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# ON DERIVED FUNCTORS OF LIMIT(1)

BY

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ABSTRACT. If $\mathfrak{A}$ is a cocomplete category with enough projectives and $\mathbf{C}$ is a $\downarrow$-finite small category, then there is a spectral sequence which shows that the cardinality of $\mathbf{C}$ and colimits over finite initial subcategories $\mathbf{C}'$ of $\mathbf{C}$ are determining factors for computation of derived functors of colimit. Applying a recent result of Mitchell to this spectral sequence we show that if the cardinality of $\mathbf{C}$ is at most $\aleph_n$, and the flat dimension of $\Delta^* Z$ (constant diagram of type $\mathbf{C}^{op}$ with value $Z$) is $k$, then the derived functors of $\lim_{\mathbf{C}}: \mathfrak{A}b^{\mathbf{C}} \to \mathfrak{A}b$ vanish above dimension $n + 1 + k$.

**Introduction.** The purpose of the paper is to study derived functors of limit. This topic was first considered by Milnor [7], Yeh [17], and Roos [14]. The results of Roos, Noebeling [11], André [1], and Laudal [6] all show that derived functors of colimit can be interpreted as the homology of a simplicial complex.

This paper introduces a spectral sequence, which isolates the cardinality of $\mathbf{C}$ and colimits over finitely generated initial subcategories $\mathbf{C}'$ of $\mathbf{C}$ as determining factors for the vanishing of derived functors of colimit (dually limit).

If $\mathfrak{A}$ is an abelian category, Stauffer [16] shows that there exists an AB5 category $D(\mathfrak{A})$, called the directed completion of $\mathfrak{A}$, and an exact, Ext-preserving, projective preserving embedding $J: \mathfrak{A} \to D(\mathfrak{A})$. $D(\mathfrak{A})$ is similar to the cocontinuous extension of $\mathfrak{A}$ studied by Hilton [4] and to Grothendieck's category of Pro-objects of $\mathfrak{A}$ [3].

If $\mathfrak{A}$ is cocomplete, we get a coreflection $U: D(\mathfrak{A}) \to \mathfrak{A}$ of $J: \mathfrak{A} \to D(\mathfrak{A})$. These two functors together give rise to a factorization

$$\operatorname*{colim}_{\mathbf{C}}: \mathfrak{A}^{\mathbf{C}} \longrightarrow \mathfrak{A} \quad \text{into} \quad \mathfrak{A}^{\mathbf{C}} \xrightarrow{\operatorname{colim}_{\mathbf{C}} J^{\mathbf{C}}} D(\mathfrak{A}) \xrightarrow{U} \mathfrak{A}.$$

When $\mathbf{C}$ is a $\downarrow$-finite small category and $\mathfrak{A}$ a cocomplete category with pro-

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jectives, we apply a well-known technique of Grothendieck [2] to the above factorization of $\operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$. This results in a first quadrant spectral sequence

$$E^2 = (L_* U)\left(L_* \operatorname*{colim}_{\mathbf{C}}\right)(J^{\mathbf{C}}(\bar{A})) \simeq L_* \operatorname*{colim}_{\mathbf{C}' \in \mathfrak{F}(\mathbf{C})} \left(L_* \operatorname*{colim}_{\mathbf{C}'}\right)(\bar{A}|\mathbf{C}')$$

which converges to $(L_* \operatorname{colim}_{\mathbf{C}})(\bar{A})$, where $\bar{A}$ is a diagram in $\mathfrak{A}$ of type $\mathbf{C}$ and $\mathfrak{F}(\mathbf{C})$ the $\downarrow$-finite directed ordered set of all finite initial subcategories $\mathbf{C}'$.

Many generalizations of ring theoretic results prove useful in applying the spectral sequence. Using a recent result of Mitchell [10], we show that if $\mathbf{C}$ is a $\downarrow$-finite small category of cardinality at most $\aleph_n$ and

$$k = \sup\{m | 0 \neq L_m \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}b^{\mathbf{C}} \to \mathfrak{A}b\},$$

then $R^r \lim_{\mathbf{C}^{op}} : \mathfrak{A}b^{\mathbf{C}^{op}} \to \mathfrak{A}b$ vanishes for $r > n + 1 + k$.

