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α_T IS FINITE FOR $\aleph_1$-CATEGORICAL T

BY

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ABSTRACT. Let T be a complete countable $\aleph_1$-categorical theory. **Definition.** If $\mathfrak{A}$ is a model of T and A is a 1-ary formula in $L(\mathfrak{A})$ then A has rank 0 if $A(\mathfrak{A})$ is finite. $A(\mathfrak{A})$ has rank n degree m iff for every set of $m+1$ formulas $B_1, \cdots, B_{m+1} \in S_1(L(\mathfrak{A}))$ which partition $A(\mathfrak{A})$ some $B_i(\mathfrak{A})$ has rank $\leq n-1$. **Theorem.** *If T is $\aleph_1$-categorical then for every $\mathfrak{A}$ a model of T and every $A \in S_1(L(\mathfrak{A}))$, $A(\mathfrak{A})$ has finite rank.* **Corollary.** *α_T is finite. The methods derive from Lemmas 9 and 11 in "On strongly minimal sets" by Baldwin and Lachlan. α_T is defined in "Categoricity in power" by Michael Morley.*

In [4] Morley assigns an ordinal α_T to each complete theory T. He conjectures that if T is $\aleph_1$-categorical α_T is finite. In this paper we prove this conjecture.

We assume familiarity with [1] and [4] but for convenience we list the principal results and definitions from those papers which are used here. Our notation is the same as in [1] with the following exceptions.

We deal with a countable first order language L. We may extend the language L in several ways. If $\mathfrak{A}$ is an L-structure there is a natural extension $L(\mathfrak{A})$ of L obtained by adjoining to L a constant a for each $a \in |\mathfrak{A}|$ (the universe of $\mathfrak{A}$). For each sentence $A(a_1, \cdots, a_n) \in L(\mathfrak{A})$ we say $\mathfrak{A}$ satisfies $A(a_1, \cdots, a_n)$ and write $\mathfrak{A} \models A(a_1, \cdots, a_n)$ if in Shoenfield's notation $\mathfrak{A}(A(a_1, \cdots, a_n)) = T$ [7, p. 19]. If $\mathfrak{A}$ is an L-structure and X is a subset of $|\mathfrak{A}|$ then $L(X)$ is the language obtained by adjoining to L a name x for each $x \in X$. $(\mathfrak{A}, X)$ is the natural expansion of $\mathfrak{A}$ to an $L(X)$-structure. A structure $\mathfrak{B}$ is an *inessential expansion* [7, p. 141] of an L-structure $\mathfrak{A}$ if $\mathfrak{B} = (\mathfrak{A}, X)$ for some $X \subseteq |\mathfrak{A}|$.

$S_n(L)$ denotes the set of formulas of L with free variables among $v_0, \cdots, v_{n-1}$. If A is a formula such that $u_1, \cdots, u_n$ in the natural order are the free variables in A, then $A(\mathfrak{A})$ is the set of n-tuples $b_1, \cdots, b_n$ such that

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$\mathfrak{A} \models A_{u_1,\cdots,u_n}(b_1, \cdots, b_n)$. If p is a unary predicate symbol we abbreviate $pv_0(\mathfrak{A})$ by $p(\mathfrak{A})$.

A consistent set of L-sentences is a *theory* in L. If T and T' are theories in L then T' extends T if $T \subseteq T'$. If T is a theory in a language L then T' is an *inessential extension* of T if there is a model $\mathfrak{A}$ of T and a subset X of $|\mathfrak{A}|$ such that $T' = Th(\mathfrak{A}, X)$ (i.e., the set of all sentences in $L(X)$ true of $(\mathfrak{A}, X)$). T' is a *principal extension* of T if T' is an inessential extension of T by a finite number of constants and a set of nonlogical axioms for T' can be obtained by adjoining a finite set of sentences to a set of nonlogical axioms for T.

Let Γ be a subset of $S_k(L)$. Then Γ is a *k-type* in T if there is some model $\mathfrak{A}$ of T and elements $a_1, \cdots, a_k \in |\mathfrak{A}|$ such that $\mathfrak{A} \models A(a_1, \cdots, a_k)$ if and only if $A \in \Gamma$. If $\mathfrak{A}$ is a model of T and $X \subseteq |\mathfrak{A}|$ then a k-type Γ *is realized in* X if there exists $x_1, \cdots, x_k \in X$ such that $\mathfrak{A} \models A(x_1, \cdots, x_k)$ for each $A \in \Gamma$. A k-type Γ is a principal k-type in T if there is a formula $A \in S_k(L(\mathfrak{A}))$ such that, for each formula B in Γ, $\mathfrak{A} \models \forall v_0, \cdots, \forall v_{k-1}(A \rightarrow B)$. Since T is complete there is one 0-*type* truth.

Following Morley [4] we assume that each $T = \Sigma^*$ for some Σ and thus that each n-ary formula Φ is equivalent in T to an n-ary relation A. $\mathfrak{N}(T)$ is a set of all substructures of models of T. The following summarizes with slight changes in notation the second paragraph of §2 in [4]. If $\mathfrak{A}$ is an L-structure $\mathfrak{D}(\mathfrak{A})$ is the set of all open sentences in $L(\mathfrak{A})$ which are true in $(\mathfrak{A}, |A|)$. If $\mathfrak{A} \in \mathfrak{N}(T)$, $T(\mathfrak{A}) = \mathfrak{D}(\mathfrak{A}) \cup T$ is a complete theory in $L(\mathfrak{A})$. Let $S_k(\mathfrak{A})$ denote the Boolean algebra whose elements are the equivalence classes into which $S_k(L(\mathfrak{A}))$ is partitioned by the relation of equivalence in $T(\mathfrak{A})$, and whose operations of intersection, union, and complementation are those induced by conjunction, disjunction and negation respectively. The Stone space of $S_1(\mathfrak{A})$, the set of dual prime ideals of $S_1(\mathfrak{A})$, is a topological space denoted $S(\mathfrak{A})$. A dual prime ideal of $S_k(\mathfrak{A})$ is a k-type of $T(\mathfrak{A})$. This is a special case of the definition of k-type in the preceding paragraph. Note that, if $p \in S(\mathfrak{A})$ and $\mathfrak{A}'$ is an inessential expansion of $\mathfrak{A}$, p is naturally a member of $S(\mathfrak{A}')$.

In [4] Morley makes the following definition. For each ordinal α and each $\mathfrak{A} \models \in \mathfrak{N}(T)$, subspaces $S^\alpha(\mathfrak{A})$ and $\mathrm{Tr}^\alpha(\mathfrak{A})$ of $S(\mathfrak{A})$ are defined inductively by

(1) $S^\alpha(\mathfrak{A}) = S(\mathfrak{A}) - \bigcup_{\beta<\alpha}\mathrm{Tr}^\beta(\mathfrak{A})$,

(2) $p \in \mathrm{Tr}^\alpha(\mathfrak{A})$ if (i) $p \in S^\alpha(\mathfrak{A})$ and (ii) for every map $(f^*: S(\mathfrak{B}) \rightarrow S(\mathfrak{A}))$ where $\mathfrak{B} \in \mathfrak{N}(T)$ and f is a monomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, $f^{*-1}(p) \cap S^\alpha(\mathfrak{B})$ is a set of isolated points in $S^\alpha(\mathfrak{B})$. (See [4, p. 519] for the definition of f^*.)

If $i_{\mathfrak{A}\mathfrak{B}}$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$ then $i^*_{\mathfrak{A}\mathfrak{B}}$ maps $S(\mathfrak{B})$ onto $S(\mathfrak{A})$. Note that $q \in i^{*-1}_{\mathfrak{A}\mathfrak{B}}(p)$ is equivalent to $q \cap S_1(L(\mathfrak{A})) = p$.

An element p of $S(\mathfrak{A})$ is algebraic if $p \in \mathrm{Tr}^0(\mathfrak{A})$; p is transcendental in rank α if $p \in \mathrm{Tr}^\alpha(\mathfrak{A})$. If $A \in S_1(L(\mathfrak{A}))$, $U_A = \{p \mid p \in S(\mathfrak{A}) \wedge A \in p\}$.

The following definitions are originally due to Marsh [3]. Let $\mathfrak{A}$ be an L-structure and X a subset of $|\mathfrak{A}|$. The *algebraic closure* of X, denoted by $\mathrm{cl}(X)$, is the union of all finite subsets of $|\mathfrak{A}|$ definable in $(\mathfrak{A}, X)$. X *spans* Y if $Y \subseteq \mathrm{cl}(X)$. X is *independent* if for each $x \in X$, $x \notin \mathrm{cl}(X - \{x\})$. X is a *basis* for Y if X is an independent subset of Y which spans Y. If every basis for Y has the same cardinality μ, we define the *dimension* of Y to be μ and write $\dim(Y) = \mu$.

