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THE AUTOMORPHISM GROUP OF AN ABELIAN p-GROUP AND ITS NORMAL p-SUBGROUPS

BY

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ABSTRACT. Let Γ be the automorphism group of a nonelementary reduced abelian p-group, $p \geq 5$. It is shown that every noncentral normal subgroup of Γ contains a noncentral normal subgroup Δ of Γ such that $\Delta^p = 1$. Furthermore, every cyclic normal subgroup of Γ is contained in the center of Γ.

1. **The results.** Throughout this article, G denotes a reduced abelian p-group for some prime $p \geq 5$ and AG its automorphism group. It is well known that 1 is the only normal p-subgroup of AG if G is elementary abelian (see [6, p. 409]). However, if $pG \neq 0$ and G is not cyclic, then AG does in fact possess a large amount of noncentral (i.e. not contained in the center of AG) normal p-subgroups. This follows from our investigations in [7] where we determined the class of all (not necessarily reduced) abelian p-groups G such that every noncentral normal subgroup of AG contains a noncentral normal p-subgroup of AG.

The purpose of this article is to improve earlier results (obtained in [7]) for the special case of reduced groups. We shall prove the following theorem.

Theorem A. *For a noncyclic reduced abelian p-group G, where $p \geq 5$, the following conditions are equivalent.*

(i) $pG \neq 0$.

(ii) *AG contains a normal p-subgroup $\neq 1$.*

(iii) *AG contains noncentral normal p-subgroups.*

(iv) *Every noncentral normal subgroup of AG contains a noncentral normal subgroup of AG of exponent p.*

The exponent of a group X is defined to be the least positive integer n such that $x^n = 1$ for all $x \in X$.

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The proof of Theorem A will be easily completed once we have established the following result which is quite interesting by itself.

Theorem B. *If G is a reduced abelian p-group, $p \geq 5$, then every cyclic normal subgroup of AG is contained in the center of AG.*

2. **Preliminaries.** Our notation and terminology concerning abelian groups will be standard (see e.g. [3]). Mappings will be written to the right. If there is no danger of confusion we may not distinguish between different identity mappings and write 1 instead of 1_G. The following symbols will be used.

$G[p] = \{g \in G \mid pg = 0\}$ [socle of G],

$o(x)$ = order of x,

$\langle x \rangle$ = cyclic subgroup generated by x,

$H \oplus K$ = direct sum of H and K,

AG = automorphism group of G,

zAG = center of AG,

$c\Delta = \{\alpha \in AG \mid \alpha\delta = \delta\alpha \text{ for all } \delta \in \Delta\}$ [centralizer of $\Delta \leq AG$ in AG],

$\alpha|S$ = restriction of $\alpha \in AG$ to $S \leq G$,

R_p^* = group of units in the ring of p-adic integers.

A subgroup S of a group X is called *noncentral* if S is not contained in the center of X. For n an integer, X^n denotes the subgroup of X generated by all x^n where $x \in X$. The exponent of X is the least positive integer n such that $X^n = 1$.

Throughout this article, G will denote a reduced abelian p-group, $p \geq 5$.

In the course of our proofs, a number of well-known facts on the automorphism group of G will be used constantly. They are collected here for the convenience of the reader.

(2.1) *The center of* AG. The center of AG consists precisely of the multiplication with p-adic units, i.e.

$$zAG = R_p^* \cdot 1_G$$

[1, pp. 110, 111]. If G is unbounded then $zAG \simeq R_p^*$, and the center of AG contains no element of order p. If G is bounded of exponent p^{m+1}, where $m \geq 0$, then zAG is a cyclic group of order $p^m(p-1)$. An automorphism α of G belongs to zAG if and only if $x\alpha \in \langle x \rangle$ for all cyclic direct summands $\langle x \rangle$ of G [5, p. 201]. Furthermore, AG is commutative if and only if G is cyclic (cf. [3, p. 222]).

(2.2) *The normal p-subgroups of* AG. Let

$$F^nG = (p^nG)[p]/(p^{n+1}G)[p]$$

denote the nth Ulm-factor of G. The maximal normal p-subgroup $P_{\max}$ of AG consists of all torsion elements $\alpha \in AG$ such that α induces the 1-automorphism in F^nG for all integers $n \geq 0$ [6, p. 412]. If α induces the identity mapping in *all* Ulm-factors of G, then the conditions $\alpha^{p^n} = 1$ and $\alpha|p^nG = 1$ are equivalent [10, p. 101]. In particular, if $p^nG = 0$, then every automorphism of G fixing $G[p]$ elementwise belongs to $P_{\max}$ and, therefore, has order a power of p.

