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REPRESENTATIONS OF JORDAN TRIPLES

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ABSTRACT. Some standard results on representations of quadratic Jordan algebras are extended to Jordan triples. It is shown that the universal envelope of a finite-dimensional Jordan triple is finite-dimensional, and that it is nilpotent if the Jordan triple is radical. A permanence principle and a duality principle are proved which are useful in deriving identities.

Introduction. A *Jordan triple* is a module V over a commutative ring k together with a composition $(x,y) \mapsto P(x)y$ which is quadratic in x and linear in y and satisfies certain identities (see (1)–(3) below). A typical example is the space of $p \times q$-matrices over k with $P(x)y = x({}^t y)x$. If J is a quadratic Jordan algebra with quadratic operators U_x then J is also a Jordan triple with $P(x)y = U_x y$. Thus Jordan triples are a natural generalization of quadratic Jordan algebras. For a systematic theory of Jordan triples see [2] and [5].

In this note, we extend to Jordan triples certain standard results from the representation theory of quadratic Jordan algebras (see [4]). Our main results concern the case where V is finite-dimensional over a field k. Then the universal envelope of V is also finite-dimensional (Theorem 2.4), and it is nilpotent in case V is radical (Theorem 3.3). The latter result is due to C.T. Anderson in case char $k \neq 2$. We also prove a permanence principle and a duality principle which are useful in deriving identities.

In [6], K. Yamaguti also defines representations of Jordan triple systems. However, his concept of Jordan triple system is different from ours (the Jordan triple systems of type II considered in [6] are a generalization of our Jordan triples).

1. Representations.

1.1. *Jordan triples*. Let k be a commutative ring with unit and let V and W be unital k-modules. A map P: $V \to W$ is called *quadratic* if $P(\alpha x) = \alpha^2 P(x)$ for all $\alpha \in k$, $x \in V$, and if $P(x,y) = P(x+y) - P(x) - P(y)$ is bilinear in x and y. If R is any commutative associative k-algebra then there is a unique quadratic map P_R: $V \otimes_k R \to W \otimes_k R$ of R-modules making the diagram

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$$\begin{array}{ccc} V \otimes_k R & \xrightarrow{P_R} & W \otimes_k R \\ \uparrow & & \Vert\uparrow \\ V & \xrightarrow[P]{} & W \end{array}$$

commutative (see [1]). In case $W = \mathrm{End}_k V$, we denote the composition $V \otimes_k R \to (\mathrm{End}_k V) \otimes_k R \to \mathrm{End}_R(V \otimes_k R)$ also by P_R.

Let now $P\colon V \to \mathrm{End}_k V$ be a quadratic map. We set

$$\{xyz\} = L(x,y)z = P(x,z)y.$$

Then $(x,y,z) \mapsto \{xyz\}$ is a k-trilinear map from $V \times V \times V$ into V such that $\{xyz\} = \{zyx\}$ and $\{xyx\} = 2P(x)y$. The pair (V,P) is called a ***Jordan triple*** if the identities

$$L(x,y)P(x) = P(x)L(y,x) = P(P(x)y,x), \tag{1}$$

$$L(P(x)y,y) = L(x,P(y)x), \tag{2}$$

$$P(P(x)y) = P(x)P(y)P(x) \tag{3}$$

hold in V and in all scalar extensions (V_R, P_R) of (V,P) (equivalently, if all linearizations of (1)–(3) hold in V).

A k-linear map $f\colon V \to W$ of Jordan triples is called a ***homomorphism*** if $f(P(x)y) = P(f(x))f(y)$ for all $x, y \in V$. An ***ideal*** of V is a k-submodule I satisfying $P(I)V + P(V)I + \{VVI\} \subset I$. For the general theory of Jordan triples see [2], [5].

1.2. *Identities.* By linearizing (1) we obtain

$$\begin{aligned} L(x,y)P(x,z) + L(z,y)P(x) &= P(x,z)L(y,x) + P(x)L(y,z) \\ &= P(\{xyz\},x) + P(P(x)y,z). \end{aligned} \tag{4}$$

We apply this to an element $u \in V$, regard it as a function of z and change u to z. Then we have

$$L(x,y)L(x,z) + L(P(x)z,y) = L(x,\{yxz\}) + P(x)P(y,z). \tag{5}$$

We linearize (2) with respect to x and y and obtain

$$L(\{xyz\},y) = L(z,P(y)x) + L(x,P(y)z), \tag{6}$$

$$L(x,\{yxz\}) = L(P(x)y,z) + L(P(x)z,y). \tag{7}$$

Again we apply this to an element of V and regard it as a function of z and obtain, after a change of notation,

$$L(z,y)L(x,y) = P(x,z)P(y) + L(z,P(y)x), \tag{8}$$

$$P(x,z)L(y,x) = P(P(x)y,z) + L(z,y)P(x). \tag{9}$$

Subtract (9) from (4) to obtain

$$P(x)L(y,z) + L(z,y)P(x) = P(x,\{xyz\}). \tag{10}$$

Addition of (5) and (7) gives

$$L(x,y)L(x,z) = L(P(x)y,z) + P(x)P(y,z). \tag{11}$$

1.3. **Definition.** Let V be a Jordan triple over k, and let A be a unital associative k-algebra. A *representation* of V in A is a pair (l, p) of maps where l: $V \times V \to A$ is bilinear and p: $V \to A$ is quadratic, such that the following identities hold in all scalar extensions.

$$l(x,y)p(x) = p(x)l(y,x) = p(x,P(x)y), \tag{12}$$

$$p(x)l(y,z) + l(z,y)p(x) = p(x,\{xyz\}), \tag{13}$$

$$l(x,y)l(x,z) = l(P(x)y,z) + p(x)p(y,z), \tag{14}$$

$$l(z,x)l(y,x) = l(z,P(x)y) + p(y,z)p(x), \tag{15}$$

$$p(P(x)y) = p(x)p(y)p(x). \tag{16}$$

If A has an involution $a \mapsto a^*$ such that $l(x,y)^* = l(y,x)$ and $p(x)^* = p(x)$ for all $x, y \in V$ then (l, p) is called a *$*$-representation.* In this case, (15) is a consequence of (14).