Let $\mathfrak{A}$ be an L-structure. A subset X of $|\mathfrak{A}|$ is *minimal* in $\mathfrak{A}$ if X is infinite, definable in $\mathfrak{A}$, and for any subset Y of $|\mathfrak{A}|$ which is definable in $\mathfrak{A}$ either $Y \cap X$ or $X - Y$ is finite.

If $D \in S_1(L(\mathfrak{A}))$ and $X = D(\mathfrak{A})$ then X is *strongly minimal in* $\mathfrak{A}$ if for any elementary extension $\mathfrak{B}$ of $\mathfrak{A}$, $D(\mathfrak{B})$ is minimal in $\mathfrak{B}$. Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be models of a complete theory T. Since up to isomorphism any two models of T have a common elementary extension, $D(\mathfrak{A}_0)$ is strongly minimal in $\mathfrak{A}_0$ if and only if $D(\mathfrak{A}_1)$ is strongly minimal in $\mathfrak{A}_1$. Thus, without ambiguity we define a formula $D \in S_1(L)$ to be *strongly minimal in* T if there is a model $\mathfrak{A}$ of T such that $D(\mathfrak{A})$ is strongly minimal in $\mathfrak{A}$.

We will refer to the following theorem which is Theorem 5 in [1].

Theorem 0. *If $\mathfrak{A}$ is a model of an $\aleph_1$-categorical theory T then $\mathfrak{A}$ is homogeneous.*

Our first step in the proof of Morley's conjecture is to introduce a concept of the rank of a formula in a model of a theory. We will compare this notion with three other sorts of rank.

If $\mathfrak{A}$ is an L-structure and A is an element of $S_1(L(\mathfrak{A}))$ then we defined A to be minimal in $\mathfrak{A}$ if $A(\mathfrak{A})$ is infinite and, for each formula $B \in S_1(L(\mathfrak{A}))$, $(B \wedge A)(\mathfrak{A})$ or $(\sim B \wedge A)(\mathfrak{A})$ is finite. We will define a notion of rank of a formula in a model such that minimal formulas have rank one.

Well order the class X consisting of $\{-1\}$ and the direct product of the class of all ordinals with the positive integers by placing -1 first in the order and then following the natural lexicographic order. For each L-structure $\mathfrak{A}$ define $f_{\mathfrak{A}}$: $X \to 2^{S_1(L(\mathfrak{A}))}$ by induction

$$f_{\mathfrak{A}}(-1) = \{A \in S_1(L(\mathfrak{A})) \mid A(\mathfrak{A}) = \varnothing\}.$$

$A \in f_{\mathfrak{A}}(\langle \alpha, k\rangle)$ if and only if $A \notin f(x)$ for any $x < \langle \alpha, k\rangle$ and if for any set of $k+1$ formulas $B_1, \cdots, B_{k+1}$ from $S_1(L(\mathfrak{A}))$ such that the sets $B_i(\mathfrak{A})$ partition $A(\mathfrak{A})$ there exists an $x < \langle \alpha, 1\rangle$ with one of the $B_i \in f(x)$.

Let T be totally transcendental, $\mathfrak{A}$ a model of T, and $A \in S_1(L(\mathfrak{A}))$. Call a formula A *rankless* if A is not in the range of $f_{\mathfrak{A}}$. We claim there is no formula $A \in S_1(L(\mathfrak{A}))$ such that A is rankless. For, if so, we can construct for each finite binary sequence σ a formula A_σ such that (1) A_σ is rankless and (2) if $\sigma' = \sigma \cup \langle \mathrm{dm}\,\sigma, 0\rangle$ and $\sigma'' = \sigma \cup \langle \mathrm{dm}\,\sigma, 1\rangle$ then $A_{\sigma'} = \sim A_{\sigma''}$. Let X be the set of constants from $|\mathfrak{A}|$ which occur in any A_σ. Then X is countable but $S(X)$ is uncountable contrary to the hypothesis that T is totally transcendental.

Thus if $\mathfrak{A}$ is a model of a totally transcendental theory we may define for each $A \in S_1(L(\mathfrak{A}))$ the rank of $A(\mathfrak{A})$ (the rank of A in $\mathfrak{A}$) which we denote by $R_{\mathfrak{A}}(A)$. $R_{\mathfrak{A}}(A)$ is -1 if $A \in f_{\mathfrak{A}}(-1)$. $R_{\mathfrak{A}}(A)$ is $\langle \alpha, k\rangle$ if $A \in f_{\mathfrak{A}}(\langle \alpha, k\rangle)$.

Notice that if $\mathfrak{A} \preceq \mathfrak{B}$ and $A \in S_1(L(\mathfrak{A}))$ then $R_{\mathfrak{A}}(A) \le R_{\mathfrak{B}}(A)$. If $\mathfrak{A}$ is a saturated model and $\mathfrak{B} \geq \mathfrak{A}$ then $R_{\mathfrak{A}}(A) = R_{\mathfrak{B}}(A)$. If $A(\mathfrak{A}) \subseteq B(\mathfrak{A})$ then $R_{\mathfrak{A}}(A) \le R_{\mathfrak{A}}(\mathfrak{B})$. Finally if $R_{\mathfrak{A}}(A) = (\alpha, k)$ and $(\beta, m) < (\alpha, k)$ then there is a formula $B \in S_1(L(\mathfrak{A}))$ such that $B(\mathfrak{A}) \subseteq A(\mathfrak{A})$ and $R_{\mathfrak{A}}(\mathfrak{B}) = (\beta, k)$. Let $\mathfrak{A}$ be a structure with one binary relation R such that R is an equivalence relation and for each n there is a unique equivalence class with exactly n elements but there are no infinite equivalence classes in $\mathfrak{A}$. Then $Th(\mathfrak{A})$ is totally transcendental and $R_{\mathfrak{A}}(v_0 = v_0) = (1, 1)$. But for each positive integer k there is an elementary extension $\mathfrak{B}_k$ of $\mathfrak{A}$ with $R_{\mathfrak{B}_k}(v_0 = v_0) = (1, k)$ and there is an elementary extension $\mathfrak{B}$ with $R_{\mathfrak{B}}(v_0 = v_0) = (2, 1)$. It is an immediate consequence of Theorem 2 that if $\mathfrak{A}$ is a model of an $\aleph_1$-categorical theory T, $A \in S_1(L(\mathfrak{A}))$, and $\mathfrak{B} \succeq \mathfrak{A}$ then $R_{\mathfrak{B}}(A) = R_{\mathfrak{A}}(A)$. In fact this remark appears to be equivalent to Theorem 2.

In [4], Morley introduced for a countable first order theory T, $X \in \mathfrak{N}(T)$, and $p \in S(X)$ the concept of the transcendental rank of p. In [2] Lachlan interprets this notion in terms of the rank of a formula A in $S_1(L(\mathfrak{A}))$ as follows

$$r_{\mathfrak{A}}(A) = \begin{cases} -1 & \text{if } A(\mathfrak{A}) = \varnothing, \\ \sup\{\alpha \mid (\exists p) p \in U_A \wedge p \in \mathrm{Tr}^{\alpha}(\mathfrak{A})\} & \text{otherwise.} \end{cases}$$

We relate $r_{\mathfrak{A}}(A)$ to $R_{\mathfrak{A}}(A)$ in the following theorem.

Theorem 1. *Let $\mathfrak{A}$ be a model of a totally transcendental theory T and $A \in S_1(L(\mathfrak{A}))$.*

(i) $r_{\mathfrak{A}}(A) \geq \sup\{\alpha \mid \exists \mathfrak{B}, \exists k (\mathfrak{B} \succeq \mathfrak{A} \wedge R_{\mathfrak{B}}(A) = \langle \alpha, k\rangle)\}$.

(ii) *For some $\mathfrak{B}$ an elementary extension of $\mathfrak{A}$ and some integer k,* $R_{\mathfrak{B}}(A) = (r_{\mathfrak{A}}(A), k)$.

(iii) $r_{\mathfrak{A}}(A) = \sup\{\alpha \mid \exists \mathfrak{B}, \exists k (\mathfrak{B} \succeq \mathfrak{A} \wedge R_{\mathfrak{B}}(A) = (\alpha, k))\}$.

(iv) *For some elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ and some positive integer k,* $R_{\mathfrak{B}}(A) = \sup\{R_{\mathfrak{C}}(A) \mid \mathfrak{C} \succeq \mathfrak{A}\} = (r_{\mathfrak{A}}(A), k)$.

(v) *If* $R_{\mathfrak{A}}(A) = (\alpha, k)$ *there is an elementary extension* $\mathfrak{B}$ *of* $\mathfrak{A}$ *and a formula* $B \in S_1(L(\mathfrak{B}))$ *such that* $B(\mathfrak{B}) \subseteq A(\mathfrak{B})$ *and* $R_{\mathfrak{A}}(B) = (\alpha, 1) = \sup\{R_{\mathfrak{C}}(B) \mid \mathfrak{C} \succeq \mathfrak{B}\}$.

