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CROSS-SECTIONS OF SYMPLECTIC STIEFEL MANIFOLDS

BY

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ABSTRACT. The cross-section problem for the symplectic Stiefel manifolds is solved, using the now-proved Adams conjecture.

Introduction. Let $\mathrm{Sp}(n)$ be the symplectic group and $W_{n,k} = \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$ the symplectic Stiefel manifold. For $k > m$ one has an obvious map $p\colon W_{n,k} \to W_{n,m}$, which is a fiber map with fiber $W_{n-m,k-m}$. The purpose of this paper is to give a complete description of the values of n, k and m for which the map p has a cross-section.

The corresponding problem for the orthogonal Stiefel manifolds is already completely solved (Adams 1962, Eckmann-Whitehead 1963), as is the unitary case (Adams-Walker 1965, Suter 1966), the most famous contribution being Adams' solution of the vector field problem on spheres [1]. For the symplectic Stiefel manifolds, the explicit results previously known can be stated as follows:

(i) The map $p\colon W_{n,2} \to W_{n,1} = S^{4n-1}$ has a cross-section if and only if n is a multiple of 24. This result is due to I. M. James [10].

(ii) For $k > m \geq 2$, the map $p\colon W_{n,k} \to W_{n,m}$ does not have a cross-section. For this result see [12, p. 203].

We shall dispose of the remaining cases, i.e. the cases with $m = 1$, $k > 2$, by the following theorem.

Theorem. *The symplectic Stiefel fibring $p\colon W_{n,k} \to W_{n,1}$ has a cross-section if and only if one of the following two equivalent conditions holds*:

(I) *For each integer j with $0 \leq j \leq k-1$ the coefficient a_j of z^j in*

$$\left[\frac{2}{\sqrt{z}}\cdot \mathrm{sh}^{-1}\left(\frac{\sqrt{z}}{2}\right)\right]^{2n} = \sum_{j=0}^{\infty} a_j z^j = 1 - \frac{n}{12}z + \cdots$$

is an integer if j is even and an even integer if j is odd. (sh^{-1} *denotes the inverse of the hyperbolic sine.*)

(II) *n is a multiple of the integer c_k, called quaternionic James number, which is defined by its decomposition into prime powers as follows*:

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$$\nu_2(c_k) = \max_s (2k-1,\ 2s + \nu_2(s)), \qquad 1 \le s \le k-1,$$

$$\nu_p(c_k) = \max_t (t + \nu_p(t)), \qquad 1 \le t \le \left[\frac{2k-1}{p-1}\right],\ p \text{ odd } < 2k,$$

$$\nu_p(c_k) = 0, \qquad p \text{ odd } > 2k.$$

(The integer $\nu_p(q)$ is the exponent of the prime p in the prime power decomposition of q.)

We observe in particular that c_k is divisible by all primes less than $2k$. The number c_2 is equal to 24, this is James' result mentioned above. Comparing c_k with the (known) complex James number b_{2k} one obtains that c_k is either equal to b_{2k} or to $\frac{1}{2}b_{2k}$. A closer study of the relation between c_k and b_{2k} shows: for k odd, c_k is always equal to $\frac{1}{2}b_{2k}$. On the other hand if k is even, there are cases with $c_k = b_{2k}$ and such with $c_k = \frac{1}{2}b_{2k}$. In approximately 63% of all cases we have $c_k = \frac{1}{2}b_{2k}$.

The proof of the theorem is based on techniques developed by Adams and Walker in the unitary case [3]. Denoting by $KO(X)$ the real K-theory of a finite CW-complex X and by $J: KO(X) \to J(X)$ Atiyah's J-homomorphism, we state the following two theorems, which are the starting point of our study.

Theorem (James [9, 1.4]). *There exists a positive integer c_k, such that the Stiefel fibring $p: W_{n,k} \to S^{4n-1}$ has a cross-section if and only if n is a multiple of c_k.*

Theorem (Atiyah [4, 6.5]). *Let $\xi \in KO(HP^{k-1})$ be the canonical real (4-dimensional) Hopf bundle over the $(k-1)$-dimensional quaternionic projective space HP^{k-1}. The James number c_k is the order of $J(\xi)$ in the group $J(HP^{k-1})$.*

We first show that condition I of our theorem expresses the fact that n is a multiple of the order of $J(\xi)$ in $J(HP^{k-1})$. This is done in §1 and §2, using the group $J'(HP^{k-1})$ [3]. (Using Quillens proof of the Adams conjecture [12] we get an isomorphism $J(HP^{k-1}) \cong J'(HP^{k-1})$.) In §3 we then show that condition I is equivalent to condition II. The final section is devoted to an investigation of the relationship between c_k and b_{2k}.