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1. **Preliminaries.** If $\mathbf{C}$ is a small category, let $|\mathbf{C}|$ denote the *set* of objects of $\mathbf{C}$ and $\mathbf{C}(p,q)$ the set of morphisms from $p$ to $q$. If $\alpha$ is a morphism of $\mathbf{C}$, then $d\alpha$ and $r\alpha$ will denote the domain and range of $\alpha$, respectively. Let $\|\mathbf{C}\|$ represent the cardinality of the set $\mathbf{C}$. Then $\mathbf{C}$ is said to be an *n-category* if $\|\mathbf{C}\| \leq \aleph_n$ for $n > 0$, and a *finite* category if $\|\mathbf{C}\| < \aleph_0$.

A subcategory $\mathbf{C}'$ of $\mathbf{C}$, denoted by $\mathbf{C}' \leq \mathbf{C}$, will be called *initial* if $\alpha \in \mathbf{C}$ with $r\alpha \in |\mathbf{C}'|$ implies $\alpha \in \mathbf{C}'$ (and consequently $d\alpha \in |\mathbf{C}'|$). It is clear that any initial subcategory is full. Let $\mathbf{C}(p)$ denote the smallest initial subcategory containing $p$. Then if $\mathbf{C}(p,q) \neq \varnothing$, it is clear that $\mathbf{C}(p) \leq \mathbf{C}(q)$. Also, $\mathbf{C}'$ initial implies $\mathbf{C}' = \bigcup_{p' \in |\mathbf{C}'|} \mathbf{C}(p')$ and $\mathbf{C}(p') \leq \mathbf{C}'$ for every $p' \in |\mathbf{C}'|$.

**Definition 1.1.** A small category $\mathbf{C}$ is said to be *downward finite*, $\downarrow$-finite, if $\mathbf{C}(p)$ is finite for every $p \in |\mathbf{C}|$.

Let $\mathfrak{F}(\mathbf{C})$ represent the collection of all finitely-generated initial subcategories $\mathbf{C}'$ of $\mathbf{C}$. If $\mathbf{C}$ is $\downarrow$-finite, then clearly $\mathfrak{F}(\mathbf{C})$ satisfies the following conditions:

(i) $\mathfrak{F}(\mathbf{C})$ is a directed ordered set under the natural ordering of inclusion of categories, with initial element the empty subcategory $\varnothing$.

(ii) $\mathfrak{F}(\mathbf{C})$ is $\downarrow$-finite, i.e. any finitely-generated initial subcategory has a finite number of initial subcategories.

(iii) If $\mathbf{C}$ is a $n$-category, then so is $\mathfrak{F}(\mathbf{C})$, i.e. $\|\mathbf{C}\| \leq \aleph_n$ implies $\|\mathfrak{F}(\mathbf{C}\| \leq \aleph_n$.

(iv) For every $p \in |\mathbf{C}|$, $\mathbf{C}(p) \in \mathfrak{F}(\mathbf{C})$.

If $\mathfrak{A}$ is an abelian category, then $\mathfrak{A}^{\mathbf{C}}$ will denote the abelian category of all diagrams of type $\mathbf{C}$, i.e. covariant functors $\bar{A} : \mathbf{C} \to \mathfrak{A}$, with $\mathfrak{A}^{\mathbf{C}}(\bar{A}, \bar{B})$ the abelian group of natural transformations from $\bar{A}$ to $\bar{B}$. In particular, let $\Delta A : \mathbf{C} \to \mathfrak{A}$

represent the constant functor with value $A$ and $\Delta^*A : \mathbf{C}^{\mathrm{op}} \to \mathfrak{A}$ the dual diagram. If $F : \mathfrak{A} \to \mathfrak{B}$ is any functor, let $F^{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{B}^{\mathbf{C}}$ denote the canonical functor given by $F^{\mathbf{C}}(\overline{A})_p = F(A_p)$.

It is well known [8] that if $\mathfrak{A}$ is a cocomplete abelian category with enough projectives and/or injectives, then so is $\mathfrak{A}^{\mathbf{C}}$. For example, if $\mathfrak{A} = \mathfrak{A}b$, the category of abelian groups, then $\mathfrak{A}b^{\mathbf{C}}$ is an AB5 category with enough projectives and injectives.

When $\mathfrak{A}$ is cocomplete, there is a functor $W : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}^{\mathfrak{F}(\mathbf{C})}$ defined by $(W\overline{A})_{\mathbf{C}'} = \mathrm{colim}_{\mathbf{C}'} \overline{A} \mid \mathbf{C}'$ with $(W\overline{A})^{\mathbf{C}'}_{\mathbf{C}''} : (W\overline{A})_{\mathbf{C}'} \to (W\overline{A})_{\mathbf{C}''}$ the canonical map of colimits induced by the inclusion $\mathbf{C}' \leq \mathbf{C}''$.