Example. (a) The *regular representation* (L, P) of V in $\mathrm{End}_k V$.

(b) The *regular $*$-representation* of V in $E = \mathrm{End}_k V \times (\mathrm{End}_k V)^{\mathrm{op}}$ given by

$$l(x,y) = (L(x,y), L(y,x)) \quad \text{and} \quad p(x) = (P(x), P(x)).$$

The interchange $(f,g) \mapsto (g,f)$ is an involution of E making (l, p) a $*$-representation.

1.4. **Lemma.** *If (l, p) is a representation of V in A then the following formulas hold.*

$$l(P(x)y,y) = l(x,P(y)x), \tag{17}$$

$$p(x,z)l(y,x) = l(z,y)p(x) + p(P(x)y,z), \tag{18}$$

$$l(x,y)p(x,z) = p(x)l(y,z) + p(P(x)y,z). \tag{19}$$

Proof. (17) follows by setting $y = z$ in (14) and (15) and subtracting. We linearize (12):

$$\begin{aligned} l(z,y)p(x) + l(x,y)p(x,z) &= p(x)l(y,z) + p(x,z)l(y,x) \\ &= p(x,\{xyz\}) + p(z,P(x)y), \end{aligned}$$

subtract (13) and obtain (18) and (19).

1.5. *Split null extensions.* If M is a k-module and (l, p) is a representation of V in $\operatorname{End}_k M$ then we say M is a V-module. As in the case of quadratic Jordan algebras (see [4]) we have

Proposition. *$V \oplus M$ becomes a Jordan triple with*

$$P(x \oplus m)(y \oplus n) = P(x)y \oplus [p(x)n + l(x,y)m]$$

$(x,y \in V,\ m,n \in M)$, the split null extension of V by M.

Proof. If we use the fact that any product in $V \oplus M$ containing more than one element from M is zero, as well as the identities (12)–(19), the verification of (1) in $V \oplus M$ amounts to

$$\begin{aligned} p(x)p(y,z) + l(x,\{yxz\}) &= l(x,y)l(x,z) + l(P(x)z,y) \\ &= l(P(x)y,z) + l(x,z)l(x,y). \end{aligned}$$

But this is an easy consequence of (14) and (17). Similarly, (2) follows without difficulty from (15), (17), and (18), and (3) comes down to showing

$$l(P(x)y,z)p(x) = p(x)l(y,P(x)z) \tag{20}$$

and

$$p(x)p(y)l(x,z) + l(x,P(y)P(x)z) = l(P(x)y,z)l(x,y). \tag{21}$$

By (13), (12), and (17) we have

$$\begin{aligned} p(x)l(y,P(x)z) + l(P(x)z,y)p(x) &= p(x,\{x,y,P(x)z\}) \\ &= p(x,P(x)\{yxz\}) = l(x,\{yxz\})p(x) \\ &= l(P(x)y,z)p(x) + l(P(x)z,y)p(x) \end{aligned}$$

which proves (20). For (21), we use (19) and (15) and get

$$\begin{aligned} &p(x)p(y)l(x,z) + l(x,P(y)P(x)z) \\ &\quad = p(x)l(y,x)p(y,z) \\ &\qquad - p(x)p(P(y)x,z) + l(x,y)l(P(x)z,y) - p(x)l(z,x)p(y) \\ &\quad = l(x,y)l(x,z)l(x,y) - p(x)p(P(y)x,z) - p(x)l(z,x)p(y) \\ &\qquad\qquad \text{(by (12) and (14))} \\ &\quad = l(P(x)y,z)l(x,y) + p(x)[p(y,z)l(x,y) - p(P(y)x,z) - l(z,x)p(y)] \\ &\qquad\qquad \text{(by (14))} \\ &\quad = l(P(x)y,z)l(x,y) \quad \text{(by (18))}. \end{aligned}$$

We remark that the discussion in [4] concerning the cohomology of quadratic Jordan algebras can be carried over word for word to the Jordan triple case, so we omit the details.

1.6. Let (JT_k) denote the class of all Jordan triples over k. As in [4], we obtain from 1.5 the following

Permanence principle. *If F is any identity in the $L(x,y)$'s and $P(z)$'s which is valid for the regular representation of all $V \in (JT_k)$ then the identity obtained from F by replacing L, P by l, p is valid for all representations of all $V \in (JT_k)$.*

Indeed, to prove F for a representation (l, p) of V in A, consider A as a V-module by composing (l, p) with the left regular representation of A. Then F is valid for the regular representation of the split null extension $V \oplus A$, and by restricting to A and applying F to the unit element of A the assertion follows.

