and let  $B$  be a circled, convex,  $\sigma(E, F)$ -bounded subset of  $E$  such that  $\phi$  is bounded by 1 on  $B^\circ$ . Then there exists a positive infinitesimal  $\delta$  and  $x \in (1 + \delta)^*B$  such that  $\phi(f) = f(x)$  for all  $f \in F$ .*

**Proof.** By Helly's theorem [5, Problem 16H, p. 151] for each finite dimensional subspace  $S$  of  $F$  and each real  $r > 0$  the set  $G(S, r) = \{x \in E: x \in (1 + r)S \text{ and } \phi(f) = f(x) \text{ for all } f \in S\}$  is nonempty. Further the family  $G$  of all such sets has the finite intersection property. Let  $x$  belong to the monad of  $G$ . Then  $\phi(f) = f(x)$  for all  $f \in F$ . Further  $\{0 < \delta \in {}^*R: x \in (1 + \delta)^*B\}$  is internal and contains the positive real numbers. Thus  $x \in (1 + \delta)^*B$  for some positive infinitesimal  $\delta$ .

Having established Lemma 4.1 we now assume that  $\mathfrak{G}$  denotes a covering of  $E'$  by strongly bounded subsets which satisfy the conditions (I) and (II) stated in §2: thus the polars of  $S^\circ$  of the sets  $S \in \mathfrak{G}$  form a basis of 0-neighbourhoods in  $E''$ . Initially we do not assume that the  $\mathfrak{G}$ -topology on  $E$  is consistent with the duality  $\langle E, E' \rangle$ .

**4.2. Definition.** Let  $E$  be a LCTVS. We say  $E$  is  $\mathfrak{G}$ -semireflexive if  $E$  is dense in  $E''(\mathfrak{G})$ .

**4.3. Theorem.** *Let  $E$  be a LCTVS. Then  $E$  is  $\mathfrak{G}$ -semireflexive if and only if each bounded set of  $E$  is  $\mathfrak{G}\sigma(E, E')$ -compact.*

**Proof.** Suppose first that  $E$  is  $\mathfrak{G}$ -semireflexive and let  $B$  be a bounded set of  $E$ . For an arbitrary  $x \in {}^*B$  it is sufficient, by Theorem 2.2, to show that  $x$  is  $\mathfrak{G}\sigma(E, E')$ -near-standard. We define  $x'' \in E''$  by

$$x''(f) = {}^\circ[f(x)] \quad \text{for all } f \in E'.$$

Now let  $S \in \mathfrak{G}$ . By assumption there exists an  $x \in E$  such that  $|f(x) - x''(f)| < 1$  for all  $f \in S$ . But this implies that  $|f(x) - f(x)| \leq 1$  for all  $f \in S$ , and consequently that  $x$  is  $\mathfrak{G}\sigma(E, E')$ -near-standard.



Conversely, let us suppose each bounded set of  $E$  is  $\mathfrak{G}\sigma(E, E')$ -compact. Accordingly, each bounded point  $x \in {}^*E$  is  $\mathfrak{G}\sigma(E, E')$ -near-standard. Consider an arbitrary element  $x''$  of  $E''$ . By Lemma 4.1 there exists a bounded point  $x \in {}^*E$ , such that  $f(x) = x''(f)$  for all  $f \in E'$ . Let  $S \in \mathfrak{G}$ . As  $x$  is  $\mathfrak{G}\sigma(E, E')$ -near-standard there exists an  $x \in E$  such that  $|f(x) - f(x)| < 1$  for all  $f \in S$ . This implies that  $|f(x) - x''(f)| \leq 1$  for all  $f \in S$ . Therefore, as  $S$  was chosen arbitrarily, and the family of polars  $\{S^\circ: S \in \mathfrak{G}\}$  forms a basis of 0-neighbourhoods of  $E''(\mathfrak{G})$ ,  $E$  is dense in  $E''(\mathfrak{G})$ .

Suppose now that the  $\mathfrak{G}$ -topology on  $E$  is consistent with duality  $\langle E, E' \rangle$ . Corollary 2.8 then implies that  $\mathfrak{G}\sigma(E, E')$ -compact sets are  $\overline{\mathfrak{G}}\sigma(E, E')$ -compact. It is, therefore, an easy consequence of Lemma 2.9 that if  $E$  is  $\mathfrak{G}$ -semireflexive and  $x'' \in E''$  we can choose a bounded set  $\{x(S)\}$  in  $E$  convergent to  $x''$  in the  $\mathfrak{G}$ -topology. It follows that if  $E$  is  $\mathfrak{G}$ -semireflexive and quasi-complete in the  $\mathfrak{G}$ -topology then  $E$  is semireflexive. Indeed it is true that if the quasi-completion of  $E(\mathfrak{G})$  is semireflexive then  $E$  is  $\mathfrak{G}$ -semireflexive. The converse is true if  $E$  is distinguished.