**Lemma 1.2.** *If $\mathfrak{A}$ is a cocomplete abelian category and $\mathbf{C}$ is a $\downarrow$-finite small category, then*

$$\begin{array}{ccc} \mathfrak{A}^{\mathbf{C}} & \xrightarrow{W} & \mathfrak{A}^{\mathfrak{F}(\mathbf{C})} \\ & {\scriptstyle \mathrm{colim}_{\mathbf{C}}} \searrow \quad \swarrow {\scriptstyle \mathrm{colim}_{\mathfrak{F}(\mathbf{C})}} & \\ & \mathfrak{A} & \end{array}$$

*commutes up to an isomorphism.*

This follows easily from the definitions.

Furthermore, when $\mathfrak{A}$ cocomplete, there are two associated functors between $\mathfrak{A}$ and $\mathfrak{A}^{\mathbf{C}}$ for each $p \in |\mathbf{C}|$. The first is the canonical evaluation functor $\mathrm{ev}_p : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ defined by $\mathrm{ev}_p(\overline{A}) = A_p$, where $\overline{A} \in \mathfrak{A}^{\mathbf{C}}$. It is exact since exactness in $\mathfrak{A}^{\mathbf{C}}$ is "pointwise". The second functor is $E_p : \mathfrak{A} \to \mathfrak{A}^{\mathbf{C}}$ which is constructed in the following way. For each $X \in \mathfrak{A}$ and $q \in |\mathbf{C}|$, let $(E_pX)_q = \coprod_{p \xrightarrow{\alpha} q} X$, and let $(E_pX)(\beta) : (E_pX)_q \to (E_pX)_{q'}$, $\beta : q \to q'$ in $\mathbf{C}$, be the canonical morphism such that $(E_pX)(\beta) i_\alpha = i_{\beta \cdot \alpha}$, $i_\alpha : X \to \coprod_{p \xrightarrow{\alpha} q} X$ being the natural inclusion into the coproduct. Similarly, for each morphism $f : X \to Y$ in $\mathfrak{A}$, there is a natural transformation $(E_pf) : (E_pX) \to (E_pY)$ defined by $(E_pf) i_\alpha = i_\alpha \cdot f$.

**Proposition 1.3.** *If $\mathfrak{A}$ is cocomplete and abelian, then*

(i) *$E_p : \mathfrak{A} \to \mathfrak{A}^{\mathbf{C}}$ is the coadjoint of $\mathrm{ev}_p : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$.*

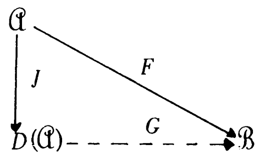
(ii) *$E_p : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ is right exact and also preserves projectives (since $\mathrm{ev}_p : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ is exact).*

(iii) *When $\mathfrak{A}$ has enough projectives $\mathfrak{A}^{\mathbf{C}}$ has enough canonical projectives of the form $\coprod_{q \in |\mathbf{C}|} E_qP_q$, $P_q$ projective in $\mathfrak{A}$. If $\overline{A} \in \mathfrak{A}^{\mathbf{C}}$ and for each $q \in |\mathbf{C}|$, $P_q \to A_q$ is an epimorphism with $P_q$ projective, then $\coprod_{q \in |\mathbf{C}|} E_qP_q \to \overline{A}$ is an epimorphism in $\mathfrak{A}^{\mathbf{C}}$.*

## 2. $D(\mathcal{A})$ and the spectral sequence.

**Theorem 2.1.** *Associated with any abelian category $\mathcal{A}$ there is an* AB5 *category $D(\mathcal{A})$ (called the directed completion of $\mathcal{A}$), and a natural embedding $J:\mathcal{A}\to D(\mathcal{A})$ such that $J:\mathcal{A}\to D(\mathcal{A})$ is exact, full, projective-preserving and* Ext*-preserving (i.e.* $\mathrm{Ext}^*(J(A), J(B)) \simeq \mathrm{Ext}^*(A,B)$). *Furthermore, $J:\mathcal{A}\to D(\mathcal{A})$ and $D(\mathcal{A})$ together satisfy the following universal extension property:*

(i) *If $\mathcal{B}$ is any cocomplete abelian category and $F:\mathcal{A}\to\mathcal{B}$ is right exact, then there exists a unique cocontinuous (i.e. colimit-preserving) functor $G: D(\mathcal{A})\to\mathcal{B}$ such that*




*commutes up to isomorphism.*

(ii) *If $\mathcal{B}$ is* AB5 *and $F:\mathcal{A}\to\mathcal{B}$ is exact, then $G: D(\mathcal{A})\to\mathcal{B}$ is cocontinuous and exact.*

For the details of the proof see Stauffer [16].