To prove this theorem we need the following extension of a lemma in [2].

Lemma 1. *Let* T *be a first order theory,* $\mathfrak{A}$ *a model of* T, $A \in S_1(L(\mathfrak{A}))$ *and suppose* $r_{\mathfrak{A}}(A) = \alpha$ *then for each* $\beta < \alpha$ *there exists an elementary extension* $\mathfrak{B}$ *of* $\mathfrak{A}$ *such that* $i^{*-1}_{\mathfrak{A}\mathfrak{B}}(U_A) \cap \mathrm{Tr}^{\beta}(\mathfrak{B})$ *is infinite.*

Proof. If the lemma is false there exists a model of T and a formula $A \in S_1(L(\mathfrak{A}))$ with $r_{\mathfrak{A}}(A) = \alpha$ and some $\beta < \alpha$ such that, for each $\mathfrak{B} \succeq \mathfrak{A}$, $i^{*-1}_{\mathfrak{A}\mathfrak{B}}(U_A) \cap \mathrm{Tr}^{\beta}(\mathfrak{B})$ is finite. Suppose $q \in \mathrm{Tr}^{\beta+1}(\mathfrak{B})$. Then for each $\mathfrak{C} \succeq \mathfrak{B}$, $i^{*-1}_{\mathfrak{B}\mathfrak{C}}(q) \cap S^{\beta+1}(\mathfrak{C})$ is a set of isolated points in $S^{\beta+1}(\mathfrak{C})$. But then if $A \in q$, $i^{*-1}_{\mathfrak{B}\mathfrak{C}}(q) \cap S^{\beta}(\mathfrak{C})$ is a set of isolated points in $S^{\beta}(\mathfrak{C})$ since $i^{*-1}_{\mathfrak{A}\mathfrak{C}}(U_A) = i^{*-1}_{\mathfrak{B}\mathfrak{C}}(U_A)$ and $i^{*-1}_{\mathfrak{A}\mathfrak{C}}(U_A) \cap \mathrm{Tr}^{\beta}(\mathfrak{C})$ is finite. Thus $q \in \mathrm{Tr}^{\beta}(\mathfrak{B})$ but q was chosen in $\mathrm{Tr}^{\beta+1}(\mathfrak{B})$ so this is impossible. Hence $i^{*-1}_{\mathfrak{A}\mathfrak{B}}(U_A) \cap \mathrm{Tr}^{\beta+1}(\mathfrak{B})$ is empty and by induction for each $\gamma \geq \beta + 1$, for each $\mathfrak{C} \succeq \mathfrak{A}$, $\mathrm{Tr}^{\gamma}(\mathfrak{C}) \cap i^{*-1}_{\mathfrak{A}\mathfrak{C}}(U_A)$ is empty. So $r_{\mathfrak{A}}(A) \neq \alpha$.

Proof of Theorem 1. (i) The proof proceeds by induction on $r_{\mathfrak{A}}(A)$. If $r_{\mathfrak{A}}(A) = -1$ then $\models_T \sim \exists v_0 A$ and so the theorem holds. Suppose, as the induction hypothesis, the theorem holds for a formula A if $r_{\mathfrak{A}}(A) = \gamma$ is less than α. We first prove that, for each $\mathfrak{B} \succeq \mathfrak{A}$, $R_{\mathfrak{B}}(A) < (\alpha + 1, 1)$. If not, there is some $\mathfrak{B}_1 \succeq \mathfrak{A}$ with $R_{\mathfrak{B}_1}(A) \geq (\alpha + 1, 1)$. Then there exists a sequence of formulas $(A_i)_{i<\omega}$ each $A_i \in S_1(L(\mathfrak{B}_1))$ such that $A_i(\mathfrak{B}_1) \subseteq A(\mathfrak{B}_1)$, $(A_i \wedge A_j)(\mathfrak{B}_1) = \varnothing$ if $i \neq j$, and $R_{\mathfrak{B}_1}(A_i) = (\alpha, 1)$. Now we show that for each natural number i there is a 1-type $p_i \in U_{A_i} \cap S^{\alpha}(\mathfrak{B}_1)$.

Case 1. α is a successor ordinal, say $\alpha = \lambda + 1$. Since $R_{\mathfrak{B}_1}(A_i) = (\lambda+1, 1)$ there exists a sequence of formulas $(A_{ij})_{j<\omega}$ each $A_{ij} \in S_1(L(\mathfrak{B}_1))$, such that $A_{ij}(\mathfrak{B}_1) \subseteq A_i(\mathfrak{B}_1)$, $(A_{ij} \wedge A_{ik})(\mathfrak{B}_1) = \varnothing$ if $j \neq k$, and $R_{\mathfrak{B}_1}(A_{ij}) = (\lambda, 1)$. Then by induction, for each j, $r_{\mathfrak{B}_1}(A_{ij}) \geq \lambda$ so there exists $p_{ij} \in U_{A_{ij}} \cap \mathrm{Tr}^{\lambda}(\mathfrak{B}_1)$. Then for each i, since $S(\mathfrak{B}_1)$ is compact and U_{A_i} is closed, there exists p_i, an accumulation point of the p_{ij}, such that $p_i \in U_{A_i} \cap S^{\lambda+1}(\mathfrak{B}_1)$.

Case 2. α is a limit ordinal. α has cofinality ω since $\alpha < \omega_1$ [2]. Then there exists a sequence of ordinals $(\alpha_j)_{j<\omega}$ and a sequence of formulas $(A_{ij})_{j<\omega}$, each $A_{ij} \in S_1(L(\mathfrak{B}_1))$, such that $A_{ij}(\mathfrak{B}_1) \subseteq A_i(\mathfrak{B}_1)$, $(A_{ij} \wedge A_{ik})(\mathfrak{B}_1) = \varnothing$ if $j \neq k$, $R_{\mathfrak{B}_1}(A_{ij}) = (\alpha_j, 1)$ for each j, and the α_j increase monotonically to α. Then by induction $r_{\mathfrak{B}_1}(A_{ij}) \geq \alpha_j$ so there exists a type $p_{ij} \in U_{A_{ij}} \cap \mathrm{Tr}^{\alpha_j}(\mathfrak{B}_1)$. Since U_{A_i} is closed and $S(\mathfrak{B}_1)$ is compact there exists p_i an accumulation point of the p_{ij} for each i. But $p_i \notin \mathrm{Tr}^{\gamma}(\mathfrak{B}_1)$ for any $\gamma < \alpha$ so $p_i \in U_{A_i} \cap S^{\alpha}(\mathfrak{B}_1)$.

Since U_A is closed there exists p, an accumulation point of the p_i and

$p \in U_A \cap S^{\alpha+1}(\mathfrak{B}_1)$ since each $p_i \in U_A \cap \mathrm{Tr}^{\alpha}(\mathfrak{B}_1)$. Hence $i^*_{\mathfrak{A}\mathfrak{B}_1}(p) \in U_A \cap S^{\alpha+1}(\mathfrak{A})$. But then $r_{\mathfrak{A}}(A) \geq \alpha + 1$ so (i) is proved.

(ii) Now we show that there exists $\mathfrak{B} \succeq \mathfrak{A}$ such that for some k, $R_{\mathfrak{B}}(A) = (\alpha, k)$. By Lemma 1 since $r_{\mathfrak{A}}(A) = \alpha$, for each $\gamma < \alpha$ there exists an elementary extension $\mathfrak{A}_\gamma$ of $\mathfrak{A}$ such that $i^{*-1}_{\mathfrak{A}\mathfrak{A}_\gamma}(U_A) \cap \mathrm{Tr}^{\gamma}(\mathfrak{A}_\gamma)$ is infinite. Hence there exists a sequence of formulas $(A_i^\gamma)_{i<\omega}$, each $A_i^\gamma \in S_1(L(\mathfrak{A}_\gamma))$, such that $(A_i^\gamma \wedge A_j^\gamma)(\mathfrak{A}_\gamma) = \varnothing$ if $i \neq j$, $A_i^\gamma(\mathfrak{A}_\gamma) \subseteq A(\mathfrak{A}_\gamma)$ and $r_{\mathfrak{A}_\gamma}(A_i^\gamma) = \gamma$. So by induction there exists $\mathfrak{A}_{\gamma,i}$ such that for each γ and i $R_{\mathfrak{A}_{\gamma,i}}(A_i^\gamma) = (\gamma, k)$ for some k. Without loss of generality we may assume $(|\mathfrak{A}_{\gamma,i}| - |\mathfrak{A}|) \cap (|\mathfrak{A}_{\delta,j}| - |\mathfrak{A}|) = \varnothing$ if $(\gamma, i) \neq (\delta, j)$. There exists a model $\mathfrak{C}$ such that for each (γ, i), $\mathfrak{C} \succeq \mathfrak{A}_{\gamma,i}$ by the compactness theorem. Then for each $\gamma < \omega$ there is a k such that $R_{\mathfrak{C}}(A_k^\gamma) \geq (\gamma, k)$ and $(A_i^\gamma \wedge A_j^\gamma)(\mathfrak{C}) = \varnothing$ if $i \neq j$. So $R_{\mathfrak{C}}(A) \geq (\alpha, 1)$. Since for each $\mathfrak{B} \succeq \mathfrak{A}$, $R_{\mathfrak{B}}(A) < (\alpha + 1, 1)$ by (i), for some k, $R_{\mathfrak{C}}(A) = (\alpha, k)$ and $\mathfrak{C}$ is the required model.