1. Preliminaries. In this section we put together all the facts we need about ordinary cohomology and K-theory of the q-dimensional quaternionic right projective space HP^q.

Let CP^m be the m-dimensional complex projective space. The classical Hopf map $S^{4q+3} \to HP^q$ factors through CP^{2q+1} and gives rise to an S^2-fiber bundle

(1.0) $$S^2 \xrightarrow{i} CP^{2q+1} \xrightarrow{g} HP^q.$$

It will turn out that g induces *injections* in ordinary cohomology as well as in K-theory. So we might regard $H^*(HP^q; Z)$, $H^*(HP^q; Q)$, $KU(HP^q)$ and $KO(HP^q)$ as subrings of the corresponding rings of CP^{2q+1}, which are well known [3].

Let $b \in H^2(CP^{2q+1}; Z)$ be the canonical generator and let $a \in H^4(HP^q; Z)$ be a generator; then for $R = Z$ or Q

(1.1) $$H^*(CP^{2q+1}; R) = R[b] \pmod{b^{2q+2}}, \qquad H^*(HP^q; R) = R[a] \pmod{a^{q+1}}.$$

The fiber S^2 of (1.0) is totally nonhomologous to zero (i.e. i^* is onto). By the usual spectral-sequence argument, it follows that $g^*: H^*(HP^q; R) \to H^*(CP^{2q+1}; R)$ is *injective* and that we can choose the generator a such that $g^*(a) = b^2$.

We write β for the canonical complex line bundle over CP^{2q+1} and $\bar{\beta}$ for its complex-conjugate. The canonical quaternionic left line bundle over HP^q is denoted by ζ. (The bundle ζ is the dual of the quaternionic right line bundle associated to the principal H^*-bundle $(H^{q+1} - 0) \to HP^q$; see [4, §4].) Let α be the 2-dimensional complex vector bundle underlying ζ. For the induced bundle $g^*(\alpha)$ one has

(1.2) $$g^*(\alpha) \cong \beta \oplus \bar{\beta}.$$

This fact is well known and can be proved by direct computation of the total spaces of the bundles involved (see also [7, 9.6]).

We turn now to the computation of the complex K-theory of HP^q and set

$$\mu = \beta - 1 \in \widetilde{KU}(CP^{2q+1}), \quad \bar{\mu} = \bar{\beta} - 1 \in \widetilde{KU}(CP^{2q+1}), \quad \nu = \alpha - 2 \in \widetilde{KU}(HP^q).$$

By [1, 7.2] we have

(1.3) $$KU(CP^{2q+1}) = Z[\mu] \pmod{\mu^{2q+2}}.$$

(1.4) **Proposition.** (i) *The ring* $KU(HP^q)$ *is generated by* ν *subject to the relation* $\nu^{q+1} = 0$, *i.e.* $KU(HP^q) = Z[\nu] \pmod{\nu^{q+1}}$.

(ii) *The homomorphism* $g^!: KU(HP^q) \to KU(CP^{2q+1})$ *is injective and given by* $g^!(\nu) = \mu + \bar{\mu}$.

Proof. Since HP^q is torsion free we get with [5] that $KU(HP^q)$ is a free abelian group of the same rank as $H^{\text{even}}(HP^q; Z)$, which by (1.1) is $q + 1$.

With (1.2) we deduce $g^!(\nu) = \mu + \bar{\mu} = \mu^2$ + higher powers of μ ($\bar{\mu} = -\mu + \mu^2 -$

$\cdots$, see [1, 7.2]). Looking at (1.3) we see that the elements $1, g^!(\nu), \cdots, g^!(\nu^q)$ generate a direct summand of rank $q+1$ in the free abelian group $KU(CP^{2q+1})$, and the proposition is proved.

Next we determine the real K-theory of HP^q. We shall prove that the complexification $c\colon KO(HP^q) \to KU(HP^q)$ is injective, i.e. $KO(HP^q)$ can be regarded as subring of $KU(HP^q)$.

To begin with, note that for the element $\nu = \alpha - 2 \in \widetilde{KU}(HP^q)$ one has

$$2\nu,\ \nu^2 \in c(\widetilde{KO}(HP^q)). \tag{1.5}$$

This is seen as follows. The bundle α is by definition equal to $c'(\zeta)$, where we write c' for the map associating to a quaternionic vector bundle its underlying complex vector bundle. Hence α is self-conjugate, i.e. $\alpha = \bar{\alpha}$. Denoting by r the realification of complex vector bundles we get $c(r(\alpha)) = \alpha \oplus \bar{\alpha} = 2\alpha$ and deduce $2\nu = c(r(\nu))$. The tensor product $\zeta \otimes \zeta$ is a *real* (4-dimensional) vector bundle and we have $c(\zeta \otimes \zeta) = c'(\zeta) \otimes c'(\zeta) = \alpha \otimes \alpha$. This implies $\nu^2 \in c(\widetilde{KO}(HP^q))$. (A reference for the above remarks is [8, §3].)