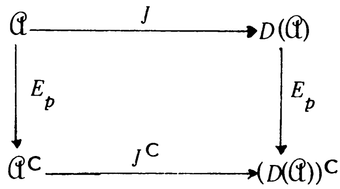
In particular, when $\mathcal{A}$ itself is cocomplete there exists a unique cocontinuous (and consequently right exact) functor $U: D(\mathcal{A})\to\mathcal{A}$ such that $U\cdot J\simeq \mathrm{id}_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}$. Thus $\mathcal{A}$ can be considered as a coreflective subcategory of $D(\mathcal{A})$. The next proposition follows easily from the facts that $U: D(\mathcal{A})\to\mathcal{A}$ is cocontinuous and $U\cdot J\simeq\mathrm{id}_{\mathcal{A}}$.

**Proposition 2.2.** *If $\mathbf{C}$ is any small category and $\mathcal{A}$ is cocomplete and abelian, then $U(\mathrm{colim}_{\mathbf{C}}\, J^{\mathbf{C}}(\overline{A}))\simeq \mathrm{colim}_{\mathbf{C}}(\overline{A})$ for all $\overline{A}\in\mathcal{A}^{\mathbf{C}}$.*

By Proposition 2.2, $\mathrm{colim}_{\mathbf{C}}:\mathcal{A}^{\mathbf{C}}\to\mathcal{A}$ is factored into $\mathrm{colim}_{\mathbf{C}}:\mathcal{A}^{\mathbf{C}}\to D(\mathcal{A})$ and $U: D(\mathcal{A})\to\mathcal{A}$. This factorization, for $\mathbf{C}$ a $\downarrow$-finite small category and a cocomplete abelian category with enough projectives, will yield the spectral sequence which is the major tool of this paper. As a first step, we prove a series of lemmas to show that $J^{\mathbf{C}}:\mathcal{A}^{\mathbf{C}}\to D(\mathcal{A})^{\mathbf{C}}$ preserve canonical projectives.

For the remainder of this section, $\mathcal{A}$ will be assumed to be a cocomplete abelian category with enough projectives.

**Lemma 2.3.** *For every $p\in|\mathbf{C}|$, the following diagram commutes.*



**Proof.** It suffices to show that, for each $q \in |\mathbf{C}|$, $J^{\mathbf{C}}(E_p X)_q = E_p(J(X))_q$. By definition, $J^{\mathbf{C}}(E_p X)_q = J(E_p X)_q = J(\amalg_{p \xrightarrow{\alpha} q} X)$. Since $\mathbf{C}$ is $\downarrow$-finite, $(E_p X)_q = \amalg_{p \xrightarrow{\alpha} q} X$ is a finite coproduct. $J : \mathfrak{A} \to D(\mathfrak{A})$ additive insures that $J(\amalg_{p \xrightarrow{\alpha} q} X) = \amalg_{p \xrightarrow{\alpha} q} J(X) = E_p(J(X))_q$, and the lemma follows.

Using $\downarrow$-finiteness of $\mathbf{C}$, a proof similar to the above yields the next lemma.

**Lemma 2.4.** *Let* $\mathbf{C}$ *be any* $\downarrow$*-finite small category,* $\{X_p\}_{p \in |\mathbf{C}|}$ *any collection of objects in* $\mathfrak{A}$. *Then*

$$J^{\mathbf{C}}\left(\coprod_{p \in |\mathbf{C}|} E_p X_p\right) = \coprod_{p \in \mathbf{C}} E_p(J(X_p)).$$

**Corollary 2.5.** $J^{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to D(\mathfrak{A})^{\mathbf{C}}$ *preserves canonical projectives.*

**Proof.** That $J^{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to D(\mathfrak{A})^{\mathbf{C}}$ preserves projectives follows immediately from Lemma 2.4, the definition of a canonical projective (1.3) and the fact that both $E_p : \mathfrak{A} \to \mathfrak{A}^{\mathbf{C}}$ and $J : \mathfrak{A} \to D(\mathfrak{A})$ preserve projectives.

**Theorem 2.6** (**Spectral sequence**). *If* $\mathbf{C}$ *is a* $\downarrow$*-finite ordered set,* $\mathfrak{A}$ *is cocomplete with projectives, and* $\overline{A} \in \mathfrak{A}^{\mathbf{C}}$, *then there is a first quadrant spectral sequence*

$$E^2_{pq} = (L_p U)\left(L_q \operatorname*{colim}_{\mathbf{C}}\right)(J^{\mathbf{C}}(\overline{A}))$$

*converging to* $(L_{p+q} \operatorname{colim}_{\mathbf{C}})(\overline{A})$.