Another useful device in deriving identities is the

Duality principle. *If F is any identity in $l(x,y)$'s and $p(z)$'s which is valid for every representation of all $V \in (JT_k)$ then its dual F^*, obtained by replacing $l(x,y)$ by $l(y,x)$ and reversing the order of the $l(x,y)$'s and $p(z)$'s, is also valid for every representation.*

Indeed, F holds in particular for the regular $*$-representation of V in $E = \mathrm{End}_k V \times (\mathrm{End}_k V)^{\mathrm{op}}$. Applying the involution of E and projecting onto the first factor $\mathrm{End}_k V$ of E, we see that F^* holds for all regular representations. By the permanence principle, F^* holds for all representations.

1.7. *Homotopes.* Let $a \in V$. With the operations

$$U_x y = P(x)P(a)y, \qquad x^2 = P(x)a,$$

the k-module V becomes a quadratic Jordan algebra V_a, the *homotope* of V with respect to a (cf. [5]). Let $\hat{V}_a = k.1 \oplus V_a$ be the unital quadratic Jordan algebra obtained from V_a by adjoining a unit element. Recall that a *unital quadratic representation* of a unital quadratic Jordan algebra J in a unital associative algebra A is a quadratic map $\mu\colon J \to A$ satisfying the following identities in all scalar extensions (cf. [4]):

$$\mu(1) = 1, \tag{22}$$

$$\mu(U_x y) = \mu(x)\mu(y)\mu(x), \tag{23}$$

$$\nu(x,y)\mu(x) = \mu(x)\nu(y,x) = \mu(U_x y, x), \tag{24}$$

where $\nu(x,y) = \mu(x,1)\mu(y,1) - \mu(x,y)$.

Proposition. *Let (l, p) be a representation of the Jordan triple V in A. Then for every $a \in V$,*

$$\mu(\alpha.1 + x) = \alpha^2.1 + \alpha l(x,a) + p(x)p(a)$$

defines a unital quadratic representation of $\hat{V}_a$ in A.

Proof. Let $\bar{\mu}(\alpha.1 + x) = U(\alpha.1 + x)|V_a = \alpha^2 \,\mathrm{Id}_V + \alpha L(x,a) + P(x)P(a)$. Since V_a is an ideal of $\hat{V}_a$, this is a unital quadratic representation of $\hat{V}_a$, and the validity of (23) and (24) for $\bar{\mu}$ is equivalent to certain identities in L's and P's. By the permanence principle, the same identities hold with L, P replaced by l, p, i.e., for μ. Hence μ satisfies (23) and (24). Since it is obviously quadratic and $\mu(1) = 1$, the proposition follows.

Recall that a pair $(x,a) \in V \times V$ is called quasi-invertible if $1 - x$ is invertible in $\hat{V}_a$ (cf. [5]).

Corollary. *If* (x,a) *is quasi-invertible then*

$$b(x,a) = 1 - l(x,a) + p(x)p(a)$$

is invertible in A.

Indeed, $b(x,a) = \mu(1 - x)$, and μ maps invertible elements of $\hat{V}_a$ into invertible elements of A.

1.8. **Definition.** A *λ-representation* of V in a unital associative algebra A is a bilinear map $l\colon V \times V \to A$ such that the identities

$$l(P(x)y,y) = l(x,P(y)x), \tag{17}$$

$$l(x,y)l(u,v) - l(u,v)l(x,y) = l(\{xyu\},v) - l(u,\{yxv\}), \tag{25}$$

$$\begin{aligned} l(x,y)^4 &= l(x,y)^2 l(P(x)y,y) + l(P(x)y,y)l(x,y)^2 \\ &\quad + l(P(x)y,y)^2 - l(P(P(x)y)y,y) - l(x,P(P(y)x)x) \end{aligned} \tag{26}$$

hold in all scalar extensions.

A *π-representation* of V in A is a quadratic map $p\colon V \to A$ such that the identities

$$p(x+y) = p(x) + p(y), \tag{27}$$

$$p(P(x)y) = p(x)p(y)p(x) \tag{16}$$

hold in all scalar extensions.

Lemma. (a) *If* (l, p) *is a representation of* V *then* l *is a λ-representation.*
(b) *If* p *is a π-representation then* $2p(x) = 0$ *for all* $x \in V$ *and*

$$p(\{xyz\}) = p(x)p(y)p(z) + p(z)p(y)p(x). \tag{28}$$

Proof. (a) (25) follows from the corresponding identity for the regular representation (see [5]) by the permanence principle. If we set $y = z$ in (14) we get $l(x,y)^2 - l(P(x)y,y) = 2p(x)p(y)$. By squaring this and using (16) and (17) we get (26).

(b) By (27), $2p(x) = p(2x) = 4p(x)$, and (28) follows from

$$\begin{aligned} p(\{xyz\}) &= p(P(x+z)y - P(x)y - P(z)y) \\ &= p(P(x+z)y) + p(P(x)y) + p(P(z)y) \\ &= p(x+z)p(y)p(x+z) + p(x)p(y)p(x) + p(z)p(y)p(z) \\ &= p(x)p(y)p(z) + p(z)p(y)p(x) + 2p(x)p(y)p(x) + 2p(z)p(y)p(z). \end{aligned}$$

2. Universal envelopes.

2.1. We first construct a universal object for quadratic maps. Let V be a k-module, let $q\colon V \to X$ be a bijection of V onto a set X, and let F be the free k-module generated by X. We set $q(x,y) = q(x+y) - q(x) - q(y)$, and let R be the submodule of F generated by

$$q(\alpha x) - \alpha^2 q(x), \qquad q(\alpha x, y) - \alpha q(x,y),$$

$$q(x+y,z) - q(x,z) - q(y,z),$$

where $\alpha \in k$, $x, y, z \in V$. We set $V^{\text{II}} = F/R$ and denote the image of $q(x)$ under the canonical map by x^{II}. We also set $\langle x,y\rangle = (x+y)^{\text{II}} - x^{\text{II}} - y^{\text{II}}$. Then $x \mapsto x^{\text{II}}$ is a quadratic map, V^{II} is generated by $\{x^{\text{II}} : x \in V\}$, and for any quadratic map $Q\colon V \to W$ there is a unique linear map $f\colon V^{\text{II}} \to W$ such that $Q(x) = f(x^{\text{II}})$. Also it is easily seen that V^{II} is functorial in V and compatible with extensions of the ring of scalars.