The following proposition contains the information we will need about the ring $KO(HP^q)$.

(1.6) **Proposition.** (i) *The complexification homomorphism* $c\colon KO(HP^q) \to KU(HP^q)$ *is injective.*

(ii) *The subring* $c(KO(HP^q))$ *of* $KU(HP^q)$ *is generated as an abelian group by the elements* $1, 2\nu, \nu^2, \cdots, e_j\nu^j, \cdots, e_q\nu^q$, *where the integer* e_j *is equal to* 1 *if* j *is even and equal to* 2 *if* j *is odd.*

Proof. Working with the cofibration $HP^{q-1} \xrightarrow{i} HP^q \xrightarrow{p} S^{4q}$ we proceed by induction on q. For $HP^1 = S^4$ the proposition is a consequence of [6, (3.15)]. We then consider the exact cohomology sequence

$$\begin{array}{ccccccccccccc}
\widetilde{KO}^{-1}(S^{4q}) & \to & \widetilde{KO}^{-1}(HP^q) & \to & \widetilde{KO}^{-1}(HP^{q-1}) & \xrightarrow{\delta} & \widetilde{KO}(S^{4q}) & \to & \widetilde{KO}(HP^q) & \to & \widetilde{KO}(HP^{q-1}) & \xrightarrow{\delta} & \widetilde{KO}^1(S^{4q}) \\
\wr\wr & & & & & & \wr\wr & & & & & & \| \\
0 \text{ or } Z_2 & & & & & & Z & & & & & & 0
\end{array}$$

and deduce first inductively, that $\widetilde{KO}^{-1}(HP^q)$ is finite. This gives the short sequence $0 \to \widetilde{KO}(S^{4q}) \to \widetilde{KO}(HP^q) \to \widetilde{KO}(HP^{q-1}) \to 0$ and by induction on q one proves $\widetilde{KO}(HP^q)$ is torsion free. The relation $r \circ c = 2$ [8, Proposition 3.1] implies that c is injective. We turn now to the proof of (ii). The commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \widetilde{KO}(S^{4q}) & \to & \widetilde{KO}(HP^q) & \to & \widetilde{KO}(HP^{q-1}) & \to & 0 \\ & & \downarrow c & & \downarrow c & & \downarrow c & & \\ 0 & \to & \widetilde{KU}(S^{4q}) & \xrightarrow{p^!} & \widetilde{KU}(HP^q) & \xrightarrow{i^!} & \widetilde{KU}(HP^{q-1}) & \to & 0 \end{array}$$

has exact rows and all the groups are free abelian. We write ν_q and ν_{q-1} for the generator of $\widetilde{KU}(HP^q)$ and $\widetilde{KU}(HP^{q-1})$ respectively (see 1.4). One has $i^!(\nu_q) = \nu_{q-1}$. Let e_j be equal to 1 if j is even and equal to 2 if j is odd. Then by (1.5) the elements $e_j\nu_q^j$, $j = 1, \cdots, q$, belong to $c(\widetilde{KO}(HP^q))$. We have to show that these elements form a basis of $c(\widetilde{KO}(HP^q))$. The images $i^!(e_j\nu_q^j) = e_j\nu_{q-1}^j$, $j = 1, \cdots, q-1$, are a basis of $c(\widetilde{KO}(HP^{q-1}))$ by inductive hypothesis and it remains to prove that $e_q\nu_q^q = p^!(c(\gamma))$ for a generator $\gamma \in \widetilde{KO}(S^{4q}) = Z$. But $c(\gamma) = e_q\gamma_C$ for a suitable generator $\gamma_C \in \widetilde{KU}(S^{4q})$ (see [8, (3.15)]) and since $\ker i^! = \operatorname{im} p^!$ is generated by ν_q^q the proof is complete.

2. **The J-calculation.** The general reference for this section is the paper of Adams and Walker, *On complex Stiefel manifolds* [3].

Let $J(HP^{k-1})$ be the group of equivalence classes of orthogonal sphere bundles over HP^{k-1} with respect to stable fibre homotopy type and let $J\colon KO(HP^{k-1}) \to J(HP^{k-1})$ be the canonical epimorphism [4]. According to Atiyah [4, 6.5] the fibration $W_{n,k} \to S^{4n-1}$ has a cross-section if and only if in $J(HP^{k-1})$ one has

$$n \cdot J(r\alpha) = 0, \tag{2.0}$$

where $r\alpha = r \circ c'(\zeta)$ is the real vector bundle underlying the canonical quaternionic line bundle ζ (see §1). To determine the integers n which satisfy (2.0) we use the groups $J''(HP^{k-1})$ and $J'(HP^{k-1})$ defined in [3], which constitute a "computable" upper, respectively lower, bound of $J(HP^{k-1})$. (Note in particular that we deal with the J' as defined in [3].)