**Proof.** Both $\operatorname{colim}_{\mathbf{C}} : D(\mathfrak{A})^{\mathbf{C}} \to D(\mathfrak{A})$ and $J^{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to D(\mathfrak{A})^{\mathbf{C}}$ (by Corollary 2.5) preserve projectives. Hence, the hypotheses of the "Grothendieck Two Functor Theorem" [2] are satisfied since $U \circ \operatorname{colim}_{\mathbf{C}} J^{\mathbf{C}} \simeq \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$, $U : D(\mathfrak{A}) \to \mathfrak{A}$ is right exact and $\operatorname{colim}_{\mathbf{C}} J^{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to D(\mathfrak{A})$ preserves projectives. Applying this theorem of Grothendieck yields a spectral sequence with $E^2_{pq} \simeq (L_p U)(L_q \operatorname{colim}_{\mathbf{C}} J^{\mathbf{C}})(\overline{A})$ converging to $(L_{p+q} \operatorname{colim}_{\mathbf{C}})(\overline{A})$. But since $J^{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to D(\mathfrak{A})^{\mathbf{C}}$ is both exact and projective-preserving,

$$\left(L_p \operatorname*{colim}_{\mathbf{C}} J^{\mathbf{C}}\right)(\overline{A}) \simeq \left(L_p \operatorname*{colim}_{\mathbf{C}}\right)(J^{\mathbf{C}}(\overline{A})),$$

giving the required form.

Also, $D(\mathfrak{A})$, AB5, and $J^{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to D(\mathfrak{A})^{\mathbf{C}}$ exact yield the next corollary.

**Corollary 2.7.** *If* $\Lambda$ *is a* $\downarrow$*-finite directed ordered set, and* $\overline{A} \in \mathfrak{A}^{\Lambda}$, *then* $(L_p U)(\operatorname{colim}_{\Lambda} J^{\Lambda}(\overline{A})) \simeq (L_p \operatorname{colim}_{\Lambda})(\overline{A})$ *for every* $p > 0$.

Recall that $W: \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}^{\mathcal{F}(\mathbf{C})}$ is the functor defined by $(W\overline{A})_{\mathbf{C}'} = \operatorname{colim}_{\mathbf{C}'} \overline{A} \mid \mathbf{C}'$, where $\mathcal{F}(\mathbf{C})$ is the $\downarrow$-finite directed ordered set consisting of all finitely-generated initial subcategories $\mathbf{C}'$.

**Lemma 2.8.** *If* $\mathbf{C}$ *is a* $\downarrow$*-finite small category, then*

$$\left(L_p \operatorname*{colim}_{\mathbf{C}} J^{\mathbf{C}}\right)(\overline{A}) \simeq \operatorname*{colim}_{\mathcal{F}(\mathbf{C})} J^{\mathcal{F}(\mathbf{C})}(L_p W)(\overline{A})$$

*for every* $\overline{A} \in \mathfrak{A}^{\mathbf{C}}$.

**Proof.** Since $J: \mathfrak{A} \to D(\mathfrak{A})$ is exact, it commutes with finite colimits, and therefore $J((W\overline{A})_{\mathbf{C}'}) \simeq W(J^{\mathbf{C}}(A))_{\mathbf{C}'}$, for every $\mathbf{C}' \in \mathcal{F}(\mathbf{C})$. But by Lemma 1.2, $\operatorname{colim}_{\mathbf{C}} J^{\mathbf{C}}(\overline{A}) \simeq \operatorname{colim}_{\mathcal{F}(\mathbf{C})} W(J^{\mathbf{C}}(A))$, and thus $\operatorname{colim}_{\mathbf{C}} J^{\mathbf{C}}(\overline{A}) \simeq \operatorname{colim}_{\mathbf{C}' \in \mathcal{F}(\mathbf{C})} J(W\overline{A})_{\mathbf{C}'}$ $\equiv \operatorname{colim}_{\mathcal{F}(\mathbf{C})} J^{\mathcal{F}(\mathbf{C})}(W\overline{A})$. Since $\mathcal{F}(\mathbf{C})$ is a $\downarrow$-finite directed ordered set and $D(\mathfrak{A})$ is AB5, $\operatorname{colim}_{\mathcal{F}(\mathbf{C})} J^{\mathcal{F}(\mathbf{C})}: \mathfrak{A}^{\mathcal{F}(\mathbf{C})} \to D(\mathfrak{A})$ is exact and therefore commutes with homology. Consequently, $(L_* \operatorname{colim}_{\mathbf{C}}) J^{\mathbf{C}}(\overline{A}) \simeq \operatorname{colim}_{\mathcal{F}(\mathbf{C})} J^{\mathcal{F}(\mathbf{C})} L_*(W\overline{A})$.

