

PRODUCTS OF DECOMPOSITIONS OF E^n

BY

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ABSTRACT. In this paper we give a sufficient condition for the existence of a homeomorphism $h: E^m/G \times E^n/H \rightarrow E^{m+n}$, where G and H are u.s.c. decompositions of Euclidean space. This condition is then shown to hold for a wide class of examples in which the decomposition spaces E^m/G and E^n/H may fail to be Euclidean.

It is well known that manifolds can be written as the product of topological spaces which may themselves fail to be manifolds. In [4], Bing gives a factorization of Euclidean 4-space, $E^4 \cong E^3/G \times E^1$, in which the factor E^3/G is not Euclidean. This factor is the so-called dogbone decomposition of E^3 . In [2], Andrews and Curtis give a simpler example in which the collection G consists of a single arc. This example was generalized in [8] to give the following example: for arcs $\alpha \subset E^m$ and $\beta \subset E^n$ it is true that $E^m/\alpha \times E^n/\beta \cong E^{m+n}$. Hence for badly embedded arcs α and β , there exist factorizations of E^{m+n} , neither factor being Euclidean. It is the purpose of this paper to show that this phenomenon occurs for fairly general decompositions of E^m and E^n . In particular if G is a decomposition of E^m and if H is a decomposition of E^n , then relatively mild conditions on G and H imply that

$$E^m/G \times E^n/H \cong E^{m+n}.$$

By relatively mild it is meant only that G and H satisfy certain conditions which many examples in the literature are known to possess.

The term decomposition will always mean a monotone, upper-semicontinuous decomposition. If G is a decomposition of E^m , then H_G denotes the union of the nondegenerate elements of G and E^m/G denotes the decomposition space of G . Notice that the nondegenerate elements of a decomposition form an upper-semicontinuous collection of sets in E^n . For the moment it is such u.s.c. collections which we investigate.

Let $A = \{\alpha\}$ be a collection of continua in E^m . For convenience we will let $A^* = \bigcup \{\alpha \mid \alpha \in A\}$. If $A = \{\alpha\}$ and $B = \{\beta\}$ are collections of continua with

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$A^* \subset E^m$ and $B^* \subset E^n$, then let $A \times B = \{\alpha \times \beta \mid \alpha \in A \text{ and } \beta \in B\}$, which is a collection in $E^m \times E^n$. For example, if $A^* \subset E^m$ and if we think of E^n as a collection of points, then

$$A \times E^n = \{\alpha \times w \mid \alpha \in A \text{ and } w \in E^n\}.$$

Let $G = \{\gamma\}$ be a collection of continua, $G^* \subset E^n$. Then the collection G is said to be *shrinkable* if and only if for each $\epsilon > 0$ there exists a homeomorphism $b_\epsilon: E^n \rightarrow E^n$ such that

- (1) $b = 1$ outside $N_\epsilon(G^*)$, the ϵ -neighborhood of G^* ,
- (2) for each $\gamma \in G$, diameter $b(\gamma) < \epsilon$,
- (3) for each $\gamma \in G$, there exists $\gamma' \in G$ such that $\gamma \cup b(\gamma) \subset N_\epsilon(\gamma')$, and
- (4) for each point $p \in E^n$, either $b(p) = p$, or $p \cup b(p) \subset N_\epsilon(\gamma)$ for some $\gamma \in G$.

In the sequel we shall always use the notation b_ϵ, f_δ , etc., to denote a homeomorphism as above, satisfying conditions (1)–(4) with respect to the number ϵ, δ , etc. These maps will be referred to as *shrinking homeomorphisms*.

Generally we use the shrinkability of a collection G in the following manner. Inductively we define a sequence of shrinking homeomorphisms, take their composition and passing to the limit we get a map $b: E^n \rightarrow E^n$. If the point-inverses of b coincide with the point-inverses of the quotient map $\pi: E^n \rightarrow E^n/G$, then the composition $b \cdot \pi^{-1}$ is continuous. A necessary and sufficient condition that $b \cdot \pi^{-1}$ be a homeomorphism is that b be a *proper map*, i.e., $b^{-1}(K)$ is compact whenever K is compact.