(2.1) **Lemma.** *For the space HP^q one has the isomorphisms*

$$J''(HP^q) \overset{\theta''}{\cong} J(HP^q) \overset{\theta'}{\cong} J'(HP^q).$$

Proof. The first isomorphism follows from the now-proved Adams conjecture [11]. To show that $\theta' \circ \theta''$ is an isomorphism we copy the proof of [3, Lemma 6.1]. For $HP^1 = S^4$, in general for S^{4q}, we have $J''(S^{4q}) \overset{\theta' \circ \theta''}{\cong} J'(S^{4q})$. (This fact follows in exactly the same way as the corresponding statement for the space CP^{2q}/CP^{2q-2} (see [3, 4.10 and 5.3]).) The cofibration $HP^{q-1} \xrightarrow{i} HP^q \xrightarrow{p} S^{4q}$ induces a commutative diagram:

$$\begin{array}{ccccccc} J''(S^{4q}) & \longrightarrow & J''(HP^q) & \longrightarrow & J''(HP^{q-1}) & \longrightarrow & 0 \\ \Big\downarrow & & \Big\downarrow (\theta' \circ \theta'')_{HP^q} & & \Big\downarrow (\theta' \circ \theta'')_{HP^{q-1}} & & \\ J'(S^{4q}) & \xrightarrow{J'(p)} & J'(HP^q) & \longrightarrow & J'(HP^{q-1}) & & \end{array}$$

The upper row of this diagram is exact [2, 3.12], $J'(p)$ is a monomorphism [3, 4.9] and the verticals are epimorphisms. By induction on q and easy diagram-chasing one shows now that $(\theta' \circ \theta'')_{HP^q}$ is an isomorphism for all q and the lemma is proved.

Lemma (2.1) enables us to work with J' instead of J. The condition (2.0) is satisfied if and only if $n \cdot J'(r\alpha) = 0$. By definition [3, §4] we have $J'(HP^{k-1}) = KO(HP^{k-1})/V(HP^{k-1})$, where $V(HP^{k-1})$ is the subgroup of elements $\delta \in KO(HP^{k-1})$, such that $\mathrm{sh}(\delta) = \mathrm{ch} \circ c(1+y)$ for some element $y \in \widetilde{KO}(HP^{k-1})$. (Here $\mathrm{sh}\colon KO(X) \to 1 + \Sigma_{s>0}^{\infty} H^{4s}(X; Q)$ is the characteristic class corresponding to the power series of $(e^{y/2} - e^{y/2})/y = (2/y) \cdot \mathrm{sh}(y/2)$, $c\colon KO(X) \to KU(X)$ is complexification and $\mathrm{ch}\colon KU(X) \to H^*(X; Q)$ is the Chern character.) The condition $n \cdot J'(r\alpha) = J'(n \cdot r\alpha) = 0$ becomes then

$$\mathrm{sh}(-n \cdot r\alpha) \in \mathrm{ch} \circ c(KO(HP^{k-1})).$$

(We change n to $-n$ for convenience.) Since $g^*\colon H^*(HP^{k-1}; Q) \to H^*(CP^{2k-1}; Q)$ is injective (see §1) we conclude

$$(2.2) \qquad n \cdot J(r\alpha) = 0 \Leftrightarrow g^* \circ \mathrm{sh}(-n \cdot r\alpha) \in g^* \circ \mathrm{ch} \circ c(KO(HP^{k-1})).$$

With (1.2) we compute $g^*(r\alpha) = r \circ g^*(\alpha) = r(\beta + \overline{\beta}) = 2 \cdot r\beta$. The naturality and the exponential property of sh imply $g^* \circ \mathrm{sh}(-n \cdot r\alpha) = \mathrm{sh}(-2n \cdot r\beta) = [\mathrm{sh}(r\beta)]^{-2n}$. But $\mathrm{sh}(r\beta) = (e^{b/2} - e^{-b/2})/b = (2 \cdot \mathrm{sh}(b/2))/b$ [3, §4] and hence

$$(2.3) \qquad g^* \circ \mathrm{sh}(-n \cdot r\alpha) = \left[\frac{b}{2\,\mathrm{sh}(b/2)}\right]^{2n} \in H^*(CP^{2k-1}; Q) = Q[b] \pmod{b^{2k}}$$

(see 1.1). The image of the homomorphism $g^* \circ \mathrm{ch} \circ c = \mathrm{ch} \circ g^! \circ c\colon KO(HP^{k-1}) \to H^*(CP^{2k-1}; Q)$ is generated by

$$(2.4) \qquad e_j \cdot [2 \cdot \mathrm{sh}(b/2)]^{2j} \pmod{b^{2k}}, \quad j = 0, 1, \cdots, k-1, \text{ where } e_j = \begin{cases} 1, & \text{if } j \text{ even}, \\ 2, & \text{if } j \text{ odd}. \end{cases}$$

This follows from (1.6), (1.4) and $\mathrm{ch}(\mu + \overline{\mu}) = (e^b - 1 + e^{-b} - 1) = [2 \cdot \mathrm{sh}(b/2)]^2$.