3. **The proofs.** We start out with a proof of Theorem B for a special case. In the following lemma no restrictions are imposed upon p.

Lemma 3.1. *Every normal subgroup of AG of order p is contained in the center of AG.*

Proof. Let N be a normal subgroup of AG of order p. Then $N = \langle \eta \rangle$, $o(\eta) = p$, is cyclic, and the automorphism group AN of N has order $p - 1$. Since AG/cN is isomorphic to a group of automorphisms of N, it follows that

$$\eta \cdot \alpha^{p-1} = \alpha^{p-1} \cdot \eta \quad \text{for all } \alpha \in AG, \tag{3.1}$$

and

$$\eta\alpha = \alpha\eta \quad \text{for all } \alpha \in AG \text{ of } p\text{-power order}. \tag{3.2}$$

Assume, by way of contradiction, that

$$\eta \notin zAG. \tag{3.3}$$

In [5] we have shown that (3.1) is valid for $\eta \notin zAG$ only if G is bounded (p. 206) and, also, that $\eta \notin zAG$ implies $x\eta \notin \langle x \rangle$ for some cyclic direct summand $\langle x \rangle$ of G (cf. (2.1)). Hence (3.3) implies the existence of a decomposition $G = \langle x \rangle \oplus \langle y \rangle \oplus C$ such that

$$x\eta = kx + ly + c, \qquad c \in C, \tag{3.4}$$

for some integers k and l, and

$$ly \neq 0. \tag{3.5}$$

We construct an endomorphism σ of G as follows. If $o(x) > o(y)$, choose any $x' \in \langle x \rangle$ such that $o(x') = o(y)$; if $o(x) \leq o(y)$, let $x' = x$. In either case, G has an endomorphism σ such that

$$x\sigma = 0, \qquad y\sigma = x', \qquad C\sigma = 0. \tag{3.6}$$

Since $\sigma^2 = 0$, it follows that $\alpha = 1_G + \sigma$ is an automorphism of G the order of which is equal to the (additive) order of σ and therefore a power of p (cf. [3, p. 221]). From (3.2) we obtain

(3.7) $$\eta\alpha = \alpha\eta.$$

If $o(x) > o(y)$, then $o(y) = o(x')$ by construction, and $lx' \neq 0$ because of (3.5). This together with (3.4) and (3.6) implies

$$x\eta\alpha = x\eta + lx' \neq x\eta = x\alpha\eta,$$

contradicting (3.7). Suppose that $o(x) \leq o(y)$. Then $x' = x$ and, because of (3.6) and (3.7),

$$y\eta + x\eta = y\alpha\eta = y\eta\alpha = y\eta(1_G + \sigma) = y\eta + y\eta\sigma.$$

Consequently,

$$x\eta = y\eta\sigma \in G\sigma = \langle x' \rangle = \langle x \rangle,$$

contrary to (3.4) and (3.5). Thus assumption (3.3) has led to a contradiction, proving the lemma.

We are now in a position to give a proof of Theorem B.

Proof of Theorem B. Clearly, the proposition holds true if AG is abelian. Therefore we can assume that G is not cyclic. If $pG = 0$, then G is a vector space over the prime field of characteristic p and, since $p \neq 2, 3$, every noncentral normal subgroup of AG contains the group of all linear transformations of G of determinant 1 (see [2, pp. 41, 45]), which is not cyclic. Therefore, we can restrict ourselves to the case $pG \neq 0$. Let $\langle \xi \rangle$ be a cyclic normal subgroup of AG and assume, by way of contradiction, that

(3.8) $$\langle \xi \rangle \not\leq \mathrm{z}AG.$$

According to [7], every noncentral normal subgroup of AG contains a noncentral normal p-subgroup of AG. Therefore, without loss of generality, we can assume that

(3.9) $$\langle \xi \rangle \text{ is a } p\text{-group.}$$

Let $\eta \in \langle \xi \rangle$ be an element of order p. Since $\langle \eta \rangle$ is a characteristic subgroup of $\langle \xi \rangle$ and $\langle \xi \rangle$ is normal in AG, it follows that $\langle \eta \rangle$ is a normal subgroup of AG of order p. From Lemma 3.1 we obtain $\eta \in \mathrm{z}AG$, and consequently

(3.10) $$\alpha^{-1}\eta\alpha = \eta \quad \text{for all } \alpha \in AG.$$

Clearly, $AG/\mathrm{c}\langle \xi \rangle$ is isomorphic to a group Φ of automorphisms of $\langle \xi \rangle$. It follows from (3.10) that every $\phi \in \Phi$ fixes the socle $\langle \eta \rangle$ of $\langle \xi \rangle$ elementwise and, therefore, the order of ϕ is a power of p (cf. (2.2)). Hence $AG/\mathrm{c}\langle \xi \rangle \simeq \Phi \leq A\langle \xi \rangle$ is a finite p-group. In particular, $AG/\mathrm{c}\langle \xi \rangle$ is finite and