In order to prove that b is well-behaved, it is frequently the case that the shrinking homeomorphisms are uniformly continuous ([4], [6]). McAuley defines shrinkability in [12], and includes a uniform property which helps assure convergence of sequences of shrinking homeomorphisms. However, there are examples due to Andrews and Rubin [14] in which such nice shrinking maps could not be produced. In their examples (and in Proposition 1 below), it is the nature of the collection which allows one to prove convergence.

Proposition 1. *Let A and B be u.s.c. collections of compact continua, A^* and B^* compact, contained in E^m and E^n respectively. If the collection*

$$D = \{A \times B\} \cup \{A \times (E^n - B^*)\} \cup \{(E^m - A^*) \times B\}$$

is shrinkable, then $E^m/A \times E^n/B \cong E^{m+n}$.

Proof. For elements of the form $(\alpha \times x) \in A \times (E^n - B^*)$, and any shrinking homeomorphism f_ϵ , if $(\alpha \times x)$ lies outside $N_\epsilon(A \times B)$ then

$$(\alpha \times x) \cup f_\epsilon(\alpha \times x) \subset N_\epsilon(\alpha' \times x') \subset N_{2\epsilon}(\alpha' \times x')$$

for some $\alpha' \in A$ and $x' \in (E^n - B^*)$. So we adjust our notation so that given any shrinking homeomorphism f_ϵ and $(\alpha \times x)$ outside of $N_\epsilon(A \times B)$, $(\alpha \times x) \cup f_\epsilon(\alpha \times x) \subset N_\epsilon(\alpha' \times x)$ for some $\alpha' \in A$. Similarly, $(y \times \beta) \cup f_\epsilon(y \times \beta) \subset N_\epsilon(y \times \beta')$ for some $\beta' \in B$. For points $(x \times y)$ not in $N_\epsilon(A \times B)$ we require that either

- $f_\epsilon|(x \times y) = 1$, or
- $(x \times y) \cup f_\epsilon(x \times y) \subset N_\epsilon(\alpha \times y)$ for some $\alpha \in A$, or
- $(x \times y) \cup f_\epsilon(x \times y) \subset N_\epsilon(x \times \beta)$ for some $\beta \in B$.

Now let $\sum_{i=1}^\infty \epsilon_i < 1/2$. Let $0 < \delta_1 < \epsilon_1$ and $b_1 = f_{\delta_1}$, where f_{δ_1} shrinks D according to the conventions above. Let $K_1 = [-1, 1]^{m+n}$ and suppose (without loss of generality) that $N_{1/2}(A \times B) \subset K_1$. Let $K_2 = [-2, 2]^{m+n}$ and note that there exists $\delta_2 < \min\{\epsilon_2, \delta_1\}$ such that if $X \subset K_2$ and diameter $X < \delta_2$, then diameter $b_1(X) < \delta_1$. Let $b_2 = b_1 f_{\delta_2}$.

Inductively we suppose that δ_i and b_i have been defined. Let $K_{i+1} = [-i-1, i+1]^{m+n}$ and choose $\delta_{i+1} < \min\{\epsilon_{i+1}, \delta_i\}$ such that if $X \subset K_{i+1}$ and diameter $X < \delta_{i+1}$, then diameter $b_i(X) < \delta_i$. Let $b_{i+1} = b_i f_{\delta_{i+1}}$.

Let $b = \lim_{j \rightarrow \infty} b_j$. To see that b is well defined and continuous, let $\gamma \in D$; say $\gamma \in K_N$. Since each f_{δ_j} cannot move points out toward infinity more than δ_j , it follows that $b_j(\gamma) \subset N_{1/2}(K_N)$ for all j . For $j > N$, diameter $b_{j+1}(\gamma) < \delta_j$. If $p \in \gamma$ is a representative point of $D^* \subset E^{m+n}$, we have

$$b_{j-1}(\gamma) \cup b_j(\gamma) \subset b_{j-1}(N_{\delta_j}(\gamma_j)) \subset N_{\delta_{j-1}}(b_{j-1}(\gamma_j)),$$

where $\gamma_j \in D$ and j is large. Therefore,

$$\text{distance}(b_{j-1}(p), b_j(p)) < \delta_{j-2} + 2\delta_{j-1}.$$

For points $p \notin D^*$, the sequence $\{b_j(p)\}$ is eventually constant. In any case, for sufficiently large integers r and s ,

$$\text{distance}(b_r(p), b_s(p)) < 3 \sum_{i=r-2}^{s-1} \delta_i.$$

Therefore b is well defined and continuous.