In $Q[b]$ $(\mathrm{mod}\, b^{2k})$ one has a unique relation

$$(R_n): \left[\frac{b}{2\cdot \mathrm{sh}(b/2)}\right]^{2n} = \sum_{j=0}^{k-1} q_j(n)\cdot e_j \cdot \left[2\cdot \mathrm{sh}\,\frac{b}{2}\right]^{2j} \quad (\mathrm{mod}\, b^{2k});$$

here $q_0(n) = 1,\ q_1(n), \cdots, q_{k-1}(n)$ are rational numbers which depend on n. Consequently we deduce from (2.2), (2.3) and (2.4),

(2.5) **Lemma.** *The following two conditions are equivalent.*

(1) $n \cdot J(r\alpha) = 0$ *in* $J(HP^{k-1})$.

(2) *The coefficients* $q_0(n) = 1,\ q_1(n), \cdots, q_{k-1}(n)$ *in* (R_n) *are integers.*

Substituting $b = 2\cdot \mathrm{sh}^{-1}(\sqrt{z}/2)$ in (R_n) we finally infer from (2.0) and (2.5) the following theorem, which is the first part of our main result.

(2.6) **Theorem.** *The quaternionic Stiefel fibring* $p\colon W_{n,k} \to S^{4n-1}$ *has a cross-section if and only if for each integer* j *with* $0 \le j \le k-1$ *the coefficient* $a_j(n)$ *of* z^j *in*

$$(S_n): \quad \left[\frac{2}{\sqrt{z}}\cdot \mathrm{sh}^{-1}\left(\frac{\sqrt{z}}{2}\right)\right]^{2n} = \sum_{j=0}^{\infty} a_j(n)\cdot z^j$$

is an integer if j *is even and an even integer if* j *is odd.*

3. **Determination of the quaternionic James number.** First let us fix some notation. For any rational number $q \neq 0$ and any prime p we define the integer $\nu_p(q)$ by

$$q = 2^{\nu_2(q)}\cdot 3^{\nu_3(q)}\cdot 5^{\nu_5(q)}\cdots$$

i.e. $\nu_p(q)$ is the exponent of p in the prime power decomposition of q. We set $\nu_p(0) = +\infty$ for all primes p.

According to our theorem (2.6) and the theorem of James [9] mentioned in the introduction, the quaternionic James number c_k is the smallest integer n such that for $j = 0, \cdots, k-1$ one has

$$(3.0)\qquad \begin{aligned} &\nu_p(a_j(n)) \ge 0, \quad \text{for } p \text{ odd},\\ &\nu_2(a_j(n)) \ge \begin{cases} 0, & \text{if } j \text{ even},\\ 1, & \text{if } j \text{ odd}. \end{cases}\end{aligned}$$

The following lemma is an essential step towards the computation of the odd part of c_k.

(3.1) **Lemma.** *Let p be an odd prime. Then the following two conditions are equivalent.*

(i) *The first k coefficients $a_0(n), \cdots, a_{k-1}(n)$ of (S_n) satisfy $\nu_p(a_j(n)) \geq 0$.*

(ii) *The first $2k$ coefficients $b_0(n), \cdots, b_{2k-1}(n)$ of $[x^{-1}\log(1+x)]^{2n} = \sum_{i=0}^{\infty} b_i(n) \cdot x^i$ satisfy $\nu_p(b_i(n)) \geq 0$.*

Proof. Setting $\sqrt{z}/2 = y$ we see that $\nu_p(a_j(n)) \geq 0$, $j = 0, \cdots, k-1$, if and only if the coefficients $d_i(n)$, $i = 0, \cdots, 2k-1$, of

$$\sum_{i=1}^{\infty} d_i(n) \cdot y^i = \left[\frac{\operatorname{sh}^{-1}(y)}{y}\right]^{2n} = \left[\frac{\log\left(y + \sqrt{1+y^2}\right)}{y}\right]^{2n}$$

satisfy $\nu_p(d_i(n)) \geq 0$.