Combining Lemma 2.8, Theorem 2.6, and Corollary 2.7 yields several equivalent forms for the spectral sequence.

**Theorem 2.9.** *If* $\mathfrak{A}$ *is cocomplete with enough projectives,* $\mathbf{C}$ *a* $\downarrow$*-finite small category, and* $\overline{A} \in \mathfrak{A}^{\mathbf{C}}$, *then there is a first quadrant spectral sequence*

$$\begin{aligned} E^2_{pq} &\simeq (L_p U)\left(L_q \operatorname*{colim}_{\mathbf{C}}\right) J^{\mathbf{C}}(\overline{A}) \simeq (L_p U)\left(\operatorname*{colim}_{\mathcal{F}(\mathbf{C})} J^{\mathcal{F}(\mathbf{C})}(L_q W(\overline{A}))\right) \\ &\simeq \left(L_p \operatorname*{colim}_{\mathcal{F}(\mathbf{C})}\right)(L_q W)(\overline{A}) \equiv \left(L_p \operatorname*{colim}_{\mathbf{C}' \in \mathcal{F}(\mathbf{C})}\right)\left(L_q \operatorname*{colim}_{\mathbf{C}'}\right)(\overline{A} \mid \mathbf{C}') \end{aligned}$$

*converging to* $(L_{p+q} \operatorname{colim}_{\mathbf{C}})(\overline{A})$.

Thus from the factorization of $\operatorname{colim}_{\mathbf{C}}: \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ into $\operatorname{colim}_{\mathbf{C}} J^{\mathbf{C}}: \mathfrak{A}^{\mathbf{C}} \to D(\mathfrak{A})$ and $U: D(\mathfrak{A}) \to \mathfrak{A}$, we get a spectral sequence which involves derived functors of colimit over a directed ordered set, namely, $\mathcal{F}(\mathbf{C})$.

3. **Applications.** In this section, we apply a recent result of Mitchell [10] to the spectral sequence. This shows the cardinality of $\mathbf{C}$ is related to the vanishing of higher derived functors of $\operatorname{colim}_{\mathbf{C}}: \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$, $\mathfrak{A}$ an AB4 category and $\mathbf{C}$ a $\downarrow$-finite small category. The method will employ generalizations of dimension theory for rings developed by Mitchell in *Rings with several objects* [9].

If $\overline{A} \in \mathfrak{A}^{\mathbf{C}}$, then the *homological (projective) dimension* of $\overline{A}$, denoted $\operatorname{hd}_{\mathbf{C}} \overline{A}$, is defined to be $\sup\{k \mid \operatorname{Ext}^k_{\mathbf{C}}(\overline{A}, \_) \neq 0\}$; or equivalently, to be the smallest integer for which there is a projective resolution

$$0 \longrightarrow \overline{P}_n \longrightarrow \cdots \longrightarrow \overline{P}_0 \longrightarrow \overline{A} \longrightarrow 0$$

when $\mathfrak{A}$ is cocomplete with projectives.

**Proposition 3.1.** $\mathrm{hd}_{\mathbf{C}}\,\Delta Z = \sup\{k \mid 0 \neq R^k \lim_{\mathbf{C}} : \mathfrak{A}b^{\mathbf{C}} \to \mathfrak{A}b\}$.

**Proof.** Let $\Delta : \mathfrak{A}b \to \mathfrak{A}b^{\mathbf{C}}$ be the full exact embedding which assigns to each $G \in \mathfrak{A}b$ the constant diagram $\Delta G$. By definition, $\Delta : \mathfrak{A}b \to \mathfrak{A}b^{\mathbf{C}}$ is the coadjoint of $\lim_{\mathbf{C}} : \mathfrak{A}b^{\mathbf{C}} \to \mathfrak{A}b$ and therefore $\mathfrak{A}b^{\mathbf{C}}(\Delta Z, \overline{A}) \simeq \mathfrak{A}b(Z, \lim_{\mathbf{C}} \overline{A}) \simeq \lim_{\mathbf{C}} \overline{A}$. Taking derived functors gives the result.