We have seen that, for points $p \in K_N$, $f_{\delta_j}^{-1}(p) \in N_{\delta_j}(K_N)$, and so $b_j^{-1}(p) \in N_{1/2}(K_N)$. Let $p_j = b_j^{-1}(p)$; since $\{p_j\} \subset K_{N+1}$, there is a limit point p' and $b(p') = p$. This shows that b is an epimorphism.

In a similar fashion, $b^{-1}(C)$ is bounded whenever $C \subset E^{m+n}$ is compact. Thus b is a proper map. The elements of D are shrunk to points, with different elements of D going to different points. Therefore $E^m/A \times E^n/B \cong E^{m+n}$.

Recall that a continuum $X \subset E^n$ is said to have property UV^∞ if for every open set U containing X , there exists an open set V , $X \subset V \subset U$, such that the

inclusion $i: V \rightarrow U$ is null homotopic. This is really a property of the embedding of X , but is a topological property of X when we restrict our attention to embeddings in ANR's (see [9]).

Let X be a compactum in the interior of a topological n -manifold M . We say that X is *definable by cells* in M if there is a sequence $\{B_i\}_1^\infty$ where each B_i consists of a finite number of disjoint n -cells in M , with $B_{i+1} \subset \text{Int } B_i$ for each i and $X = \bigcap_{i=1}^\infty B_i$. The set X is said to be *cellular* if it is connected and definable by cells.

Theorem 1. *Suppose $\alpha \subset E^m$ and $\beta \subset E^n$ are compact, UV^∞ continua such that $\alpha \times E^n$ and $\beta \times E^m$ are shrinkable. Then*

$$E^m/\alpha \times E^n/\beta \cong E^{m+n}.$$

This theorem follows directly from Theorem 2, but an independent proof is simpler and gives some insight into the proof of Theorem 2. The interested reader can supply the appropriate ϵ 's and δ 's in the following outline.

Outline of Theorem 1. From Theorem 8 in [13] and observations on these proofs made in [11], it is easy to see that $\alpha \times \beta$ is cellular in $E^m \times E^n$. Using this fact, one can construct a uniformly continuous map which shrinks $\alpha \times \beta$ to a point, is a homeomorphism off of $\alpha \times \beta$, and is the identity outside an arbitrary preassigned neighborhood of $\alpha \times \beta$. If we call this map f , then it is well known that the image of f is homeomorphic to $E^m \times E^n$.

Now the hypothesis gives shrinking homeomorphisms f' of $\alpha \times E^n$ and f'' of $E^m \times \beta$. Using Theorem 7.1 of [5], these are replaced by homeomorphisms f_1 and f_2 which are the identity near $\alpha \times \beta$, but agree with f' and f'' respectively outside a small neighborhood of $\alpha \times \beta$.

Consider the composition $f f_1 f_2$, which shrinks $\alpha \times \beta$ to a point and shrinks $\{\alpha \times w \mid w \in E^n - \beta\}$ and $\{z \times \beta \mid z \in E^m - \alpha\}$ to small sets. By passing to the limit of a sequence of such maps that shrink things smaller and smaller, we verify the conclusion of Theorem 1.

Theorem 2. *Let $A = \{\alpha\}$ and $B = \{\beta\}$ be upper-semicontinuous collections of compact continua such that $A^* \subset E^m$, $B^* \subset E^n$, A^* and B^* compact, $A \times E^n$ is shrinkable, $B \times E^m$ is shrinkable, and $A \times B$ is shrinkable. Then the collection*

$$(A \times B) \cup (A \times (E^n - B^*)) \cup ((E^m - A^*) \times B)$$

is shrinkable.

If we denote by G_A and G_B the decompositions of E^m and E^n whose non-degenerate elements consist of the elements of A and B respectively, then we get the following

Corollary. *Under the hypothesis of Theorem 2,*

$$E^m/G_A \times E^n/G_B \cong E^{m+n}.$$

In essence, Theorem 2 follows from the u.s.c. conditions on A and B . The proof involves several lemmas, all of which assume the hypotheses of Theorem 2. The lemmas simply state things about u.s.c. collections; the proofs are similar, so most are omitted.