(3.11) $$(p-1) \nmid [AG : \mathrm{c}\langle \xi \rangle].$$

In [5] we have shown that for every noncentral normal subgroup Γ of AG such that $AG/\mathrm{c}\Gamma$ is finite, $p-1$ divides the index $[AG:\mathrm{c}\Gamma]$ of the centralizer $\mathrm{c}\Gamma$ of Γ in AG (p. 214). Therefore, (3.11) implies $\langle \xi \rangle \leq \mathrm{z}AG$, which is the desired contradiction to (3.8). This completes the proof of Theorem B.

Theorem B enables us to establish the following result which is the essential part of Theorem A.

Theorem 3.2. *Let G be a reduced abelian p-group for some prime $p \geq 5$ and let $pG \neq 0$. Then every noncentral normal subgroup of AG contains a noncentral normal subgroup of AG of exponent p.*

Proof. Let N be a noncentral normal subgroup of AG. Then N contains a normal p-subgroup N_1 of AG such that

$$N_1 \not\leq \mathrm{z}AG \tag{3.12}$$

(see [7]). For elements γ in the maximal normal p-subgroup of AG, the conditions $\gamma^p = 1$ and $\gamma \mid pG = 1$ are known to be equivalent (cf. (2.2)). Hence

$$P = \{\gamma \in N_1 \mid \gamma^p = 1\} = N_1 \cap \{\alpha \in AG \mid (\alpha \mid pG) = 1\} \tag{3.13}$$

is a normal subgroup of AG of exponent p which is contained in N. Assume, by way of contradiction, that

$$P = \{\gamma \in N_1 \mid \gamma^p = 1\} \text{ is cyclic.} \tag{3.14}$$

Then $P \leq \mathrm{z}AG$ according to Lemma 3.1 and, since $P \neq 1$, it follows that $\mathrm{z}AG = R_p^* \cdot 1_G$ has elements of order p. Hence $p^n G = 0$ (see (2.1)) and therefore

$$N_1^{p^n} = 1 \tag{3.15}$$

for some positive integer n (cf. (2.2)). Consider the subgroup chains

$$G[p] \geq (pG)[p] \geq (p^2G)[p] \geq \cdots \geq (p^nG)[p] = 0, \tag{3.16}$$

$$G = G[p^n] \geq G[p^{n-1}] \geq \cdots \geq G[p] \geq 0. \tag{3.17}$$

Philip Hall has proved that a subgroup Γ of AG (G any group) is nilpotent if Γ stabilizes a finite subgroup chain of G, i.e. Γ induces the identity mapping in all factors of this chain [4, p. 787]. Since N_1 is a normal p-subgroup of AG, N_1 induces in $G[p]$ a group θ of automorphisms which stabilizes (3.16) (see (2.2)). Clearly $\theta \simeq N_1/M$, where M is the set of all $\gamma \in N_1$ such that $\gamma \mid G[p] = 1$. One verifies that M stabilizes (3.17). Applying P. Hall's result it follows that $\theta \simeq N_1/M$ and M both are nilpotent. Hence, N_1 is a solvable p-group and consequently (cf. [9, p. 190]),

$$N_1 \text{ is locally finite.} \tag{3.18}$$

It follows from (3.14) and (3.15) that every abelian subgroup of N_1 is cyclic. A finite p-group, $p \neq 2$, is cyclic if all its abelian normal subgroups are cyclic (cf. [8, p. 304]). Consequently, every finite subgroup of N_1 is cyclic, and because of (3.18), so is every finitely generated subgroup of N_1. Hence N_1 is abelian and (3.14) and (3.15) imply that N_1 is cyclic. From Theorem B we obtain that N_1 is contained in the center of AG violating (3.12). Hence, the assumption (3.14) has been contradicted and P is not cyclic. Since the maximal p-subgroup of the center of AG is cyclic (cf. (2.1)) it follows that $P \nleq zAG$ and, because of (3.13), P is a noncentral normal subgroup of AG of exponent p which is contained in N. This completes the proof.

Proof of Theorem A. The automorphism group of a reduced abelian p-group G is commutative if and only if G is cyclic (cf. (2.1)). Since this, by hypothesis, is not the case, AG is a noncentral normal subgroup of itself. Therefore, (iv) implies (iii) which in turn implies (ii). Since every normal p-subgroup of AG fixes $G[p]/(pG)[p]$ elementwise (cf. (2.2)), (i) is a consequence of (ii). Theorem 3.2 completes the proof.

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