If $\mathfrak{A}$ is any cocomplete category and $\mathbf{C}$ is any small category, there exists a covariant additive cocontinuous (colimit-preserving) bifunctor $\otimes_{\mathbf{C}} : \mathfrak{A}b^{\mathbf{C}^{\mathrm{op}}} \times \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ (whose value on the pair $(M, F)$ is denoted by $M \otimes_{\mathbf{C}} F$), such that for every $M \in \mathfrak{A}b^{\mathbf{C}^{\mathrm{op}}}$, $F \in \mathfrak{A}^{\mathbf{C}}$, and $X \in \mathfrak{A}$, $\mathfrak{A}b^{\mathbf{C}^{\mathrm{op}}}(M, \mathfrak{A}(F, X)) \simeq \mathfrak{A}(M \otimes_{\mathbf{C}} F, X)$ (where $\mathfrak{A}(F, X) : \mathbf{C}^{\mathrm{op}} \to \mathfrak{A}b$ is given by $\mathfrak{A}(F, X)_p = \mathfrak{A}(F_p, X)$). Define $\mathrm{Tor}_*^{\mathbf{C}}(M, F) \equiv H_*(P \otimes_{\mathbf{C}} F)$, where $P$ is a projective resolution for $M$. From [12], we know that when $\mathfrak{A}$ is AB4 and when $M$ has free values (for example $M = \Delta^* Z$), $\mathrm{Tor}_*^{\mathbf{C}}(M, \_)$ is the sequence of left satellites (left derived functors when $\mathfrak{A}$ has enough projectives) of $M \otimes_{\mathbf{C}} \_ : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$.

**Lemma 3.2.** *If* $\mathfrak{A}$ *is* AB4, *then* $\mathrm{Tor}_*^{\mathbf{C}}(\Delta^* Z, \_) : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ *and* $L_* \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ *are isomorphic.*

**Proof.** If $F \in \mathfrak{A}^{\mathbf{C}}$ and $X \in \mathfrak{A}$, then by definitions of $\operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ and $\lim_{\mathbf{C}^{\mathrm{op}}} : \mathfrak{A}b^{\mathbf{C}^{\mathrm{op}}} \to \mathfrak{A}b$,

$$\mathfrak{A}(\Delta^* Z \otimes_{\mathbf{C}} F, X) \simeq \mathfrak{A}b^{\mathbf{C}^{\mathrm{op}}}(\Delta^* Z, \mathfrak{A}(F, X)) \simeq \mathfrak{A}b\Big(Z, \lim_{\mathbf{C}^{\mathrm{op}}} \mathfrak{A}(F, X)\Big)$$

$$\simeq \lim_{\mathbf{C}^{\mathrm{op}}} \mathfrak{A}(F, X) \simeq \mathfrak{A}\Big(\operatorname*{colim}_{\mathbf{C}} F, X\Big).$$

By Yoneda, this composite natural equivalence must come from a natural equivalence. Hence

$$\Delta^* Z \otimes_{\mathbf{C}} F \simeq \operatorname*{colim}_{\mathbf{C}} F \quad \text{and} \quad \Delta^* Z \otimes_{\mathbf{C}} \_ \simeq \operatorname*{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \longrightarrow \mathfrak{A}.$$

Since $\mathfrak{A}$ is AB4, $L_* \operatorname{colim}_{\mathbf{C}} \simeq \mathrm{Tor}_*^{\mathbf{C}}(\Delta^* Z, \_) : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$.

If $\mathfrak{A} = \mathfrak{A}b$, we say the *weak* (*or flat*) *dimension* of $M \in \mathfrak{A}b^{\mathbf{C}^{\mathrm{op}}}$, denoted $\mathrm{wd}_{\mathbf{C}} M$, is the $\sup\{k \mid 0 \neq \mathrm{Tor}_k^{\mathbf{C}}(M, \_) : \mathfrak{A}b^{\mathbf{C}} \to \mathfrak{A}b\}$. Thus by Lemma 3.2, $\mathrm{wd}_{\mathbf{C}} \Delta^* Z = \sup\{k \mid 0 \neq L_k \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}b^{\mathbf{C}} \to \mathfrak{A}b\}$. Now when $\mathfrak{A}$ is AB5, we can use flat resolutions of $M$ to compute $\mathrm{Tor}(M, F)$. This yields the second part of the following (see [9]).

**Corollary 3.3.** (i) *If* $\mathfrak{A}$ *is* AB4 *and* $\mathrm{hd}_{\mathbf{C}^{\mathrm{op}}} \Delta^* Z = r$, *then* $0 = L_k \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ *for every* $k > r$.

(ii) *If* $\mathfrak{A}$ *is* AB5 *and* $\mathrm{wd}_{\mathbf{C}} \Delta^* Z = r$, *then* $0 = L_k \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ *for every* $k > r$.