Lemma 1. *Given $\epsilon > 0$, there exists δ' , $0 < \delta' < \epsilon$, such that if $0 < \delta \leq \delta'$, then any f_δ shrinking $A \times B$ satisfies the following: for all $\alpha \in A$, $x \in E^n$, either*

- (1) $f_\delta | (\alpha \times x) = 1$, or
- (2) $(\alpha \times x) \cup f_\delta(\alpha \times x) \subset N_\epsilon(\alpha' \times \beta')$ for some $\alpha' \in A$, $\beta' \in B$.

Lemma 2. *Given $\epsilon > 0$, there exists a $\delta' > 0$ such that if $0 < \delta \leq \delta'$ then there is a $\gamma > 0$ such that for all $\alpha \in A$, $\beta \in B$ and f_δ shrinking $A \times B$, there exist $\alpha' \in A$ and $\beta' \in B$ with*

$$N_\gamma(\alpha \times \beta) \cup f_\delta(N_\gamma(\alpha \times \beta)) \subset N_\epsilon(\alpha' \times \beta').$$

Lemma 3. *Given $\epsilon > 0$ and $\beta \in B$, there exists a $\delta'_\beta > 0$ such that $0 < \delta \leq \delta'_\beta$ implies that any f_δ shrinking $A \times E^n$ satisfies the following: for all $\alpha \in A$, there exists $\alpha' \in A$ such that*

$$(\alpha \times \beta) \cup f_\delta(\alpha \times \beta) \subset N_\epsilon(\alpha' \times \beta).$$

Lemma 4. *Given $\epsilon > 0$, there exists a $\delta' > 0$ such that $0 < \delta \leq \delta'$ implies that any f_δ shrinking $A \times E^n$ satisfies the following: for all $\alpha \in A$, $\beta \in B$, there exist $\alpha' \in A$ and $\beta' \in B$ such that*

$$(\alpha \times \beta) \cup f_\delta(\alpha \times \beta) \subset N_\epsilon(\alpha' \times \beta').$$

Lemma 5. *Given $\epsilon > 0$, there exists a $\delta' > 0$ such that $0 < \delta \leq \delta'$ implies that any f_δ shrinking $A \times E^n$ satisfies the following: for all $\beta \in B$ and $x \in E^m$, either*

- (1) $f_\delta | (x \times \beta) = 1$, or
- (2) $(x \times \beta) \cup f_\delta(x \times \beta) \subset N_\epsilon(\alpha' \times \beta')$ for some α' , β' .

Lemma 6. *Given $\epsilon > 0$ there exists a $\delta' > 0$ such that $0 < \delta \leq \delta'$ implies the existence of a $\gamma > 0$, γ depends on δ , such that for any f_δ shrinking $A \times B$, either*

$$f_\delta | N_\gamma(\alpha \times x) = 1, \text{ or}$$

$$N_\gamma(\alpha \times x) \cup f_\delta(N_\gamma(\alpha \times x)) \subset N_\epsilon(\alpha' \times \beta').$$

Lemma 7. *Given $\epsilon > 0$ there exists a $\delta' > 0$ such that $0 < \delta \leq \delta'$ implies the existence of a $\gamma > 0$, $\gamma = \gamma(\delta)$, such that for any f_δ shrinking $E^m \times B$,*

- (1) $N_\gamma(y \times \beta) \cup f_\delta(N_\gamma(y \times \beta)) \subset N_\epsilon(y \times \beta')$,
 (2) $N_\gamma(\alpha \times \beta) \cup f_\delta(N_\gamma(\alpha \times \beta)) \subset N_\epsilon(\alpha'' \times \beta'')$,
 (3) $f_\delta | N_\gamma(\alpha \times x) = 1$, or $N_\gamma(\alpha \times x) \cup f_\delta(N_\gamma(\alpha \times x)) \subset N_\epsilon(\alpha' \times \beta')$.