(iii) This follows immediately from (i) and (ii).

(iv) We must find $\mathfrak{B} \succeq \mathfrak{A}$ and a positive integer k, such that $R_{\mathfrak{B}}(A) = (r_{\mathfrak{A}}(A), k) = \sup\{R_{\mathfrak{C}}(A) | \mathfrak{C} \succeq \mathfrak{A}\}$. By (ii) choose $\mathfrak{B}_0 \succeq \mathfrak{A}$ such that, for some k, $R_{\mathfrak{B}}(A) = (r_{\mathfrak{A}}(A), k)$. Then applying (i) for each $\mathfrak{C} \succeq \mathfrak{B}_0$ there is an integer k such that $R_{\mathfrak{C}}(A) = (r_{\mathfrak{A}}(A), k)$. It suffices to show that the set of such k is bounded. If not, there exists an increasing sequence of positive natural numbers n_m and a sequence of models $\mathfrak{B}_m$ such that $\mathfrak{B}_m \succeq \mathfrak{B}_0$ and $R_{\mathfrak{B}_m}(A) = (r_{\mathfrak{A}}(A), n_m)$. We may assume that, if $m \neq l$, $(|\mathfrak{B}_m| - |\mathfrak{B}_0|) \cap (|\mathfrak{B}_l| - |\mathfrak{B}_0|) = \varnothing$. By the compactness theorem there exists a model $\mathfrak{D}$ which elementarily extends each $\mathfrak{B}_m$. But then $R_{\mathfrak{D}}(A) \geq (r_{\mathfrak{A}}(A) + 1, 1)$ contrary to (i). Hence there exists a maximum k and an elementary extension $\mathfrak{B}$ of $\mathfrak{B}_0$ such that

$$R_{\mathfrak{B}}(A) = (r_{\mathfrak{A}}(A), k) = \sup\{R_{\mathfrak{C}}(A) | \mathfrak{C} \geq \mathfrak{B}\}.$$

(v) We will construct a sequence of models $\mathfrak{B}_i$ and formulas $B_i \in S_1(L(\mathfrak{B}_{i-1}))$ such that $B_{i+1}(\mathfrak{B}_i) \subseteq B_i(\mathfrak{B}_i)$, $R_{\mathfrak{B}_i}(B_{i+1}) = (\alpha, 1)$, $R_{\mathfrak{B}_{i+1}}(B_{i+1}) = \sup\{R_{\mathfrak{C}}(B_{i+1}) | \mathfrak{C} \succeq \mathfrak{B}_{i+1}\}$ and if $R_{\mathfrak{B}_i}(B_i) > (\alpha, 1)$ then $R_{\mathfrak{B}_{i+1}}(B_{i+1}) < R_{\mathfrak{B}_i}(B_i)$. Since there is no infinite descending sequence in a well ordered set, for some i, $R_{\mathfrak{B}_i}(B_i) = (\alpha, 1)$ and letting $\mathfrak{B} = \mathfrak{B}_i$ and $B = B_i$ proves (v). Let $\mathfrak{B}_0 = \mathfrak{A}$ and $B_0 = A$. Suppose $\mathfrak{B}_i$ and B_i have been chosen for $i < n$. Let $B_n \in S_1(L(\mathfrak{B}_{n-1}))$ such that $B_n(\mathfrak{B}_{n-1}) \subseteq B_{n-1}(\mathfrak{B}_{n-1})$ and $R_{\mathfrak{B}_{n-1}}(B_{n-1}) = (\alpha, 1)$. Then by (iv) choose $\mathfrak{B}_n \succeq \mathfrak{B}_{n-1}$ such that

$$R_{\mathfrak{B}_n}(B_n) = \sup\{R_{\mathfrak{C}}(B_n) | \mathfrak{C} \succeq \mathfrak{B}_n\}.$$

If $R_{\mathfrak{B}_n}(B_n) > (\alpha, 1)$ then both $R_{\mathfrak{B}_n}(B_n \wedge B_{n+1})$ and $R_{\mathfrak{B}_n}(B_n \wedge \sim B_{n+1})$ are greater than or equal to $(\alpha, 1)$. Hence, if $R_{\mathfrak{B}_{n+1}}(B_{n+1}) \geq R_{\mathfrak{B}_n}(B_n)$, $R_{\mathfrak{B}_{n+1}}(B_n) > R_{\mathfrak{B}_n}(B_n)$ contrary to the choice of $\mathfrak{B}_n$.

At the suggestion of the referee we include the following comparison of the rank defined here with that defined by Shelah in his paper on the uniqueness of prime models [6].

Shelah chooses a sufficiently saturated model $\mathfrak{M}$ of T (for T totally transcendental a countable saturated model suffices) and defines for $A \in S_1(L(\mathfrak{M}))$,

(A) $\rho(A) = -1$ iff $\mathfrak{M} \models \sim \exists v_0 A$.

(B) $\rho(A) = \alpha$ iff

(1) $\mathfrak{M} \models \exists v_0 A$,

(2) for no $\beta < \alpha$, $\rho(A) = \beta$,

(3) for no $B \in S_1(L(\mathfrak{M}))$ do both $A \wedge B$ and $A \wedge \sim B$ satisfy (1) and (2).

(C) $\rho(A) = \infty$ if $\rho(A)$ is not defined by (A) and (B). ∞ is assumed greater than each ordinal.

Shelah proves that if T is totally transcendental then $\rho(A) < \infty$. The following theorem indicates the relation between $R_{\mathfrak{A}}(A)$ and $\rho(A)$ if $Th(\mathfrak{A})$ is totally transcendental.

Theorem 1′. *Let T be a totally transcendental theory and $\mathfrak{A}$ a saturated model of T then, for $A \in S_1(L(\mathfrak{A}))$, $R_{\mathfrak{A}}(A) = (\alpha, k)$ if and only if $\rho(A) = \omega \cdot \alpha + m$ where $2^m \leq k < 2^{m+1}$. $R_{\mathfrak{A}}(A) = 1$ if and only if $\rho(A) = -1$.*

Proof. Since $\mathfrak{A}$ is a saturated model of T we may take $\mathfrak{A}$ for $\mathfrak{M}$ in the definition of $\rho(A)$. The result is evident if $R_{\mathfrak{A}}(A) = -1$. For the rest we induct on $R_{\mathfrak{A}}(A)$. It is easy to verify that $R_{\mathfrak{A}}(A) = (0, 1)$ if and only if $\rho(A) = 0$.

Now suppose the conclusion holds for each $A \in S_1(L(\mathfrak{A}))$ with $R_{\mathfrak{A}}(A) < (\alpha, k)$ and choose an $A \in S_1(L(\mathfrak{A}))$ with $R_{\mathfrak{A}}(A) = (\alpha, k)$.

Case 1. Let $k = 1$. To show $\rho(A) \geq \omega \cdot \alpha$ it suffices by [6, Theorem 1.1 A, B], as T is totally transcendental, to show there is an increasing sequence of ordinals $\langle \gamma_i \rangle_{i<\omega}$ tending to $\omega \cdot \alpha$ and a collection of formulas $B_i \in S_1(L(\mathfrak{A}))$ such $\rho(A \wedge B_i) \geq \gamma_i$ and $\rho(A \wedge \sim B_i) \geq \gamma_i$. Let $\langle \delta_i, k_i \rangle$ be an increasing sequence tending to $(\alpha, 1)$. For each i, choose $B_i, B_i' \in S_1(L(\mathfrak{A}))$ such that $B_i(\mathfrak{A}) \subseteq A(\mathfrak{A})$, $B_i'(\mathfrak{A}) \subseteq A(\mathfrak{A})$, $B_i(\mathfrak{A}) \cap B_i'(\mathfrak{A}) = \varnothing$ and $R_{\mathfrak{A}}(B_i) = R_{\mathfrak{A}}(B_i') = (\delta_i, k_i)$. Then by induction $\rho(A_i \wedge B_i) = \omega \cdot \delta_i + m_i$ and $\rho(A \wedge \sim B_i) \geq \omega \cdot \delta_i + m_i$ where $2^{m_i} \leq k_i < 2^{m_i+1}$. Let $\gamma_i = \omega \cdot \delta_i + m_i$; we have an appropriate sequence.