The power series of $y + \sqrt{1+y^2}$ is of the form $1 + g(y)$, where $g(y)$ has the inverse $h(x) = x - \frac{1}{2}\sum_{i=2}^{\infty}(-1)^i x^i$. The coefficients of $h(x)$ and hence those of $g(y)$ are rational numbers having only powers of 2 in the denominator. Substituting $y = h(x)$ in $\sum_{i=0}^{\infty} d_i(n) \cdot y^{i+2n} = [\log(1+g(y))]^{2n}$ we deduce therefore by elementary manipulations of power series: one has $\nu_p(d_i(n)) \geq 0$, $i = 0, \cdots, 2k-1$, if and only if the coefficients $b_i(n)$, $i = 0, \cdots, 2k-1$, of $\sum_{i=0}^{\infty} b_i(n) \cdot x^{i+2n} = [\log(1+x)]^{2n}$ satisfy $\nu_p(b_i(n)) \geq 0$. This completes the proof of the lemma.

The power series of $[x^{-1}\log(1+x)]^{2n}$ has been investigated by Atiyah and Todd [6] and the following proposition is a direct consequence of [6, Proposition 6.4] and our Lemma (3.1).

(3.2) **Proposition.** *Let p be an odd prime. Then the following two conditions are equivalent.*

(i) *The first k coefficients $a_0(n), \cdots, a_{k-1}(n)$ of (S_n) satisfy $\nu_p(a_j(n)) \geq 0$.*

(ii) $$\nu_p(n) \geq \begin{cases} \max(r + \nu_p(r)), & 1 \leq r \leq [(2k-1)/(p-1)],\ p \leq 2k-1, \\ 0, & p > 2k. \end{cases}$$

Next we deal with the prime 2 and show

(3.3) **Proposition.** *The following two conditions are equivalent.*

(i) *The first k coefficients $a_0(n), \cdots, a_{k-1}(n)$ of (S_n) satisfy*

$$\nu_2(a_j(n)) \geq \begin{cases} 0, & \text{if } j \text{ even}, \\ 1, & \text{if } j \text{ odd}. \end{cases}$$

(ii) $\nu_2(n) \geq \max_s (2k-1,\ 2s+\nu_2(s)),\ 1 \leq s \leq k-1.$

We postpone the proof of (3.3) for a moment. Referring to (3.0), (3.2) and (3.3) we obtain the following theorem, which constitutes the second part of our main result stated in the introduction.

(3.4) **Theorem.** *The quaternionic James number* c_k *is determined by*

$$\nu_2(c_k) = \max_s (2k-1,\ 2s+\nu_2(s)), \qquad 1 \leq s \leq k-1,$$

$$\nu_p(c_k) = \max_r (r+\nu_p(r)), \qquad 1 \leq r \leq \left[\frac{2k-1}{p-1}\right],\ p \text{ odd} < 2k,$$

$$\nu_p(c_k) = 0, \qquad p \text{ odd} > 2k.$$

Now we turn to the proof of (3.3). To begin with we provide two lemmas.

(3.5) **Lemma.** *Suppose that* $\nu_2(n) \geq 2j-1$. *Then for the binomial coefficient* $\binom{n}{j}$ *one has*

$$\nu_2\binom{n}{j} = \nu_2(n) - \nu_2(j).$$

Proof. Write

$$\binom{n}{j} = \frac{n}{j}\cdot\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\ldots\cdot\frac{n-(j-1)}{j-1}$$

and note that $\nu_2((n-q)/q) = 0$ for $1 \leq q < 2^{2j-1}$.

Next we observe that the function $f(z) = [(2/\sqrt{z})\cdot \mathrm{sh}^{-1}(\sqrt{z}/2)]^{2n}$ satisfies the functional equation

$$f(z) = (1+z/4)^n \cdot f(4z+z^2).$$

This implies the following relation for the coefficients $a_0(n) = 1,\ a_1(n), \cdots, a_j(n), \cdots$ of (S_n) (see 2.6).

$$\binom{n}{j} + \sum_{i=1}^{j-1}\binom{n+i}{j-i}4^{2i}a_i(n) = 4^j(1-4^j)a_j(n), \qquad j = 1, 2, \cdots. \tag{3.6}$$

Working with the equations (3.6) we will prove

(3.7) **Lemma.** *If* $\nu_2(n) \geq 2j-1$, *then*

$$\nu_2(n) = 2j + \nu_2(j) + \nu_2(a_j(n)).$$

Proof. The proof is by induction over j. For $j = 1$ we have $a_1(n) = -n/12$ and hence $\nu_2(a_1(n)) = \nu_2(n) - 2$.