Proof of Lemma 1. Suppose not; then for each $\delta' = 1/n$ ($1/n < \epsilon$), there exist a $\delta_n \leq 1/n$ and $\alpha_n \in A, x_n \in E^n$ such that

- (1) $f_{\delta_n} | (\alpha_n \times x_n) \neq 1$, and
 (2) $(\alpha_n \times x_n) \cup f_{\delta_n}(\alpha_n \times x_n) \not\subset N_\epsilon(\alpha \times \beta)$ for any $\alpha \in A, \beta \in B$.

Condition (1) assures us that all the $(\alpha_n \times x_n)$ are in some compact neighborhood of $A \times B$. Since A^* is compact, let $\alpha_n \in \alpha_n$, for each n , and it follows that $\{\alpha_n\}$ has a limit point, say a . By passing to a subsequence we may suppose that $a \in \underline{\lim} \alpha_n$. Since A^* is compact, $a \in \alpha$ for some $\alpha \in A$. Since A is u.s.c., it follows that $\underline{\lim} \alpha_n \subset \alpha$.

Passing to subsequences when necessary, we may suppose, w.l.o.g., that $\{x_n\}$ converges to x . Condition (1) provides that $x \in \beta$ for some $\beta \in B$. For otherwise, there exists $N > 0$ such that $N_{\delta_N}(\alpha \times x) \cap N_{\delta_N}(A \times B) = \emptyset$. Thus for sufficiently large n , $f_{\delta_n} | (\alpha_n \times x_n) = 1$. So let $x \in \beta \in B$.

Consider $\alpha \times \beta$. For each n , there exists $\alpha'_n \in A$ and $\beta'_n \in B$ such that $(\alpha \times \beta) \cup f_{\delta_n}(\alpha \times \beta) \subset N_{1/n}(\alpha'_n \times \beta'_n)$. Evidently $\underline{\lim} \alpha'_n \neq \emptyset$ and $\overline{\lim} \alpha'_n \subset \alpha$. Similarly $\overline{\lim} \beta'_n \subset \beta$.

Hence there exists N such that $n > N$ implies that

- (1) $\alpha_n \times x_n \subset N_\epsilon(\alpha \times \beta)$, and
 (2) $N_{1/n}(\alpha'_n \times \beta'_n) \subset N_\epsilon(\alpha \times \beta)$.

Hence $(\alpha_n \times x_n) \cup f_{\delta_n}(\alpha_n \times x_n) \subset N_\epsilon(\alpha \times \beta)$. From this contradiction, the lemma follows.

Proof of Theorem 2. Let $\epsilon > 0$ be given. A homeomorphism $b: E^m \times E^n \rightarrow E^m \times E^n$ which shrinks the collection

$$\{A \times B\} \cup \{A \times (E^n - B^*)\} \cup \{(E^m - A^*) \times B\}$$

is constructed as follows: First shrink $A \times B$. Select $\delta' < \epsilon$ satisfying Lemmas 1, 2, and 6. Let $\delta_1 \leq \delta'$ be positive. Using Lemmas 2 and 6 we can choose $\delta_2 > 0$ such that

$$N_{\delta_2}(\alpha \times \beta) \cup f_{\delta_1}(N_{\delta_2}(\alpha \times \beta)) \subset N_\epsilon(\alpha' \times \beta'),$$

either $f_{\delta_1} | N_{\delta_2}(\alpha \times x) = 1$, or

$$N_{\delta_2}(\alpha \times x) \cup f_{\delta_1}(N_{\delta_2}(\alpha \times x)) \subset N_\epsilon(\alpha'' \times \beta'')$$

and either $f_{\delta_1} | N_{\delta_2}(y \times \beta) = 1$, or

$$N_{\delta_2}(y \times \beta) \cup f_{\delta_1}(N_{\delta_2}(y \times \beta)) \subset N_\epsilon(\alpha''' \times \beta''').$$

Since f_{δ_1} is the identity off of a compact set, it is uniformly continuous, so we may impose additional requirements on δ_2 . We require that diameter $X < \delta_2$ imply that diameter $f_{\delta_1}(X) < \delta_1$, and diameter $f_{\delta_1}(N_{\delta_2}(a \times \beta)) < \delta_1$ for all $a \times \beta$ in $A \times B$.