But for each formula $B \in S_1(L(\mathfrak{A}))$ either $R_{\mathfrak{A}}(A \wedge B) < (\alpha, 1)$ or $R_{\mathfrak{A}}(A \wedge \sim B) < (\alpha, 1)$. Say $R_{\mathfrak{A}}(A \wedge B) = (\beta, k) < (\alpha, 1)$. Then by induction $\rho(A \wedge B) = \omega \cdot \beta + m < \omega \cdot \alpha$ where $2^m \leq k < 2^{m+1}$. Hence $\rho(A) \leq \omega \cdot \alpha$ so $\rho(A) = \omega \cdot \alpha$.

Case 2. Suppose $k > 1$, and $2^m \leq k < 2^{m+1}$. Let $B \in S_1(L(\mathfrak{A}))$, then either $R_{\mathfrak{A}}(A \wedge B) < (\alpha, 2^m)$ or $R_{\mathfrak{A}}(A \wedge \sim B) < (\alpha, 2^m)$ since $R_{\mathfrak{A}}(A \wedge B) \geq (\alpha, 2^m)$ and $R_{\mathfrak{A}}(A \wedge \sim B) \geq (\alpha, 2^m)$ implies $R_{\mathfrak{A}}(A) \geq (\alpha, 2^{m+1}) > (\alpha, k)$. Hence by induction $\rho(A \wedge B) < \omega \cdot \alpha + m$ or $\rho(A \wedge \sim B) < \omega \cdot \alpha + m$. Thus $\rho(A) \leq \omega \cdot \alpha + m$.

There exist formulas $B_1, \cdots, B_k \in S_1(L(\mathfrak{A}))$ such that the $B_i(\mathfrak{A})$ partition $A(\mathfrak{A})$ and each $R_{\mathfrak{A}}(B_i) = (\alpha, 1)$. Let $B = \bigvee_{i=1}^{2m-1} B_i$. Then by induction $\rho(A \wedge B) = \omega \cdot \alpha + (m-1)$ and $\rho(A \wedge \sim B) \geq \omega \cdot \alpha + (m-1)$ so by [6, Theorem 1.1B] $\rho(A) \geq \omega \cdot \alpha + m$. Thus $\rho(A) = \omega \cdot \alpha + m$.

Corollary to Main Theorem. *If T is $\aleph_1$-categorical, $\mathfrak{A} \models T$ and $A \in S_1(L(\mathfrak{A}))$, $\rho(A) < \omega \cdot \omega$.*

Proof. This is immediate from Theorem 1′ and Theorem 3.

We now restrict our attention to $\aleph_1$-categorical theories. In particular, we will deal with an $\aleph_1$-categorical theory T with a specified strongly minimal formula D such that, for each model $\mathfrak{B}$ of T, $D(\mathfrak{B}) \cap \mathrm{cl}(\emptyset)$ is infinite.

We want to assign to each formula $B \in S_1(L(\mathfrak{A}))$ a formula B^* which "witnesses" the rank of B. In order to do this we consider formulas $A \in S_{l+1}(L)$ for each l. To each A and for each n we assign a class $\Gamma_A^{(n)}$ of possible witnesses. Each $\Gamma_A^{(n)}$ is a set of l-ary formulas such that there is a positive integer k with $R_{\mathfrak{A}}(A(a_1, \cdots, a_l)) = (n, k)$ if and only if, for some $A^* \in \Gamma_B^{(n)}$, $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$. The simplest cases are as follows. If $A(\mathfrak{A})$ is finite, A^* tells how many elements are in $A(\mathfrak{A})$. If A is strongly minimal A^* expresses A as a "uniform union of finite sets" over the fixed strongly minimal set D. In the following definition A^* will be in $\Phi_A^{(n)}$ just when $R_{\mathfrak{A}}(A) = (n, 1)$. The definition of $\Theta_A^{(n)}$ arises from the intuition that $R_{\mathfrak{A}}(A) = (n, k)$ when $A(\mathfrak{A})$ is a union of finitely many definable sets with rank $(n, 1)$.

For each natural number l, for each $A \in S_{l+1}(L)$ and to -1 and each natural number n assign a set of formulas as follows

$$\Gamma_A^{(-1)} = \{\sim \exists v_0 A\},$$

$$\Phi_A^{(0)} = \{\exists v_0 A \wedge \exists^{\leq k} v_0 A \mid 0 < k < \omega\},$$

$$\begin{aligned}\Phi_A^{(n)} = \{\exists v_{l+1}, \cdots, \exists v_k (\forall v_0 (A \leftrightarrow \exists v_{k+1}(C \wedge D(v_{k+1}) \wedge C^*))\\ \wedge (\forall v_0 (A \rightarrow \exists^{\leq p} v_{k+1}(C \wedge D(v_{k+1}))))\\ \wedge (\exists^{\leq p} v_{k+1} \exists v_0 (D(v_{k+1}) \wedge C \wedge (\sim A \vee \sim C^*))))\mid\\ 0 < p < \omega,\ l \leq k < \omega,\ C \in S_{k+2}(L), \text{ and } C^* \in \Gamma_C^{(n-1)}\},\end{aligned}$$

$$\begin{aligned}\Theta_A^{(n)} = \{\exists v_{l+1}, \cdots, \exists v_k (\forall v_0 (A \leftrightarrow (A_1 \vee \cdots \vee A_s)) \wedge A_1^* \wedge \cdots \wedge A_s^*) \mid\\ l \leq k < \omega,\ A_i \in S_{k+1}(L),\ s < \omega \text{ each } A_i^* \in \bigcup_{r<n} \Gamma_{A_i}^{(r)} \cup \Phi_{A_i}^{(n)}\\ \text{and some } A_i^* \in \Phi_{A_i}^{(n)}\},\end{aligned}$$

$$\Gamma_A^{(n)} = \Phi_A^{(n)} \cup \Theta_A^{(n)}.$$

Note that if $A \in S_{l+1}(L)$ and $A^* \in \Gamma_A^{(n)}$ for some n, then A^* has free variables $v_1, \cdots, v_l$. Thus when we write $A^*(a_1, \cdots, a_l)$ we mean the result of substituting a_i for v_i for $i = 1, 2, \cdots, l$. We abbreviate $A_{v_1,\cdots,v_l}(a_1, \cdots, a_l)$ by $A(a_1, \cdots, a_l)$. Thus $A(a_1, \cdots, a_l) \in S_1(L(\{a_1, \cdots, a_l\}))$.

Theorem 2. *Let T be an $\aleph_1$-categorical theory and D a strongly minimal formula in T such that, in each model B of T, $D(B) \cap \mathrm{cl}(\emptyset)$ is infinite. Let $\mathfrak{A}$ be a model of T, $m \in \{-1\} \cup \omega$, $A \in S_{l+1}(L)$, and $a_1, \cdots, a_l \in |\mathfrak{A}|$. The following two propositions are equivalent.*

(i) *There exists a formula $A^* \in \Gamma_A^{(n)}$ such that $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$.*

(ii) *For some k $R_{\mathfrak{A}}(A(v_0, a_1, \cdots, a_l)) = (m, k)$ if $m \geq 0$. If $m = -1$, $R_{\mathfrak{A}}(A(v_0, a_1, \cdots, a_l)) = -1$.*

Notice that there is no loss of generality in this theorem because of our assumption that T has a strongly minimal formula D and that, for each model $\mathfrak{B}$ of T, $D(\mathfrak{B}) \cap \mathrm{cl}(\emptyset)$ is infinite. For, let T be an arbitrary $\aleph_1$-categorical theory in a first order language L. Then there is a principal extension T' of T with a strongly minimal formula D'. Let $\mathfrak{A}$ be a prime model of T'. Let X be an infinite subset of $D'(\mathfrak{A}')$. Then $Th(\mathfrak{A}', X) = T''$ is a theory of the specified kind. Suppose $\mathfrak{B}$ is a model of T'', $A \in S_{l+1}(L)$, $A^* \in \Gamma_A^{(m)}$ for some m, and $a_1, \cdots, a_l \in |\mathfrak{B}|$. Then $\mathfrak{B} \models A^*(a_1, \cdots, a_l)$ if and only if $\mathfrak{B}\,|\,L \models A^*(a_1, \cdots, a_l)$. Moreover, $R_{\mathfrak{B}|L}(A(v_0, a_1, \cdots, a)) = R_{\mathfrak{B}}(A(v_0, a_1, \cdots, a_l))$. Thus it suffices to prove the theorem for T''.