2.2. *The universal envelope.* Let V be a Jordan triple over k, let $W = V^{\text{II}} \oplus (V \otimes_k V)$, and let $\mathbf{T}(W)$ be the tensor algebra over W. The product of two elements $u, v \in \mathbf{T}(W)$ is denoted by $u \cdot v$.

Let $\mathbf{J}$ be the ideal of $\mathbf{T}(W)$ generated by the elements

$$\begin{aligned} &(x \otimes y) \cdot x^{\text{II}} - x^{\text{II}} \cdot (y \otimes x), \qquad (x \otimes y) \cdot x^{\text{II}} - \langle x, P(x)y\rangle, \\ &x^{\text{II}} \cdot (y \otimes z) + (z \otimes y) \cdot x^{\text{II}} - \langle x, \{xyz\}\rangle, \\ &(x \otimes y) \cdot (x \otimes z) - P(x)y \otimes z - x^{\text{II}} \cdot \langle y,z\rangle, \\ &(z \otimes x) \cdot (y \otimes x) - z \otimes P(x)y - \langle y,z\rangle \cdot x^{\text{II}}, \\ &(P(x)y)^{\text{II}} - x^{\text{II}} \cdot y^{\text{II}} \cdot x^{\text{II}}, \end{aligned}$$

corresponding to (12)–(16). The *universal envelope of V* is $\mathbf{U}(V) = \mathbf{T}(W)/\mathbf{J}$. We define $\tilde{l}\colon V \times V \to \mathbf{U}(V)$ by $\tilde{l}(x,y) = x \otimes y + \mathbf{J}$ and $\tilde{p}\colon V \to \mathbf{U}(V)$ by $\tilde{p}(x) = x^{\text{II}} + \mathbf{J}$.

Proposition. (a) *There is an involution $*$ of $\mathbf{U}(V)$ such that $(\tilde{l},\tilde{p})$ is a $*$-representation of V. For any representation (l, p) of V in A there is a unique homomorphism $f\colon \mathbf{U}(V) \to A$ of unital associative algebras such that $p = f \circ \tilde{p}$ and $l = f \circ \tilde{l}$. If (l, p) is a $*$-representation then f commutes with the involutions of $\mathbf{U}(V)$ and A.*

(b) *There is an augmentation* ε: $\mathbf{U}(V) \to k$ *so that* $\mathbf{U}(V) = k.1 \oplus \mathbf{U}^\circ(V)$ *where* $\mathbf{U}^\circ(V) = \operatorname{Ker} \varepsilon$ *is the augmentation ideal. Also,* $\mathbf{U}(V)$ *is functorial in* V *and is compatible with scalar extensions.*

(c) *If* I *is an ideal of* V *and* $\tilde{\mathbf{I}}$ *is the ideal of* $\mathbf{U}(V)$ *generated by* $\tilde{l}(I, V)$, $\tilde{l}(V, I)$, $\tilde{p}(I, V)$, *and* $\tilde{p}(I)$ *then* $\mathbf{U}(V/I) \cong \mathbf{U}(V)/\tilde{\mathbf{I}}$.

The proof of this proposition follows established lines and is therefore omitted. Let us just indicate how the involution $*$ of $\mathbf{U}(V)$ is defined. The k-module $W = V^{\mathrm{II}} \oplus (V \otimes V)$ possesses an endomorphism of period 2 given by $x^{\mathrm{II}} \mapsto x^{\mathrm{II}}$ and $x \otimes y \mapsto y \otimes x$. By the universal property of the tensor algebra, this endomorphism extends to an involution of $\mathbf{T}(W)$ leaving $\mathbf{J}$ invariant, and thus induces an involution $*$ of $\mathbf{U}(V)$ with the desired properties.

2.3. Similarly as in 2.2, we define the *universal* λ*-envelope* $\mathbf{U}_\lambda(V) = \mathbf{T}(V \otimes V)/\mathbf{J}_\lambda$ where $\mathbf{J}_\lambda$ is the ideal of $\mathbf{T}(V \otimes V)$ generated by the elements

$$P(x)y \otimes y - x \otimes P(y)x,$$

$$(x \otimes y) \cdot (u \otimes v) - (u \otimes v) \cdot (x \otimes y) - \{xyu\} \otimes v + u \otimes \{yxv\},$$

$$(x \otimes y)^4 - (x \otimes y)^2 \cdot (P(x)y \otimes y) - (P(x)y \otimes y) \cdot (x \otimes y)^2$$
$$-(P(x)y \otimes y)^2 + P(P(x)y)y \otimes y + x \otimes P(P(y)x)x,$$

corresponding to (17), (25), and (26), and the *universal* π*-envelope* $\mathbf{U}_\pi(V) = \mathbf{T}(V^{\mathrm{II}})/\mathbf{J}_\pi$ where $\mathbf{J}_\pi$ is the ideal of $\mathbf{T}(V^{\mathrm{II}})$ generated by the elements $\langle x,y \rangle = (x + y)^{\mathrm{II}} - x^{\mathrm{II}} - y^{\mathrm{II}}$ and $(p(x)y)^{\mathrm{II}} - x^{\mathrm{II}} \cdot y^{\mathrm{II}} \cdot x^{\mathrm{II}}$. Define $\bar{l}$: $V \times V \to \mathbf{U}_\lambda(V)$ by $\bar{l}(x,y) = x \otimes y + \mathbf{J}_\lambda$ and $\bar{p}$: $V \to \mathbf{U}_\pi(V)$ by $\bar{p}(x) = x^{\mathrm{II}} + \mathbf{J}_\pi$.