Now shrink $E^m \times B$. In Lemmas 3–5 and 7, let δ_2 be used as the $\epsilon > 0$, and select δ' which works for all these lemmas. Let $\delta_3 < \delta'$ be positive, and select $\delta_4 > 0$ (by Lemma 7) such that

$$\begin{aligned} N_{\delta_4}(y \times \beta) \cup f_{\delta_3}(N_{\delta_4}(y \times \beta)) &\subset N_{\delta_2}(y \times \beta'), \\ f_{\delta_3} | N_{\delta_4}(a \times x) &= 1, \text{ or} \\ N_{\delta_4}(a \times x) \cup f_{\delta_3}(N_{\delta_4}(a \times x)) &\subset N_{\delta_2}(\alpha'' \times \beta''), \text{ and} \\ N_{\delta_4}(a \times \beta) \cup f_{\delta_3}(N_{\delta_4}(a \times \beta)) &\subset N_{\delta_2}(\alpha''' \times \beta'''). \end{aligned}$$

Let K be a compact neighborhood containing $N_{2\epsilon}(A \times B)$ and such that $f_{\delta_1} | (E^m \times E^n) - K = 1$. We impose an additional requirement on δ_4 . If $X \subset K$ and diameter $X < \delta_4$, then diameter $f_{\delta_3}(X) < \delta_3$.

Now shrink $A \times E^n$ by a shrinking homeomorphism f_{δ_4} . We see that

$$\begin{aligned} (a \times x) \cup f_{\delta_4}(a \times x) &\subset N_{\delta_4}(\alpha' \times x), \\ (a \times \beta) \cup f_{\delta_4}(a \times \beta) &\subset N_{\delta_4}(\alpha'' \times \beta''), \\ f_{\delta_4} | y \times \beta &= 1, \text{ or} \\ (y \times \beta) \cup f_{\delta_4}(y \times \beta) &\subset N_{\delta_4}(\alpha''' \times \beta'''). \end{aligned}$$

Set $b = f_{\delta_1} f_{\delta_3} f_{\delta_4}$; we must verify that b satisfies conditions 1–4 of shrinkability. Condition 1 is easy to verify. To check conditions 2 and 3, we use the following diagram to help enumerate the various possibilities.

$$f_{\delta_4} \begin{array}{c} \text{I hit} \\ \text{II miss} \end{array} \rightarrow f_{\delta_3} \begin{array}{c} \text{III hit} \\ \text{IV miss} \end{array} \rightarrow f_{\delta_2} \begin{array}{c} \text{V hit} \\ \text{VI miss} \end{array} .$$

We will use case numbers like Case I-III-V so show how the elements of the collection $\{A \times B\} \cup \{A \times (E^n - B^*)\} \cup \{(E^m - A^*) \times B\}$ are affected by b .

We consider cases for elements of the form $(a \times x)$.

Case I-III-V.

$$\begin{aligned} (a \times x) \cup f_{\delta_4}(a \times x) &\subset N_{\delta_4}(\alpha' \times x) \text{ and diameter } f_{\delta_4}(a \times x) < \delta_4. \\ N_{\delta_4}(\alpha' \times x) \cup f_{\delta_3}(N_{\delta_4}(\alpha' \times x)) &\subset N_{\delta_2}(\alpha'' \times \beta'') \text{ and diameter } f_{\delta_3} f_{\delta_4}(a \times x) < \\ \delta_3, \\ N_{\delta_2}(\alpha'' \times \beta'') \cup f_{\delta_1}(N_{\delta_2}(\alpha'' \times \beta'')) &\subset N_\epsilon(\alpha''' \times \beta'''). \\ \text{Therefore } (a \times x) \cup b(a \times x) &\subset N_\epsilon(\alpha''' \times \beta''') \text{ and diameter } b(a \times x) < \delta_1. \end{aligned}$$

Case I-IV-V.

$(\alpha \times x) \cup f_{\delta_4}(\alpha \times x) \subset N_{\delta_4}(\alpha' \times x)$ and diameter $f_{\delta_4}(\alpha \times x) < \delta_4$,

$f_{\delta_3} | N_{\delta_4}(\alpha' \times x) = 1$ and hence diameter $f_{\delta_3} f_{\delta_4}(\alpha \times x) < \delta_2$.