Proof of theorem. The proof proceeds by induction on m. If $m = -1$, $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$ for some $A^* \in \Gamma_A^{(-1)}$ if and only if $A(v_0, a_1, \cdots, a_l)(\mathfrak{A}) = \emptyset$ which is equivalent to $R_{\mathfrak{A}}(A(v_0, a_1, \cdots, a_l)) = -1$. We assume the theorem is true for $m < n$ and prove (i) implies (ii) for $m = n$. Then we prove a lemma. Finally we assume the theorem holds for $m < n$ and prove (ii) implies (i) for $m = n$.

Tc prove (i) implies (ii) consider a formula $A \in S_{l+1}(L)$ and a formula $A^* \in \Gamma_A^{(n)}$ such that $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$ with $a_1, \cdots, a_l \in |\mathfrak{A}|$. Notice first that it suffices to prove the case in which $A^* \in \Phi_A^{(n)}$. For, suppose that (i) implies (ii) has been shown for each integer l, each $A \in S_l(L)$ and each $A^* \in \Phi_A^{(n)}$ and that $A^* \in \Theta_A^{(n)}$. Then since $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$, $A(v_0, a_1, \cdots, a_l)(\mathfrak{A}) = \bigcup_{i=1}^{s}(A_i(v_0, a_1, \cdots, a_k)(\mathfrak{A}))$ for some $a_{l+1}, \cdots, a_k$ in $|\mathfrak{A}|$ and some $A_1, \cdots, A_s$. Moreover, for each i, $\mathfrak{A}$ satisfies $A_i^*(a_1, \cdots, a_k)$ and each $A_i^* \in \bigcup_{i=1}^{n-1}\Gamma_{A_i}^{(n-1)} \cup \Phi_{A_i}^{(n)}$ and some $A_i^* \in \Phi_{A_i}^{(n)}$. So for each i there exists $n_i \leq n$ and a k_i such that $R_{\mathfrak{A}}(A_i(v_0, a_1, \cdots, a_l)) = (n_i, k_i)$ and for some i there exists k such that $R_{\mathfrak{A}}(A_i(a_1, \cdots, a_l)) = (n, k)$, by induction and the assumption that the theorem holds for each $B^* \in \Phi_B^{(n)}$. But then $R_{\mathfrak{A}}(A(a_1, \cdots, a_l)) = (n, m)$ for some integer m.

Thus to prove (i) implies (ii) when $m = n$, let $A \in S_{l+1}(L)$ and suppose $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$ where $A^* \in \Phi_A^{(n)}$. Letting $A' = A(v_0, a_1, \cdots, a_l)$ we wish to prove that, for some q, $R_{\mathfrak{A}}(A') = (n, q)$. From the definition of $\Phi_A^{(n)}$ we see A^* has the form

$$\exists v_{l+1}, \cdots, \exists v_k(\forall v_0(A \leftrightarrow \exists v_{k+1}(C \wedge D(v_{k+1}) \wedge C^*))$$
$$\wedge (\forall v_0(A \rightarrow \exists^{\leq p} v_{k+1}(C \wedge D(v_{k+1}))))$$
$$\wedge \exists^{\leq p} v_{k+1} \exists v_0(D(v_{k+1}) \wedge C \wedge (\sim A \vee \sim C^*)))$$

where p is a positive integer, $l \leq k < \omega$, C is in $S_{k+2}(L)$ and C^* is in $\Gamma_C^{(n-1)}$. Since $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$ there exist $a_{l+1}, \cdots, a_k \in |\mathfrak{A}|$ such that, for all but p elements b of $D(\mathfrak{A})$, $\mathfrak{A} \models C^*(a_1, \cdots, a_k, b)$. Thus, for any $\mathfrak{A}_1 \succeq \mathfrak{A}$ and $d \in D(\mathfrak{A}_1) - D(\mathfrak{A})$, $\mathfrak{A}_1 \models C^*(a_1, \cdots, a_k, d)$.

By induction, for some s, $R_{\mathfrak{A}_1}(C'_{v_{k+1}}(d)) = (n-1, s)$ where $C' = C(v_0, a_1, \cdots, a_k, v_{k+1})$. Then $R_{\mathfrak{A}}(A')$ is $\leq (n, s)$. For, if not there exist L-formulas $B_1, \cdots, B_{s+1}$ where each B_i has free variables $v_0, v_{k+2}, \cdots, v_m$ with the following properties. There exist constants $a^i_{k+2}, \cdots, a^i_m \in |\mathfrak{A}|$ such that if $B'_i = B_i(v_0, a^i_{k+2}, \cdots, a^i_m)$, $B'_i(\mathfrak{A}) \subseteq A'(\mathfrak{A})$, $B'_i(\mathfrak{A}) \cap B'_j(\mathfrak{A}) = \emptyset$ if $i \neq j$, and $R_{\mathfrak{A}}(B'_i) \geq (n, 1)$. We will show that this condition implies for each elementary extension $\mathfrak{A}_1$ of $\mathfrak{A}$, each $d \in D(\mathfrak{A}_1) - D(\mathfrak{A})$, and each i that $R_{\mathfrak{A}_1}(B'_i \wedge C'_{v_{k+1}}(d)) \geq (n-1, 1)$. This in turn implies $R_{\mathfrak{A}_1}(C'_{v_{k+1}}(d)) > (n-1, s)$ which is a contradiction allowing us to conclude that $R_{\mathfrak{A}}(A') \leq (n, s)$.

Suppose $R_{\mathfrak{A}}(B'_i) \geq (n, 1)$ and for some $\mathfrak{A}_1 \succeq \mathfrak{A}$ and some $d \in D(\mathfrak{A}_1) - D(\mathfrak{A})$, $R_{\mathfrak{A}_1}(B'_i \wedge C'_{v_{k+1}}(d)) < (n-1, 1)$. By induction there exists a formula $(B_i \wedge C)^* \in \Gamma^{(r)}_{B_i \wedge C}$ for some $r < n-1$ such that $\mathfrak{A}_1 \models (B_i \wedge C)^*(a_1, \cdots, a_k, d, a^i_{k+2}, \cdots, a^i_m)$. Since D is strongly minimal, there exists $p_1 \in \omega$ which may be assumed larger than p such that, for all but p_1 members of $D(\mathfrak{A})$, $\mathfrak{A}_1 \models (B_i \wedge C)^*(a_1, \cdots, a_k, b, a^i_{k+2}, \cdots, a^i_m)$. Consider the formulas

$$F = \exists v_{k+1}(D(v_{k+1}) \wedge (B_i \wedge C) \wedge (B_i \wedge C)^*),$$

$$G = (\forall v_0(F \leftrightarrow F)) \wedge (\forall v_0(F \rightarrow \exists^{\leq p_1} v_{k+1}(D(v_{k+1}) \wedge (B_i \wedge C))))$$
$$\wedge (\exists^{\leq p_1} v_{k+1} \exists v_0(D(v_{k+1}) \wedge (B_i \wedge C) \wedge (\sim F \vee \sim (B_i \wedge C)^*))),$$

$$H = \exists v_0 F.$$

If $r = -1$ let $F^* = H$; otherwise let $F^* = G$. Then $F^* \in \Gamma^r_F \cup \Gamma^{r+1}_F$ and $\mathfrak{A} \models F^*(a_1, \cdots, a_k, a^i_{k+2}, \cdots, a^i_m)$ so if F' is the formula $F(v_0, a_1, \cdots, a_k, a^i_{k+2}, \cdots, a^i_m)$ by induction there is an integer q such that $R_{\mathfrak{A}}(F') = (r+1, q) < (n, 1)$. For each element $c \in B'_i(\mathfrak{A})$ there exists an element b in $D(\mathfrak{A})$ such that $\mathfrak{A} \models B'_i(c) \wedge C'(b, c) \wedge C^*(a_1, \cdots, a_k, b)$ since $B'_i(\mathfrak{A}) \subseteq A'(\mathfrak{A})$ and $\mathfrak{A} \models A^*(a_1, \cdots, a_k)$. Let

$b_1, \cdots, b_q$ be an enumeration of the elements $b \in D(\mathfrak{A})$ such that

$$\mathfrak{A} \models C^*(a_1, \cdots, a_k, b) \wedge \sim (B_i \wedge C^*)(a_1, \cdots, a_k, b, a^i_{k+2}, \cdots, a^i_m).$$

We know there are only finitely many such b from above. Then $R_{\mathfrak{A}}(B'_i \wedge C'_{v_{k+1}}(b)) \leq R_{\mathfrak{A}}(C'_{v_{k+1}}(b)) = (n-1, u)$ for some $u < \omega$ by induction. But

$$\mathfrak{A} \models \forall v_0 \left(B'_i \leftrightarrow F' \vee \bigvee_{j=1}^{q} (B_i \wedge C_{v_{k+1}}(b_j)) \right).$$

So $B'_i(\mathfrak{A})$ is the union of a finite number of definable sets each with rank less than $(n, 1)$ and thus $R_{\mathfrak{A}}(B'_i) < (n, 1)$ contrary to assumption. Thus we conclude as outlined above $R_{\mathfrak{A}}(A') \leq (n, s)$. Since $\mathfrak{A} \models \forall v_0 \exists^{\leq p_1} v_{k+1}(C')$, $R_{\mathfrak{A}}(A') \geq (n, 1)$. Therefore there exists an l, $1 \leq l \leq s$, such that $R_{\mathfrak{A}}(A') = (n, l)$. We have shown (i) implies (ii) when $m = n$.