Proposition. (a) $\bar{l}$ (*resp.* $\bar{p}$) *is a* λ*-representation* (*resp. a* π*-representation*) *of* V *in* $\mathbf{U}_\lambda(V)$ (*resp. in* $\mathbf{U}_\pi(V)$), *and any* λ- (*resp.* π-) *representation may be factored via* $\mathbf{U}_\lambda(V)$ (*resp.* $\mathbf{U}_\pi(V)$).

(b) $\mathbf{U}_\lambda(V)$ *and* $\mathbf{U}_\pi(V)$ *are functorial in* V *and compatible with scalar extensions. There are augmentations* $\mathbf{U}_\lambda(V) \to k$ *and* $\mathbf{U}_\pi(V) \to k$ *so that* $\mathbf{U}_\lambda(V) = k.1 \oplus \mathbf{U}_\lambda^\circ(V)$ *and* $\mathbf{U}_\pi(V) = k.1 \oplus \mathbf{U}_\pi^\circ(V)$.

(c) *If* I *is an ideal of* V *and if* $\mathbf{I}_\lambda$ (*resp.* $\mathbf{I}_\pi$) *is the ideal of* $\mathbf{U}_\lambda(V)$ (*resp.* $\mathbf{U}_\pi(V)$) *generated by* $\bar{l}(I, V)$ *and* $\bar{l}(V, I)$ (*resp.* $\bar{p}(I)$) *then* $\mathbf{U}_\lambda(V/I) \cong \mathbf{U}_\lambda(V)/\mathbf{I}_\lambda$ (*resp.* $\mathbf{U}_\pi(V/I) \cong \mathbf{U}_\pi(V)/\mathbf{I}_\pi$).

(d) $\mathbf{U}_\lambda(V)$ *and* $\mathbf{U}_\pi(V)$ *possess involutions* $*$ *such that* $\bar{l}(x,y)^* = \bar{l}(y,x)$ *and* $\bar{p}(x)^* = \bar{p}(x)$. *Also,* $\mathbf{U}_\pi(V) = \mathbf{U}_\pi^+(V) \oplus \mathbf{U}_\pi^-(V)$ *is* $\mathbf{Z}_2$*-graded, the gradation is invariant under* $*$, *and* $\bar{p}(x) \in \mathbf{U}_\pi^-(V)$ *for* $x \in V$.

The proof of (a), (b), and (c) is again straightforward. To prove (d), let $*$ denote the involutions of $\mathbf{T}(V \otimes V)$ (resp. $\mathbf{T}(V^{\mathrm{II}})$) induced by $x \otimes y \mapsto y \otimes x$ (resp. the identity on V^{II}). Then $\mathbf{J}_\lambda$ and $\mathbf{J}_\pi$ are invariant under $*$, and the statement about the involutions follows. Finally, $\mathbf{J}_\pi$ is generated by elements of odd degree (with respect to the natural gradation of $\mathbf{T}(V^{\mathrm{II}})$), and the gradation of $\mathbf{T}(V^{\mathrm{II}})$ is invariant under $*$.

2.4. **Theorem.** *If k is a field and V is finite-dimensional over k then so are $\mathbf{U}(V)$, $\mathbf{U}_\lambda(V)$, and $\mathbf{U}_\pi(V)$.*

Proof. Let $x_1, \ldots, x_n$ be a basis of V. First we show that $\mathbf{U}_\lambda(V)$ is finite-dimensional. As an algebra, $\mathbf{U}_\lambda(V)$ is generated by 1 and $\{\bar{l}(x_i, x_j): i,j = 1, \ldots, n\}$. We number the $\bar{l}(x_i, x_j)$ consecutively: $y_1 = \bar{l}(x_1, x_1), y_2 = \bar{l}(x_1, x_2), \ldots, y_{n^2} = \bar{l}(x_n, x_n)$.

Lemma 1. $\mathbf{U}_\lambda(V)$ *is spanned by the monomials* $y_{i_1} \cdots y_{i_r}$ *where* $0 \leq r \leq 3n^2$ *and* $i_1 \leq i_2 \leq \cdots \leq i_r$.

Proof. Let $\mathbf{X}_r$ be the subspace of $\mathbf{U}_\lambda(V)$ spanned by the monomials $y_{i_1} \cdots y_{i_s}$ where $s \leq r$. Clearly, $\mathbf{X}_0 = k.1$, $\mathbf{X}_r \subset \mathbf{X}_{r+1}$, and $\mathbf{X}_r \cdot \mathbf{X}_s \subset \mathbf{X}_{r+s}$. We claim that $\mathbf{X}_r = \mathbf{X}_{r-1}$ if $r > 3n^2$. Indeed, because of (25) we have

$$(29) \qquad y_i y_j \equiv y_j y_i \quad \mod \mathbf{X}_1.$$

In a monomial $y_{i_1} \cdots y_{i_r}$ where $r > 3n^2$, at least one of the y_i, say y_1, occurs at least 4 times. By (29), we get $y_{i_1} \cdots y_{i_r} \equiv y_1^4 y_{j_1} \cdots y_{j_{r-4}}$ mod $\mathbf{X}_{r-1}$, and (26) shows $y_1^4 \in \mathbf{X}_3$. This proves our assertion. Since $\mathbf{U}_\lambda(V)$ is the union of the $\mathbf{X}_r$, we have $\mathbf{U}_\lambda(V) = \mathbf{X}_{3n^2}$. Finally, the ordered monomials suffice because of (29).