Using other generalizations of ring theoretic results of Osofsky [13], Mitchell [10] proves the next result.

**Theorem 3.4.** *Let* $\aleph_n$ *be the smallest cardinal number of a cofinal subset of the directed (upward) ordered set* $\Lambda$ $(-1 \le n \le \infty)$. *Then* $\mathrm{hd}_{\Lambda^{\mathrm{op}}} \Delta^* Z = n+1$.

This and the above corollary immediately imply that $L_p \operatorname{colim}_\Lambda : \mathfrak{A}^\Lambda \to \mathfrak{A}$ vanish for $p$ above $n+1$ whenever $\mathfrak{A}$ is AB4, e.g. $\mathfrak{A} = \mathfrak{A}b$.

Using these preliminary results, we now consider the spectral sequence.

**Theorem 3.5.** *Suppose* $\mathfrak{A}$ *is* AB4 *category with projectives, and* $\mathbf{C}$ *is small* $\downarrow$*-finite n-category with* $\mathrm{wd}_{\mathbf{C}} \Delta^* Z = k$. *Then* $L_r \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ *vanishes whenever* $r > n + 1 + k$.

**Proof.** By Theorem 2.9, there exists a first quadrant spectral sequence

$$E^2_{pq} = (L_p U)\left(L_q \operatorname*{colim}_{\mathbf{C}}\right)(J^{\mathbf{C}}(\overline{A})) \simeq \left(L_p \operatorname*{colim}_{\mathfrak{F}(\mathbf{C})}\right)(L_q W)(\overline{A})$$

converging to $(L_{p+q} \operatorname{colim}_{\mathbf{C}})(\overline{A})$ for every $\overline{A} \in \mathfrak{A}^{\mathbf{C}}$. We first hold $p$ constant. Since $\mathfrak{D}(\mathfrak{A})$ is AB5, Corollary 3.3(ii) and $\mathrm{wd}_{\mathbf{C}} \Delta^* Z = k$ insure that $(L_p U)$ $\cdot (L_q \operatorname{colim}_{\mathbf{C}})(J^{\mathbf{C}}(\overline{A}))$ is zero for $q > k$. Next, let $q$ be held constant. $\mathbf{C}$ an $n$-category implies $\mathfrak{F}(\mathbf{C})$, the directed set of all finite initial subcategories, is also a $n$-category, i.e. $\|\mathfrak{F}(\mathbf{C})\| < \aleph_n$. Therefore, by Proposition 3.4, $\mathrm{hd}_{\mathfrak{F}(\mathbf{C})^{\mathrm{op}}} \Delta^* Z = n+1$ and $(L_p \operatorname{colim}_{\mathfrak{F}(\mathbf{C})})(L_q W)(\overline{A}) = 0$ for $p > n+1$. Combining these together yields $(L_r \operatorname{colim}_{\mathbf{C}})(\overline{A}) = 0$ for $r > n + 1 + k$.

The dual statement is the following.

**Theorem 3.6.** *If* $\mathfrak{A}$ *is* AB4$^*$ *with injectives and* $\mathbf{C}$ *is a* $\downarrow$*-finite small n-category with* $\mathrm{wd}_{\mathbf{C}} \Delta^* Z = k$, *then* $R^r \lim_{\mathbf{C}^{\mathrm{op}}} : \mathfrak{A}^{\mathbf{C}^{\mathrm{op}}} \to \mathfrak{A}$ *is zero for* $r > n + 1 + k$.

In the case when $\mathfrak{A} = \mathfrak{A}b$, the following corollary holds.

**Corollary 3.7.** *If* $\mathbf{C}$ *is a* $\downarrow$*-finite small n-category with* $\mathrm{wd}_{\mathbf{C}} \Delta^* Z = k$, *then* $\mathrm{hd}_{\mathbf{C}^{\mathrm{op}}} \Delta^* Z \le n + 1 + k$.

This follows from Lemma 3.1.

Lastly, putting Corollary 3.3 and Corollary 3.7 together, we can drop the hypothesis of Corollary 3.7 that $\mathfrak{A}$ have enough projectives.

**Theorem 3.9.** *If* $\mathfrak{A}$ *is an* AB4 *category and* $\mathbf{C}$ *is a* $\downarrow$*-finite small n-category with* $\mathrm{wd}_{\mathbf{C}} \Delta^* Z = k$, *then* $L_r \operatorname{colim}_{\mathbf{C}} : \mathfrak{A}^{\mathbf{C}} \to \mathfrak{A}$ *vanishes for* $r > n + 1 + k$.

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