**4.4. Theorem.** *Suppose the LCTVS  $E$  is distinguished. Then  $E$  is  $\mathfrak{G}$ -semireflexive if and only if the quasi-completion of  $E(\mathfrak{G})$  is semireflexive.*

**Proof.** Let  $\overline{E}$  denote the quasi-completion of  $E(\mathfrak{G})$  and assume  $\overline{E}$  is semireflexive. Then since  $E'' \subset \overline{E}''$  it follows that  $E$  is semireflexive. We next prove the necessity of the condition. Note first that as  $E$  is distinguished  $E''$  is the quasi-completion of  $E$  [6, p. 306]. Therefore, since  $E$  is  $\mathfrak{G}$ -semireflexive,  $E''$  is the quasi-completion of  $E(\mathfrak{G})$ . Furthermore the strong topologies  $\beta(E', E)$  and  $\beta(E', E'')$  are identical on  $E'$  for  $E'$  is barrelled [6, p. 306]. Thus  $E''$  is semireflexive establishing that the quasi-completion of  $E(\mathfrak{G})$  is semireflexive.

Let  $S$  be a subset of  $E'$  and  $F$  be a subspace of  $E$ . Suppose the set of restrictions of functionals in  $S$  to  $F$  is denoted by  $S_F$ . Then the  $\mathfrak{G}$ -topology on  $E$  induces a topology on  $F$  which is the  $\mathfrak{G}_F$ -topology, where  $\mathfrak{G}_F = \{S_F: S \in \mathfrak{G}\}$ . If  $F$  is  $\mathfrak{G}_F$ -semireflexive we agree to say that  $F$  is  $\mathfrak{G}$ -semireflexive. With this notation we prove the following generalization of a result of Fleming [2, Theorem 4.1].

**4.5. Theorem.** *Let  $E$  be a LCTVS. Then  $E$  is  $\mathfrak{G}$ -semireflexive if and only if every separable subspace is  $\mathfrak{G}$ -semireflexive.*

**Proof.** We prove the necessity of the condition first. Suppose that  $F$  is any subspace of  $E$ . Let  $B$  be a bounded set in  $F$  and let  $x \in {}^*B$ . As  $E$  is  $\mathfrak{G}$ -semireflexive  $x$  is  $\mathfrak{G}\sigma(E, E')$ -near-standard and so, by Lemma 2.9, for each  $S \in \mathfrak{G}$  there is a point  $x$  belonging to the convex hull of  $B$  such that  $|f(x) - x| \leq 1$

for all  $f \in S$ . This implies that  $x$  is  $\mathfrak{G}_F$ - $\sigma(F, F')$ -near-standard so that  $B$  is  $\mathfrak{G}_F$ - $\sigma(F, F')$ -compact. That  $F$  is  $\mathfrak{G}$ -semireflexive therefore follows by Theorem 4.3.

Next we prove the sufficiency of the condition. Suppose in fact that  $E$  is not  $\mathfrak{G}$ -semireflexive. Then there is a bounded subset  $B$  of  $E$  which is not  $\mathfrak{G}$ - $\sigma(E, E')$ -compact. Thus Theorem 3.2 implies there is a sequence  $\{x_n\}$  in  $B$  such that  $\{x_n\}$  has no weak limit point in  $E''(\mathfrak{G})$ . Let  $F$  be the linear span of  $\{x_n\}$ . Then  $F$  is a separable space and  $\{x_n\}$  is a bounded sequence in  $F$ . Suppose that  $\{x_n\}$  has a weak limit point  $y''$  in  $F''(\mathfrak{G})$ . We define an element  $x'' \in E''$  by

$$x''(f) = y''(f/F) \quad \text{for all } f \in E'.$$

It follows that  $x''$  is a weak limit point of  $\{x_n\}$  in  $E''(\mathfrak{G})$ , which is a contradiction.

If  $\mathfrak{G}$  generates the Mackey topology the notion of  $\mathfrak{G}$ -semireflexivity is of special interest.

**4.6. Definition.** Let  $\mathfrak{R}$  be the family of circled, convex,  $\sigma(E', E)$ -compact subsets of  $E'$ . If  $E$  is  $\mathfrak{R}$ -semireflexive we say  $E$  is *nearly semireflexive*.

As a consequence of Theorem 4.5 we have the following.

**4.7. Corollary.** *Let  $E$  be a LCTVS and suppose that  $E$  is quasi-complete in the Mackey topology. Then  $E$  is semireflexive if and only if each separable subspace is nearly semireflexive.*

**Proof.** The necessity of the condition is obvious. The sufficiency is an immediate consequence of Theorem 4.5 and the comment preceding Theorem 4.4.