Lemma 2. *The subalgebra* $\mathbf{U}'$ *of* $\mathbf{U}(V)$ *generated by* $\tilde{l}(V, V)$ *and* $\tilde{p}(V, V)$ *is a finite-dimensional ideal of* $\mathbf{U}(V)$.

Proof. Let $\mathbf{L}$ be the subalgebra of $\mathbf{U}(V)$ generated by $\tilde{l}(V, V)$. Since $\tilde{l}$: $V \times V \to \mathbf{U}(V)$ is a λ-representation, $\mathbf{L}$ is a homomorphic image of $\mathbf{U}_\lambda^\circ(V)$ and therefore finite-dimensional by Lemma 1. Let $\mathbf{P}$ be the subalgebra of $\mathbf{U}(V)$ generated by $\tilde{p}(V, V)$, and let $\mathbf{P}_+$ be the subalgebra of $\mathbf{P}$ generated by $\{\tilde{p}(u, v)\tilde{p}(x, y): u, v, x, y \in V\}$. From (14) we obtain by linearizing

$$\tilde{p}(u, v)\tilde{p}(x, y) = \tilde{l}(v, y)\tilde{l}(u, x) + \tilde{l}(u, y)\tilde{l}(v, x) - \tilde{l}(\{uyv\}, x)$$

which implies $\mathbf{P}_+ \subset \mathbf{L}$. Also $\mathbf{P} = \mathbf{P}_+ + \tilde{p}(V, V) + \mathbf{P}_+\tilde{p}(V, V)$ shows that $\mathbf{P}$ is finite-dimensional. From (13) we get by linearizing

$$\tilde{l}(z, y)\tilde{p}(x, w) = \tilde{p}(x, \{wyz\}) + \tilde{p}(w, \{xyz\}) - \tilde{p}(x, w)\tilde{l}(y, z)$$

which implies by induction

$$(30) \qquad \mathbf{LP} \subset \mathbf{PL} + \mathbf{P}.$$

This shows that $\mathbf{U}' = \mathbf{L} + \mathbf{PL} + \mathbf{P}$ is finite-dimensional. Finally, it follows from (14), (15), (18) and (19) that $\mathbf{U}'$ is an ideal of $\mathbf{U}(V)$. Note that $\mathbf{U}' = \mathbf{U}^\circ(V)$ if char $k \neq 2$.

Now let $z_i = \bar{p}(x_i) \in \mathbf{U}_\pi(V)$.

Lemma 3. *$\mathbf{U}_\pi(V)$ is spanned by the monomials $z_{i_1} \cdots z_{i_r}$ where $0 \leq r \leq 2n$.*

Proof. Similarly to the proof of Lemma 1, let $\mathbf{X}_r$ be the subspace of $\mathbf{U}_\pi(V)$ spanned by the monomials $z_{i_1} \cdots z_{i_s}$ where $0 \leq s \leq r$. If $z_{i_1} \cdots z_{i_r}$ is a monomial with $r > 2n$ then at least one of the z_i, say z_1, occurs at least 3 times. Because of (28) we have $z_i z_j z_k \equiv z_k z_j z_i \mod \mathbf{X}_1$. Using this repeatedly, we see that $z_{i_1} \cdots z_{i_r}$ is congruent, modulo $\mathbf{X}_{r-2}$, to a monomial of the form $\cdots z_1^3 \cdots$ or $\cdots z_1 z_i z_1 \cdots$. But $z_1^3 \in \mathbf{X}_1$ and $z_1 z_i z_1 \in \mathbf{X}_1$ by (16). Hence $\mathbf{X}_r = \mathbf{X}_{r-2}$ if $r > 2n$ which shows $\mathbf{U}_\pi(V) = \mathbf{X}_{2n}$.

Now we finish the proof of the theorem. From the definition of $\mathbf{U}'$ it is clear that the map p: $x \mapsto \bar{p}(x) + \mathbf{U}'$ is a π-representation of V in $\mathbf{U}(V)/\mathbf{U}'$, and that $\mathbf{U}(V)/\mathbf{U}'$ is generated by 1 and $p(V)$. Hence $\mathbf{U}(V)/\mathbf{U}'$ is a homomorphic image of $\mathbf{U}_\pi(V)$ and therefore finite-dimensional by Lemma 3. Now $\mathbf{U}(V)$ is finite-dimensional by Lemma 2.

3. **Nilpotence.**

3.1. *The radical.* Let V be a Jordan triple over the ring k. The *radical* of V is

$$\operatorname{Rad} V = \{x \in V: (x,y) \text{ is quasi-invertible, for all } y \in V\}.$$

For the basic properties of the radical we refer to [5]. In particular, $\operatorname{Rad} V$ is an ideal of V, and if f: $V \to W$ is a surjective homomorphism of Jordan triples then $\operatorname{Rad} W \supset f(\operatorname{Rad} V)$. A Jordan triple with $\operatorname{Rad} V = 0$ is called *semisimple.* Recall also that an *inner ideal* of V is a k-submodule I such that $P(I)V \subset I$, and an *absolute zero divisor* is an element $x \in V$ such that $P(x) = 0$. It is easily seen that an absolute zero divisor belongs to the radical.