Lemma 2. *Let $\mathfrak{A} \models T$, $A \in S_{l+1}(L)$, $a_1, \cdots, a_l \in |\mathfrak{A}|$, $A' = A(v_0, a_1, \cdots, a_l)$ and $\alpha \leq \omega$. Suppose the theorem holds for each $m < \alpha$ and that for each $\mathfrak{B} \succeq \mathfrak{A}$ there is some k such that $R_{\mathfrak{B}}(A') = (\alpha, k)$, then there exists $r < \alpha$ and $A^* \in \Gamma_A^{(r+1)}$ such that $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$.*

Proof. Adjoin a new unary predicate symbol q to L to form L' and a new constant symbol f to L' to form L''. Let Δ be the set of L' sentences which are true in an L' structure $\mathfrak{C}'$ just if there is an elementary substructure $\mathfrak{C}^*$ of the reduct of $\mathfrak{C}'$ to L such that $|\mathfrak{C}^*| = q(\mathfrak{C}')$. Let D^n be the L' sentence $\exists^{\geq n} v_0(D \wedge \sim q)$. Let Γ_1 be the set of sentences

$$\{\text{elementary diagram of } \mathfrak{A}\} \cup \Delta \cup \{D^n | \, n < \omega\} \cup \{q(a) | \, a \in |\mathfrak{A}|\}.$$

If $k < \omega$ and $F \in S_{k+2}(L)$ consider the following formulas.

Let $m = l + k$.

Let $F_1 \in S_{m+2}(L)$ be the formula

$$F(v_0, v_{l+1}, \cdots, v_m, v_{m+1}) \wedge A.$$

Let F_1^* be in $S_{m+1}(L)$.

Let $G(F, F_1^*) = \exists v_{m+1}(D(v_{m+1}) \wedge F_1 \wedge F_1^*)$.

Let $G^*(F, F_1^*, p)$ be

$$(\forall v_0(G(F, F_1^*) \leftrightarrow G(F, F_1^*))) \wedge (\forall v_0(G(F, F_1^*) \rightarrow \exists^{\leq p} v_{m+1}(D(v_{m+1}) \wedge F_1))$$
$$\wedge \exists^{\leq p} v_{m+1} \exists v_0(D(v_{m+1}) \wedge (\sim G(F, F_1^*) \vee \sim F_1^*))).$$

Then if F_1^* is in $\Gamma_{F_1}^{(s)}$, $G^*(F, F_1^*, p)$ is in $\Gamma_{G(F, F_1^*)}^{(s+1)}$. Let Γ_2 be the set of sentences

$$\Gamma_1 \cup \{A'(f) \wedge \sim q(f)\} \cup \Big\{ \sim (G(F, F_1^*)(f, a_1, \cdots, a_l, b_{l+1}, \cdots, b_m) \wedge G^*(F, F_1^*, p)(a_1, \cdots, a_l, b_{l+1}, \cdots, b_m)) \,\Big|\, \text{for } k \in \omega \text{ let } F \in S_{k+2}(L),\ F_1^* \in \bigcup_{u<\alpha} \Gamma_{F_1}^{(u)} b_{l+1}, \cdots, b_m, \in |\mathfrak{A}| \Big\}.$$

Now we show that Γ_2 is inconsistent by finding for each L'' structure C'' such that $C'' \models \Gamma_1$, for each element $f \in (A' \wedge \sim q)(\mathcal{C}')$ formulas F and F_1^*, an integer p, and constants $c_{l+1}, \cdots, c_m$ such that

$$\mathcal{C}'' \models G(F, F_1^*)(f, q_1, \cdots, a_l, c_{l+1}, \cdots, c_m) \wedge G^*(F, F_1^*, p)(a_1, \cdots, a_l, c_{l+1}, \cdots, c_m).$$

Let $\mathcal{C}'' \models \Gamma_1^*$ and $|\mathcal{B}| = q(\mathcal{C}'')$. Let $\mathcal{C} = \mathcal{C}'' \,|\, L$. $\mathcal{B}$ is an L-structure. Let $\mathcal{C}_1$ be an L-structure prime over $|\mathfrak{A}| \cup \{f\}$. Then $D(\mathcal{C}_1) - D(\mathcal{B}) \neq \varnothing$. For, suppose $D(\mathcal{C}_1) \subseteq D(\mathcal{B})$ and let $\mathcal{B}_1$ be prime over $D(\mathcal{C}_1)$. ($\mathcal{B}_1$, C_1 exist by 4.3 of [7].) Then $C_1 = \mathcal{B}_1$ for if not $\mathcal{B}_1 \subsetneq \mathcal{C}_1$ while $D(\mathcal{B}_1) = D(\mathcal{C}_1)$. But then $\mathcal{B}_1$ and $\mathcal{C}_1$ are models of T which satisfy the hypothesis of the two cardinal theorem so T is not $\aleph_1$-categorical. For, by the two cardinal theorem [5] there is a model $\mathfrak{A}$ of T with $\kappa(\mathfrak{A}) = \aleph_1$ and $\kappa(D((\mathfrak{A})) = \aleph_0$. But there is certainly a model $\mathcal{B}$ of T with $\kappa(\mathcal{B}) = \aleph_1$ and $\kappa(D(\mathcal{B})) = \aleph_1$. Thus there exists $d \in D(\mathcal{C}_1) \wedge \sim D(\mathcal{B})$. Let $C \in S_{k+2}(L)$ and $c_1, \cdots, c_k \in |\mathfrak{A}|$ such that $C(f, c_1, \cdots, c_k, v_{k+1})$ generates the principal 1-type in $Th(C, |\mathfrak{A}| \cup \{f\})$ realized by d. Then $C(f, c_1, \cdots, c_k, v_{k+1})(\mathcal{C})$ is finite. For if not, since D is strongly minimal and contains infinitely many algebraic points there exists an algebraic point $b \in |\mathfrak{A}|$ such that $\mathcal{C} \models C(f, c_1, \cdots, c_k, b)$. Since b is algebraic there exists a formula $B \in S_1(L)$ and an integer t such that $\mathcal{C} \models B(b) \wedge \exists^{\leq t} v_0 B$. But since $\mathcal{C} \models C(f, c_1, \cdots, c_k, b)$, $C(f, c_1, \cdots, c_k, v_{k+1})$ generates a principal type and $C(f, c_1, \cdots, c_k, v_{k+1})(\mathcal{C})$ is infinite, $B(\mathcal{C})$ is infinite. So for some $q < \omega$,

$$\mathcal{C} \models C(f, c_1, \cdots, c_k, d) \wedge \exists^{\leq q} v_{k+1} C(f, c_1, \cdots, c_k, v_{k+1}).$$

Let C_1 be the following member of $S_{m+2}(L)$.

$$C_{v_1, \cdots, v_{k+1}}(v_{l+1}, \cdots, v_{m+1}) \wedge A \wedge \exists^{\leq q} v_{m+1} C_{v_1, \cdots, v_{k+1}}(v_{l+1}, \cdots, v_{m+1}).$$

Let C_1' be obtained from C_1 by substituting $a_1, \cdots, a_l$ for $v_1, \cdots, v_l$ and $c_1, \cdots, c_k$ for $v_{l+1}, \cdots, v_m$. For any $b \in D(\mathcal{C}) - D(\mathcal{B})$, $R_{\mathcal{C}}(C'_{1_{v_{m+1}}}(b)) = R_{\mathcal{C}}(C'_{1_{v_{m+1}}}(d))$ since any such b realizes the same 1-type in