Our next result provides examples of nearly semireflexive spaces. Before stating it we need recall that a Banach space  $X$  is said to be *almost reflexive* [7] if every bounded sequence contains a weak Cauchy subsequence. Also here  $K(X)$  is used to denote the weak\* sequential closure of  $X$  in  $X''$  ( $K(X)$  is sometimes termed the *Baire subspace of class one*).

**4.8. Proposition.** *Let  $X$  be an almost reflexive Banach space. Then  $K(X)$  equipped with the  $X'$ -topology is nearly semireflexive.*

**Proof.** Let  $E$  denote  $(K(X), \sigma(K(X), X'))$ . It suffices by Theorem 4.3 to show that each bounded set in  $E$  is nearly weakly compact. If  $B$  is such a set it is bounded in the norm topology on  $K(X)$  by the uniform boundedness principle. Consequently there exists a bounded set  $A$  in  $X$  such that  $B$  is contained in the closure of  $A$  considered as a subset of  $E$ . Now  $A$  is weakly conditionally sequentially compact, so that  $A$  is a weakly sequentially compact subset in  $E$ . Corollary 3.4 implies that  $B$  is nearly weakly compact which establishes the result.

**4.9. Example.** We refer the reader to Day [1, p. 28] for the basic properties of the following spaces. Let  $\Gamma$  be an arbitrary set and let

$m(\Gamma)$  be the space of all bounded real functions on  $\Gamma$  with norm defined by  $\|x''\| = \sup \{|x''(\gamma)| : \gamma \in \Gamma\}$ ,

$m_0(\Gamma)$  be the subspace of all those  $x'''$  in  $m(\Gamma)$  which vanish except on a countable set, and

$l_1(\Gamma)$  be the space of real functions  $f$  on  $\gamma$  for which  $\|f\| = \sum_{\gamma} |f(\gamma)| < \infty$ .

If  $E$  denotes  $m_0(\Gamma)$  with the  $l_1(\Gamma)$ -topology  $E$  is nearly semireflexive by Proposition 4.8 since  $c_0(\Gamma)$  is almost reflexive. However, if  $\Gamma$  is uncountable,  $E$  is not semireflexive since  $E'_\beta = l_1(\Gamma)$  so that  $E'' = m(\Gamma)$ . On the other hand if  $G$  is a separable subspace of  $E$  the set  $\{\gamma \in \Gamma : x''(\gamma) \neq 0 \text{ for some } x'' \in G\}$  is countable. From this observation it is an easy consequence that a closed separable subspace of  $E$  is quasi-complete and hence semireflexive. This example, together with our previous results, clarifies the comment made by Fleming after the proof of his Theorem 4.1 [2, p.77].

The following is a useful characterization of nearly semireflexive spaces.

**4.10. Theorem.** *Let  $E$  be a LCTVS. Then  $E$  is nearly semireflexive if and only if the topology  $\tau(E'', E')$  coincides on  $E$  with the topology  $\tau(E, E')$ .*

**Proof.** Let us suppose firstly that  $E$  is nearly semireflexive. Then every circled, convex,  $\sigma(E', E)$ -compact set  $S$  is  $\sigma(E', E'')$ -compact and the topologies are therefore equivalent. Conversely, we know that  $E''$  is obtained from  $E$  by taking the  $\sigma(E'', E')$ -closure points of the bounded sets in  $E$ . Since these can be taken to be circled and convex it is sufficient to consider the  $\sigma(E'', E')$ -closure points. But by assumption this implies that  $E$  is nearly semireflexive.

So far we have only considered generalizations of semireflexivity. There is a natural generalization of reflexivity too.

**4.11. Definition.** Let  $E$  be a LCTVS. We say  $E$  is *nearly reflexive* if  $E$  is nearly semireflexive and  $E''$  induces the topology on  $E$ ; i.e., if  $E$  is nearly semireflexive and  $E$  is infrabarrelled (see [10, p. 144]).

It is possible to extend a number of results using this definition. We prove one here.

**4.12. Theorem.** *Suppose the strong dual of a LCTVS  $E$  is semireflexive. Then  $E_\tau$  is nearly reflexive.*

**Proof.** Let  $B$  be a strongly bounded set in  $E'$ . It follows from the semireflexivity of  $E'_\beta$  that  $B$  is compact. This implies  $E$  is infrabarrelled. We complete the proof once we show  $E$  is nearly semireflexive. Suppose  $S$  is a circled, convex  $\sigma(E', E)$ -compact set in  $E'$ , then, as  $S$  is strongly bounded,  $S$  is  $\sigma(E', E'')$ -compact. Therefore,  $E$  is nearly semireflexive by Theorem 4.10.

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