The proof of the following proposition can be found in [5].

Proposition. *If V satisfies the descending chain condition on inner ideals then V is semisimple if and only if it contains no absolute zero divisors $\neq 0$.*

3.2. **Proposition.** *Let V be a Jordan triple over the ring k satisfying the descending chain condition on inner ideals. Let R be a commutative associative k-algebra, let $V_R = V \otimes_k R$ be the scalar extension of V, and define φ: $V \to V_R$ by $\varphi(x) = x \otimes 1$. Then $\varphi(\operatorname{Rad} V) \subset \operatorname{Rad} V_R$.*

Proof. Let $I = \varphi^{-1}(\operatorname{Rad} V_R)$. Since φ is a homomorphism of Jordan triples over k, this is an ideal of V. Let I_R be the R-submodule of V_R generated by $\varphi(I)$. Then I_R is an ideal of V_R, contained in $\operatorname{Rad} V_R$. Since tensoring with R is a right exact functor, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "\phi"] & V/I \arrow[r] \arrow[d, "\bar{\phi}"] & 0 \\
& I \otimes_k R \arrow[r, "i"'] & V_R \arrow[r] & (V/I) \otimes_k R \arrow[r] & 0
\end{tikzcd}$$

Clearly, $I_R = i(I \otimes_k R)$ so that V_R/I_R may be identified with $(V/I) \otimes_k R$. We denote the canonical maps $V \to V/I$ and $V_R \to V_R/I_R$ by $x \mapsto \bar{x}$. Assume now that $\bar{x}$ is an absolute zero divisor of V/I. Then $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$ is an absolute zero divisor of V_R/I_R and thus contained in the radical of V_R/I_R which is $(\mathrm{Rad}\ V_R)/I_R$. This means $\varphi(x) \in \mathrm{Rad}\ V_R$, i.e., $x \in I$ and therefore $\bar{x} = 0$. Since the descending chain condition is inherited by V/I we have V/I semisimple by 3.1, i.e., $I \supset \mathrm{Rad}\ V$.

3.3. **Theorem.** *Let V be a finite-dimensional Jordan triple over the field k and assume that $V =$ Rad V. Then $\mathbf{U}^\circ(V)$, $\mathbf{U}^\circ_\lambda(V)$ and $\mathbf{U}^\circ_\pi(V)$ are nilpotent.*

Proof. By 3.2 and the fact that the universal envelopes are compatible with scalar extensions we may assume k algebraically closed. The crucial fact is

Lemma 1. *A finite-dimensional Jordan triple V over an algebraically closed field with* Rad $V \neq 0$ *contains an ideal $I \neq 0$ such that $P(I) = L(I, I) = 0$.*

This is proved in [3]. In case char $k \neq 2$, see also [2].

From now on, we assume $V =$ Rad V and k algebraically closed.

Lemma 2. $\mathbf{U}^\circ_\lambda(V)$ *is nilpotent.*

The proof is by induction on dim V. Let $\bar{l}\colon V \times V \to \mathbf{U}_\lambda(V)$ be the universal λ-representation of V in $\mathbf{U}_\lambda(V)$ (cf. 2.3). If dim $V = 1$, i.e., $V = k.x$, then $P(x)x = 0$, and $\mathbf{U}^\circ_\lambda(V)$ is generated by $\bar{l}(x,x)$. By (26), $\bar{l}(x,x)^4 = 0$ and hence $\mathbf{U}^\circ_\lambda(V)$ is nilpotent. Now let dim $V > 1$. Then by Lemma 1, V contains a proper ideal I such that $P(I)V = \{IIV\} = 0$. By induction, $\mathbf{U}^\circ_\lambda(I)$ is nilpotent. Let $\mathbf{I}$ be the subalgebra of $\mathbf{U}_\lambda(V)$ generated by $\bar{l}(I, I)$. Then $\mathbf{I}$ is a homomorphic image of $\mathbf{U}^\circ_\lambda(I)$ and hence is nilpotent. Also, $\mathbf{I}$ is contained in the center of $\mathbf{U}_\lambda(V)$ because of (25) and $\{IIV\} = 0$. Therefore $\mathbf{I}$ generates a nilpotent ideal $\mathbf{J} = \mathbf{I} \cdot \mathbf{U}_\lambda(V)$ of $\mathbf{U}_\lambda(V)$.

Let $\mathbf{I}'$ be the subalgebra of $\mathbf{U}_\lambda(V)$ generated by $\bar{l}(I, V)$ and $\bar{l}(V, I)$. Then it follows from (25) and $\{IVI\} = \{IIV\} = 0$ that $\mathbf{I}'/\mathbf{J} \cap \mathbf{I}'$ is commutative. Also by (17), (26), and $P(I) = 0$, we have $\bar{l}(x,y)^4 \equiv 0 \mod \mathbf{J}$ if $x \in V, y \in I$, or $x \in I$, $y \in V$. By finite-dimensionality, $\mathbf{I}'/\mathbf{J} \cap \mathbf{I}'$ is nilpotent, and hence $\mathbf{I}'$ is nilpotent.