Therefore $(\alpha \times x) \cup b(\alpha \times x) \subset f_{\delta_1}(N_{\delta_4}(\alpha' \times x)) \subset N_\epsilon(\alpha'' \times \beta'')$.

Also diameter $b(\alpha \times x) = \text{diameter } f_{\delta_1} f_{\delta_3} f_{\delta_4}(\alpha \times x) < \delta_1$.

Case I-IV-VI.

$(\alpha \times x) \cup b(\alpha \times x) \subset N_{\delta_4}(\alpha' \times x)$ and diameter $b(\alpha \times x) = \text{diameter } f_{\delta_4}(\alpha \times x) < \delta_4$.

Case I-III-VI.

Is impossible, as are all the cases which begin with II.

Consider cases for elements of the form $y \times \beta$. These are similar to the cases above. For example:

Case II-III-V.

$f_{\delta_4} | y \times \beta = 1$,

$(y \times \beta) \cup f_{\delta_3} f_{\delta_4}(y \times \beta) \subset N_{\delta_2}(y \times \beta')$ and diameter $f_{\delta_3}(y \times \beta) < \delta_3$,

$(y \times \beta) \cup f_{\delta_1} f_{\delta_3} f_{\delta_4}(y \times \beta) = (y \times \beta) \cup b(y \times \beta) \subset f_{\delta_1}(N_{\delta_2}(y \times \beta')) \subset N_\epsilon(\alpha'' \times \beta'')$ and diameter $b(y \times \beta) < \delta_1$.

For elements of the form $\alpha \times \beta$, there is only one possibility:

Case I-III-V.

$(\alpha \times \beta) \cup f_{\delta_4}(\alpha \times \beta) \subset N_{\delta_4}(\alpha' \times \beta')$,

$N_{\delta_4}(\alpha' \times \beta') \cup f_{\delta_3}(N_{\delta_4}(\alpha' \times \beta')) \subset N_{\delta_2}(\alpha'' \times \beta'')$, and $N_{\delta_2}(\alpha'' \times \beta'') \cup f_{\delta_1}(N_{\delta_2}(\alpha'' \times \beta'')) \subset N_\epsilon(\alpha''' \times \beta''')$.

Therefore $(\alpha \times \beta) \cup b(\alpha \times \beta) \subset N_\epsilon(\alpha''' \times \beta''')$ and since $b(\alpha \times \beta) \subset f_{\delta_1}(N_{\delta_2}(\alpha'' \times \beta''))$, we see that diameter $b(\alpha \times \beta) < \delta_1$.

Thus b satisfies conditions 1, 2, and 3 of shrinkability, and condition 4 follows in a fashion similar to condition 3.

We turn now to applications of Theorem 2. The literature contains many examples of collections A which satisfy the condition that $A \times E^1$ is shrinkable. This is the content of [2] and [6] for examples in which A consists of a single arc or a k -cell. Bing established this shrinkability criterion for the nondegenerate elements of the dogbone decomposition of E^3 in [4]. So, much of the hypothesis of Theorem 2 is readily satisfied. The difficulty lies in establishing the shrinkability of $A \times B$, but there are sufficient conditions for this shrinkability. The most obvious are if $A \times B$ is cellular, or if $A \times B$ is simply definable by cells.

Recall that a decomposition means a monotone, u.s.c. decomposition, and if

G is a decomposition of E^n , then H_G denotes the union of the nondegenerate elements of G . The decomposition G is said to be *compact* if $\text{Cl } H_G$ is compact and *0-dimensional* if the image of $\text{Cl } H_G$ is 0-dimensional in E^n/G .

Theorem 3. *Let G and F be compact, 0-dimensional decompositions of E^m and E^n respectively, each element of which possesses property UV^∞ . Then $\text{Cl}(H_G \times H_F)$ is definable by cells in $E^m \times E^n$.*

Corollary. $H_G \times H_F$ is shrinkable in $E^m \times E^n$.

The proof of Theorem 3 requires the following lemma which is a simple generalization of Lemma 1 in [7].