Now let $j > 1$ and assume $\nu_2(n) \geq 2j - 1$. This means in particular that $\nu_2(n) \geq 2i - 1$, $i = 1, \cdots, j - 1$, and one has by inductive hypothesis that $\nu_2(a_i(n)) = \nu_2(n) - 2i - \nu_2(i)$, $i = 1, \cdots, j = 1$. One computes then

$$\nu_2 \binom{n+i}{j-i} 4^{2i} a_i(n) \geq 4i + \nu_2(a_i(n)) = \nu_2(n) + 2i - \nu_2(i)$$
$$> \nu_2(n) + 1, \qquad i = 1, \cdots, j - 1.$$

From Lemma (3.5) we get $\nu_2\binom{n}{j} = \nu_2(n) - \nu_2(j) < \nu_2(n) + 1$ and deduce with the equation

$$\binom{n}{j} + \sum_{i=1}^{j-1} \binom{n+i}{j-i} 4^{2i} a_i(n) = 4^j(1 - 4^j) a_j(n)$$

that $\nu_2(n) - \nu_2(j) = 2j + \nu_2(a_j(n))$. Q.E.D.

Proof of Proposition (3.3). We first show that (ii) implies (i). If $\nu_2(n) = \max_{1 \leq s \leq k-1}(2k - 1, 2s + \nu_2(s))$ then one has in particular $\nu_2(n) \geq \max(2j + 1, 2j + \nu_2(j)) > 2j - 1$, $j = 1, \cdots, k - 1$, and with Lemma (3.7) one derives readily

$$\nu_2(a_j(n)) \geq \begin{cases} 0, & \text{if } j \text{ even,} \\ 1, & \text{if } j \text{ odd,} \end{cases} \qquad j = 1, \cdots, k - 1.$$

To prove (i) $\Rightarrow$ (ii) we proceed by induction on k. If $k = 2$, then $\nu_2(a_1(n)) = \nu_2(-n/12) \geq 1$ means precisely $\nu_2(n) \geq 3 = \max\ (3, 2)$. Now let $k > 2$ and assume

$$\nu_2(a_j(n)) \geq \begin{cases} 0, & \text{if } j \text{ even,} \\ 1, & \text{if } j \text{ odd,} \end{cases} \qquad j = 0, 1, \cdots, k - 1, k.$$

By inductive hypothesis we have

$$\nu_2(n) \geq \max_{1 \leq s \leq k-1} (2k - 1, 2s + \nu_2(s)) \geq 2k - 1.$$

With Lemma (3.7) we conclude

$$\nu_2(n) = 2k + \nu_2(k) + \nu_2(a_k(n)) \geq \begin{cases} 2k + \nu_2(k), & \text{if } k \text{ even,} \\ 2k + 1, & \text{if } k \text{ odd.} \end{cases}$$

Hence $\nu_2(n) \geq \max_{1 \leq s \leq k}(2k+1, 2s + \nu_2(s))$.

4. Comparison between the complex and the quaternionic case. In this section we study the relation between c_k and the known complex James number b_{2k} [3, Theorem 1.2].

(4.0) **Proposition.** *The quaternionic James number c_k is either equal to the complex James number b_{2k} or to $\frac{1}{2}b_{2k}$.*

Proof. The proposition follows at once from Theorem (3.4), [3, 1.2] and [6, 1.7].

We give a second proof which does not depend on our calculations in §3, using the functor J''. First we show that the element $J''_C(\alpha) \in J''_C(HP^{k-1})$ has the same order as $J''(r\beta) \in J''(CP^{2k-2})$ (notation as in §1 and §2). The homomorphism $g_1^*: KU(HP^{k-1}) \to KU(CP^{2k-2})$ induced by $CP^{2k-2} \to CP^{2k-1} \xrightarrow{g} HP^{k-1}$ is given by $g_1^*(\alpha) = \beta + \bar{\beta}$ (see 1.2) and the complexification $c: KO(CP^{2k-2}) \to KU(CP^{2k-2})$ is determined by $c(r\beta) = \beta + \bar{\beta}$. Since both g_1^* and c are injective (see (1.5) and [3, 2.2(iv)]) and compatible with the ψ-operations one obtains, slightly abusing notation, a ψ-ring isomorphism $c^{-1} \circ g_1^* = B: KU(HP^{k-1}) \cong KO(CP^{2k-2})$.(1) With the induced isomorphism $J''(B): J''_C(HP^{k-1}) \cong J''(CP^{2k-2})$ (see [2]) we deduce : the order of $J''_C(\alpha)$ is equal to the order of $J''(r\beta)$ which is b_{2k-1} [3]; by [6, p. 344] one has $b_{2k-1} = b_{2k}$.

The homomorphism $J''(r): J''_C(HP^{k-1}) \to J''(HP^{k-1})$ (see [3, Appendix]) maps $J''_C(\alpha)$ onto $J''(r\alpha)$ and it follows, that the order of $J''(r\alpha)$, i.e. the integer c_k, divides b_{2k}. But b_{2k} is a factor of $2c_k$ [10, 1.5] and (4.0) is proved.