$Th(\mathcal{C}, \{a_1, \cdots, a_l, c_1, \cdots, c_k\})$ as d and $\mathcal{C}$ is homogeneous by Theorem 0. Since $D(\mathcal{C}) - D(\mathcal{B})$ is infinite and $\mathfrak{A}_1 \models \forall v_0 \exists^{\leq q} v_{m+1} C'$, if $R_{\mathcal{C}}(C^1_{1_{v_{m+1}}}(d)) \geq (\alpha, 1)$ then $R_{\mathcal{C}}(A') \geq (\alpha + 1)$ contrary to hypothesis. So for some $u < \alpha$ and some k, $R_{\mathcal{C}}(C'_{1_{v_{m+1}}}(d)) = (u, k)$. Thus by hypothesis, there exists a formula $C_1^* \in \Gamma^{(u)}_{C_1}$ such that $\mathcal{C} \models C_1^*(a_1, \cdots, a_l, c_1, \cdots, c_p, d)$. Let p be the maximum of q and the cardinality of $\sim C_1^*(a_1, \cdots, a_l, c_1, \cdots, c_k)(\mathcal{C}'')$ which is a finite subset of $D(\mathcal{C}'')$. Then

$$C'' \models A'(f) \wedge \sim q(f) \wedge G(C, C_1^*)(f, a_1, \cdots, a_l, c_1, \cdots, c_k)$$
$$\wedge\ G^*(C, C_1^*, p)(a_1, \cdots, a_l, c_1, \cdots, c_k)$$

so $\mathcal{C}''$ does not model Γ_2 but C'' was an arbitrary model of Γ_1 so Γ_2 is inconsistent. By the compactness theorem, there exists $k \in \omega$, $F^1, \cdots, F^s$ in $S_{k+2}(L)$ and $F_1^{i*} \in \Gamma^{(t_i)}_{F_1^i}$ for some $t_i < \alpha$ such that

$$\Gamma_1 \vdash \left(\forall v_0 \left(A'(v_0) \wedge \sim q(v_0) \rightarrow \bigvee_1^s G(F^i, F_1^{i*})(a_1, a_l, c_1, \cdots, c_k)\right)\right).$$
$$\wedge \left(\bigwedge_1^s G^*(F^i, F_1^{i*}, p_i)(a_1, \cdots, a_l, c_1, \cdots, c_k)\right).$$

$c_1, \cdots, c_k$ list the constants occurring in some F^i and are assumed to occur in each F_i for notational convenience.

Let $B' = \bigvee_1^s G(F^i, F_1^{i*})(v_0, a_1, \cdots, a_l, c_1, \cdots, c_k)$. If $(A' \wedge \sim B')(\mathfrak{A})$ is infinite then there are models of T of arbitrarily large cardinality with $(A' \wedge \sim B')(\mathcal{B}) - (A' \wedge \sim B')(\mathfrak{A}) \neq \emptyset$. Thus there is a model $\mathcal{C}$ of Γ_1 with $(A' \wedge \sim B')(\mathcal{C}) - q(\mathcal{C}) \neq \emptyset$. But this is impossible. Let H be

$$\forall v_0 \left(A' \leftrightarrow \left(\bigvee_{i=1}^s (G(F^i, F_1^{i*})(a_1, \cdots, a_l, c_1, \cdots, c_k)) \vee (A' \wedge \sim B)\right)\right)$$
$$\wedge \left(\bigwedge_{i=1}^s (G^*(F^i, F_1^{i*}, p_i))\right) \wedge \left(\exists^{\leq j} v_0 \left(A \wedge \sim \left(\bigvee_{i=1}^s G(F_i, F_1^i)\right)\right)\right).$$

Then $\mathfrak{A} \models H$ so

$$\mathfrak{A} \models \exists v_l, \cdots, \exists v_{l+k} H_{c_1, \cdots, c_k}(v_l, \cdots, v_{l+k})$$

and

$$\exists v_l \cdots \exists v_{l+k} H_{c_1, \cdots, c_k}(v_l, \cdots, v_{l+k}) \in \Gamma_A^{(u+1)}$$

where $u = \max(u_i) < \alpha$.

We return to the proof of Theorem 2. The induction hypothesis asserts that (i) is equivalent to (ii) if $m < n$. We have already proved (i) implies (ii) if $m = n$

and now we wish to show (ii) implies (i) if $m = n$. Suppose $A \in S_{l+1}(L)$, $a_1, \cdots, a_l \in |\mathfrak{A}|$, $A' = A(a_1, \cdots, a_l)$ and, for some k, $R_{\mathfrak{A}}(A') = (n, k)$. The definition of $\Theta_A^{(n)}$ allows us to assume that $k = 1$. We will find a formula $A^* \in \Gamma_A^{(n)}$ such that $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$.

By Theorem 1 (v) there is an elementary extension of $\mathfrak{B}$ of $\mathfrak{A}$ and a formula $B' \in S_1(L(\mathfrak{B}))$ such that $B'(\mathfrak{B}) \subseteq A'(\mathfrak{B})$ and $R_{\mathfrak{B}}(B') = (n, 1) = \sup\{R_{\mathcal{C}}(B') \mid \mathcal{C} \geq \mathfrak{B}\}$. Now B' and $\mathfrak{B}$ satisfy the hypothesis of Lemma 2 so there exists $B^* \in \Gamma_B^{(k+1)}$ for some $k < n$ such that $\mathfrak{B} \models B^*(b_1, \cdots, b_s)$. If $k < n - 1$ by the induction hypothesis $R_{\mathfrak{B}}(B') < (n, 1)$ so $k = n - 1$. $\mathfrak{B} \models B^*(b_1, \cdots, b_s) \wedge \forall v_0(B(b_1, \cdots, b_s) \rightarrow A')$ and $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ so for some $c_1, \cdots, c_s \in |\mathfrak{A}|$, $\mathfrak{A} \models B^*(c_1, \cdots, c_s) \wedge \forall v_0(B(c_1, \cdots, c_s) \rightarrow A')$. Since $B^* \in \Gamma_B^{(n)}$, and we have proved (i) implies (ii) for $m = n$, for some l, $R_{\mathfrak{A}}(B(c_1, \cdots, c_s)) = (n, l)$. l must equal 1 since $B(c_1, \cdots, c_s)(\mathfrak{A}) \subseteq A'(\mathfrak{A})$ and $R_{\mathfrak{A}}(A') = (n, 1)$. If $C' = C(v_0, a_1, \cdots, a_l, c_1, \cdots, c_s) = A' \wedge \sim B(v_0, c_1, \cdots, c_s)$ then $R_{\mathfrak{A}}(C') < (n, 1)$. So by induction there exists $C^* \in \bigcup_{j=1}^{n-1} \Gamma_{\mathcal{C}}^{(j)}$ such that $\mathfrak{A} \models C^*(a_1, \cdots, a_l, c_1, \cdots, c_s)$. Hence letting

$$A^* = \exists v_{l+1}, \cdots, \exists v_{l+s}((\forall v_0(A \leftrightarrow B(v_0, v_{l+1}, \cdots, v_{l+s}) \vee C)) \wedge B^* \wedge C^*).$$

A^* is in $\Gamma_A^{(n)}$ and $\mathfrak{A} \models A^*(a_1, \cdots, a_l)$ proving the theorem.

Recall that α_T is defined to be the least ordinal such that, for all $\mathfrak{A} \in \mathfrak{N}(T)$ and $\beta > \alpha_T, S^{\alpha_T}(\mathfrak{A}) = S^{\beta}(\mathfrak{A})$. In [4] Morley proved α_T exists and is less than $(2^{\aleph_0})^+$ for every complete theory. In [2] Lachlan shows that $\alpha_T \leq \omega_1$ for each complete theory. We apply Theorem 2 to prove the following conjecture of Morley.

Theorem 3. *If T is $\aleph_1$-categorical then α_T is finite.*

Proof. If for some $\mathfrak{A}$ and some $\beta \geq \omega$ there exists $p \in S^{\beta}(\mathfrak{A})$, then since T is totally transcendental for some $\gamma \geq \beta$, $p \in \mathrm{Tr}^{\gamma}(\mathfrak{A})$ and by Lemma 1 there exists $\mathfrak{B} \succeq \mathfrak{A}$, $q \in \mathrm{Tr}^{\omega}(\mathfrak{B}) \cap i_{\mathfrak{A}\mathfrak{B}}^{*-1}(p)$ so there is a formula $A' = A(v_0, a_1, \cdots, a_l)$ in $S_1(L(\mathfrak{B}))$ with $r_{\mathfrak{B}}(A') = \omega$. By Theorem 1, there exists $\mathcal{C} \succeq \mathfrak{B}$ and an integer k such that, for every elementary extension $\mathcal{C}_1$ of $\mathcal{C}$, $R_{\mathcal{C}_1}(A) = (\omega, k)$. Now by Lemma 2 with $\alpha = \omega$, there exists an $n < \omega$ and a formula $A^* \in \Gamma_A^{(n+1)}$ such that $\mathcal{C} \models A^*(a_1, \cdots, a_l)$. By Theorem 2, for some k, $R_{\mathcal{C}}(A') = (n + 1, k)$. This is a contradiction so there is no $\mathfrak{A}$ and no $\beta \geq \omega$ and no p with $p \in S^{\beta}(\mathfrak{A})$. Hence $\alpha_T < \omega$.

This proof relied on Theorem 0 which is shown in [1] to be equivalent to Vaught's conjecture that $\aleph_1$-categorical theory has either 1 or $\aleph_0$-countable models. According to Morley this conjecture had already been verified under the assumption that α_T was finite. In fact, it is easy to deduce Lemma 13 of [1] which is crucial to the proof of Vaught's conjecture from our Theorem 3.

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