Lemma 8. *Suppose that $M_1 \subset M_2 \subset \dots \subset M_{k-r+1}$ is a sequence of finite combinatorial k -manifolds (not necessarily connected) such that each M_i is a combinatorial subspace of M_{i+1} and the inclusion of each component of M_i into M_{i+1} is homotopically trivial. If Y is any subcomplex of M_1 such that $\dim Y \leq k - r - 1$ and $r \geq 2$, then Y lies in a finite union of disjoint k -cells in M_{k-r+1} .*

In the case $r = 2$, any subcomplex of M_1 having codimension 3 lies in the union of k -cells in M_{k-1} . Suppose that $M_1 \supset M_2 \supset \dots$ is a sequence of k -manifolds such that any subcomplex $Y \subset M_{i+1}$ having codimension 3 lies in a finite number of disjoint k -cells in M_i . We will call $\{M_i\}$ a *special sequence*.

Corollary. *If $\{M_i\}$ is a sequence of finite combinatorial k -manifolds such that each M_{i+1} is a combinatorial subspace of M_i and the inclusion of each component of M_{i+1} into M_i is homotopically trivial, then $\{M_i\}$ can be refined to a special sequence.*

As in [1], the sequence H_1, H_2, \dots of compact m -manifolds-with-boundary will be called a *defining sequence* for the decomposition G of E^m provided $H_{i+1} \subset \text{Int } H_i$ for each i , and g is a nondegenerate element of G if and only if g is a nondegenerate component of $\bigcap_{i=1}^\infty H_i$. We will say that H_i has a k -spine if H_i is PL and H_i collapses to a subpolyhedron of dimension k or less.

Proof of Theorem 3. Using the techniques in [1], there exist PL defining sequences $\{H_i\}$ and $\{K_i\}$ for G and F respectively, such that

- (i) Each $H_{i+1}(K_{i+1})$ is a finite combinatorial subspace of $H_i(K_i)$.
- (ii) The inclusion of each component of $H_{i+1}(K_{i+1})$ into $H_i(K_i)$ is homotopically trivial.
- (iii) Each $H_i(K_i)$ collapses to a spine of codimension 2.

Consider $\{H_i \times K_i\}$ which is a PL defining sequence for the decomposition $G \times F$ of $E^m \times E^n$. Note the elements of $G \times F$ are $\{\alpha \times \beta \mid \alpha \in G, \beta \in F\}$. It is easy to check that:

- (i) Each $H_i \times K_i$ is a finite combinatorial $m + n$ -manifold.
- (ii) The inclusion of each component of $H_{i+1} \times K_{i+1}$ into $H_i \times K_i$ is homotopically trivial.
- (iii) Each $H_i \times K_i$ collapses to a spine of codimension four.
- (iv) W.l.o.g., $\{H_i \times K_i\}$ is a special sequence.

If we let $M_i = H_i \times K_i$ and let M_i' be the spine, it follows that there exist finitely many disjoint $m + n$ -cells B_1, \dots, B_k such that $M_{i+1}' \subset \bigcup_{i=1}^k B_i \subset M_i$. Thus there exists a PL homeomorphism, as in [16], $b: E^m \times E^n \rightarrow E^m \times E^n$ which is fixed outside of M_i such that

$$M_{i+1} \subset b \left(\bigcup_{i=1}^k B_i \right) \subset M_i.$$

Since $H_G \times H_F = \bigcap_{i=1}^{\infty} M_i$, it follows that $H_G \times H_F$ is definable by cells, which concludes the proof of Theorem 3.

In [15], Siebenmann points out that Bing's criterion (shrinkability) is a necessary condition for the existence of a homeomorphism $E^n/G \times E^m \cong E^{m+n}$ ($n + m > 4$), at least provided the elements of G have property UV^∞ .

Theorem 4. *Let G and H be compact, 0-dimensional decompositions of E^m and E^n respectively, such that each element of G (and H) has property UV^∞ . If $E^m/G \times E^n \cong E^{m+n}$, $E^n/H \times E^m \cong E^{m+n}$, then $E^m/G \times E^n/H \cong E^{m+n}$.*

Proof. If $m + n > 4$, then Siebenmann's result mentioned above [15] together with Theorem 3 imply Theorem 4. If $m = 3$ and $n = 1$, then the theorem is trivially a consequence of the hypothesis. If $m = n = 2$, then results of Moore imply the theorem.

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