Next we give a description of the distribution of the integers k for which $c_k = \frac{1}{2}b_{2k}$ and an approach to the evaluation of the corresponding density. We have $c_1 = \frac{1}{2}b_2$. If $k \geq 2$ we get the following result:

(4.1) **Theorem.** *Let $k \geq 2$. Then one has $c_k = \frac{1}{2}b_{2k}$ if and only if the integer k belongs to the set*

$$A = \{s + n \cdot 2^{2s-1} \mid s, n \in N\}.$$

(By N we denote the set of integers ≥ 1.)

Before giving the proof, let us draw some simple consequences illustrating the result.

(a) Given an integer k, it can be decided in a finite number of steps whether $c_k = \frac{1}{2}b_{2k}$. This is obvious from the description of A.

(1) This elegant way of deriving the ψ-ring isomorphism $KU(HP^{k-1}) \cong KO(CP^{2k-2})$ was communicated to us by J. F. Adams.

(b) If k is odd, $c_k = \frac{1}{2} b_{2k}$. Indeed, all odd integers belong to A.

(c) The first even value of k for which $c_k = \frac{1}{2} b_{2k}$ is $k = 10$.

Proof of Theorem (4.1). We first recall from Theorem (3.4) that

$$\nu_2(c_k) = \max_{1 \le t \le k-1} (2k-1,\ 2t + \nu_2(t)) = \max_{1 \le r \le 2k-1} (2k-1,\ r + \nu_2(r) - 1).$$

By [3, 1.2] and [6, 1.7] one has

$$\nu_2(b_{2k}) = \max_{1 \le r \le 2k-1} (r + \nu_2(r)).$$

We then show that the following four properties of an integer $k \geq 2$ are equivalent.

(i) $c_k = \frac{1}{2} b_{2k}$.

(ii) $\nu_2(b_{2k}) > 2k - 1$.

(iii) There exists an integer s with $0 < s < k$ and $k \equiv s \pmod{2^{2s-1}}$.

(iv) $k \in A$.

The equivalence (i) $\Leftrightarrow$ (ii) follows immediately from the comparison of $\nu_2(c_k)$ and $\nu_2(b_{2k})$. The equivalence (iii) $\Leftrightarrow$ (iv) is trivial.

(ii) $\rightarrow$ (iii). If $\nu_2(b_{2k}) > 2k - 1$ there exists an even integer r with $1 \leq r \leq 2k - 1$ and $r + \nu_2(r) > 2k - 1$. Putting $r = 2k - 2s$ we obtain the existence of s with $0 < s < k$ and $2k - 2s + \nu_2(2k - 2s) > 2k - 1$. This last condition is equivalent to $k - s \equiv 0 \pmod{2^{2s-1}}$.

(iii) $\rightarrow$ (ii). Taking $r = 2k - 2s$ we get

$$\nu_2(b_{2k}) \geq (2k - 2s) + \nu_2(2k - 2s) \geq (2k - 2s) + 2s > 2k - 1.$$

This proves the theorem.

We now turn to an evaluation of the density of A in N. We are greatly indebted to our friend H. Carnal for the considerations below. Any integer of A belongs to exactly one arithmetic progression $A_q = \{q + r \cdot 2^{2q-1}\}$ with $q \notin A$ (!). Moreover the set A is the disjoint union $A = \bigcup_{q \notin A} A_q$ and its density α is equal to $\sum_{m \notin A} 2^{-(2m-1)}$. We then write

$$\alpha = \sum_{m \notin A} 2^{-(2m-1)} = \sum_{m \in N} 2^{-(2m-1)} - \sum_{m \in A} 2^{-(2m-1)} = \frac{2}{3} - \sum_{n \notin A} \sum_{m \in A_n} 2^{-(2m-1)}$$

$$= \frac{2}{3} - \sum_{n \notin A} \sum_{m=1}^{\infty} 2^{-(2n-1) - m \cdot 2^{2n}} = \frac{2}{3} - \sum_{n \notin A} 2^{-(2n-1)} (2^{(2^{2n})} - 1)^{-1}$$

$$= \frac{2}{3} - \frac{1}{2(2^4 - 1)} + \frac{1}{2^3(2^{16} - 1)} + \frac{1}{2^7(2^{256} - 1)} \cdots.$$

The convergence of the series is extraordinarily rapid. Using the fact that A contains sequences of consecutive integers of arbitrary length, one deduces easily from Liouville's criterion that α is a transcendental number.

The approximate value of α is 0.63. Hence in about 63% of all cases, the James number c_k is equal to $\frac{1}{2}b_{2k}$ ("Complex vector fields on spheres can be chosen quaternionic".) It remains interesting to ask what geometric (and homotopy-theoretic) phenomena could be related to this strange algebraic fact.

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