Now let $\mathbf{I}_\lambda = \mathbf{I}' \cdot \mathbf{U}_\lambda(V)$. Then (25) implies

$$\mathbf{U}_\lambda(V) \cdot \mathbf{I}' \subset \mathbf{I}_\lambda. \tag{31}$$

Hence $\mathbf{I}_\lambda$ is an ideal of $\mathbf{U}_\lambda(V)$, in fact, the ideal generated by $\bar{l}(I, V)$ and $\bar{l}(V, I)$. Since $\mathbf{I}'$ is nilpotent so is $\mathbf{I}_\lambda$ by (31). By induction, $\mathbf{U}^\circ_\lambda(V/I) = \mathbf{U}^\circ(V)/\mathbf{I}_\lambda$ (cf. 2.3) is nilpotent. Hence $\mathbf{U}^\circ_\lambda(V)$ is nilpotent.

We denote by $\mathbf{U}'$ the ideal of $\mathbf{U}(V)$ generated by $\bar{l}(V, V)$ and $\bar{p}(V, V)$ (see Lemma 2 in 2.4).

Lemma 3. $\mathbf{U}'$ *is nilpotent.*

Proof. Let $\mathbf{L}$, $\mathbf{P}$, and $\mathbf{P}_+$ be as in the proof of Lemma 2 in 2.4. Then $\mathbf{L}$ is a homomorphic image of $\mathbf{U}_\lambda^\circ(V)$ and therefore nilpotent by Lemma 2. Also $\mathbf{P}_+$ is nilpotent since it is contained in $\mathbf{L}$. Let $\mathbf{P}_-$ be the subspace of $\mathbf{P}$ spanned by the products of an odd number of elements of $\bar{p}(V, V)$. Then we have $\mathbf{P} = \mathbf{P}_+ + \mathbf{P}_-$, $\mathbf{P}_-^2 \subset \mathbf{P}_+$, $\mathbf{P}_+\mathbf{P}_- = \mathbf{P}_-\mathbf{P}_+ \subset \mathbf{P}_-$, and induction shows $\mathbf{P}^{2n} \subset \mathbf{P}_+^n + \mathbf{P}_+^n\mathbf{P}_-$. Hence $\mathbf{P}$ is nilpotent.

From (30) we see that $\mathbf{J} = \mathbf{PL} + \mathbf{P}$ is an ideal of $\mathbf{U}'$, and by induction we have $\mathbf{J}^n \subset \mathbf{P}^n\mathbf{L} + \mathbf{P}^n$. Thus $\mathbf{J}$ is nilpotent. Since $\mathbf{U} = \mathbf{L} + \mathbf{J}$ and $\mathbf{U}'/\mathbf{J} \cong \mathbf{L}/\mathbf{L} \cap \mathbf{J}$ is nilpotent, $\mathbf{U}'$ is nilpotent.

Note. Lemma 2 and Lemma 3 (with a different proof) as well as the concept of λ-representation are due to C.T. Anderson (cf. [2], §8).

Lemma 4. *There exists an ideal $N \neq 0$ of V such that $P(N) = L(V,N) = L(N,V) = 0$.*

Proof. Let I be the ideal of Lemma 1, and consider the regular representation (L_I, P_I) of V in $\mathrm{End}_k I$, i.e., $L_I(x,y) = L(x,y) \mid I$, and $P_I(x) = P(x) \mid I$. Let f: $\mathbf{U}(V) \to \mathrm{End}_k I$ be the induced homomorphism of associative algebras. Then by Lemma 3, $f(\mathbf{U}')$ is nilpotent. Thus if $f(\mathbf{U}')^n \neq 0$ but $f(\mathbf{U}')^{n+1} = 0$ then $N = \{x \in I: f(\mathbf{U}').x = 0\} \supset f(\mathbf{U}')^n.I \neq 0$ and N is invariant under $f(\mathbf{U}(V))$ since $\mathbf{U}'$ is an ideal of $\mathbf{U}(V)$. Hence N is an ideal of V with the desired properties.

Lemma 5. $\mathbf{U}_\pi^\circ(V)$ *is nilpotent.*

Proof. We may assume char $k = 2$ since otherwise $\mathbf{U}_\pi^\circ(V) = 0$. The proof is by induction on dim V. If $V = 0$ there is nothing to prove. Let $V \neq 0$, and let N be as in Lemma 4. From $P(N)V = \{VVN\} = 0$, (16) and (28) we obtain

$$zxz = 0, \tag{32}$$

$$xyz = zyx, \tag{33}$$

for $z \in \bar{p}(N)$ and $x, y \in \bar{p}(V)$. We will show that the ideal $\mathbf{N}_\pi$ of $\mathbf{U}_\pi(V)$ generated by $\bar{p}(N)$ is nilpotent. Since $\mathbf{U}_\pi(V)$ is finite-dimensional, it suffices to show that $\mathbf{N}_\pi$ is spanned by nilpotent elements. Now $\mathbf{N}_\pi$ is spanned by the monomials $m = x_1 \cdots x_r$ where $x_i \in \bar{p}(V)$ and at least one of the x_i belongs to $\bar{p}(N)$. Using (33), we see that m^3 equals a monomial of the form $\cdots z^3 \cdots$ or $\cdots zxz \cdots$ where $z \in \bar{p}(N)$ and $x \in \bar{p}(V)$. Thus $m^3 = 0$ by (32), and $\mathbf{N}_\pi$ is nilpotent. By induction, $\mathbf{U}_\pi^\circ(V/N) = \mathbf{U}_\pi^\circ(V)/\mathbf{N}_\pi$ is nilpotent, and the lemma is proved.

Now we finish the proof of the theorem. $\mathbf{U}^\circ(V)/\mathbf{U}'$ is a homomorphic image of $\mathbf{U}_\pi^\circ(V)$ and therefore nilpotent by Lemma 5. Thus $\mathbf{U}^\circ(V)$ is nilpotent by Lemma 3.

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