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MULTIPLIERS FOR CERTAIN CONVOLUTION MEASURE ALGEBRAS

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ABSTRACT. Let $(A,*)$ be a commutative semisimple convolution measure algebra with structure semigroup Γ, and let S denote a commutative locally compact topological semigroup. Under the assumption that A possesses a weak bounded approximate identity, it is shown that there is a topological embedding of the multiplier algebra $\mathcal{M}(A)$ of A in $M(\Gamma)$. This representation leads to a proof of the commutative case of Wendel's theorem for $A = L_1(G)$, where G is a commutative locally compact topological group. It is also proved that if $l_1(S)$ has a weak bounded approximate identity of norm one, then $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$, where $\Omega(S)$ is the multiplier semigroup of S. Likewise, if S is cancellative, then $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$.

An example is provided of a semigroup S for which $l_1(\Omega(S))$ is isomorphic to a proper subset of $\mathcal{M}(l_1(S))$.

1. **Introduction.** Let $(A,*)$ be a commutative semisimple convolution measure algebra with structure semigroup Γ, and let S denote a commutative locally compact topological semigroup with multiplier semigroup $\Omega(S)$ (details concerning notation, definitions, and background results are given in §2). In §3 we study the multiplier algebra $\mathcal{M}(A)$ of $(A,*)$ when A possesses a weak bounded approximate identity. Theorem 3.1 asserts that $\mathcal{M}(A)$ is topologically embeddable in $M(\Gamma)$. We apply Theorem 3.1 to the particular convolution measure algebra $A = M(S)$ and proceed in Corollary 3.8 to characterize those measures in $M(\Gamma)$ that give rise to multipliers of $M(S)$. Finally, §4 is devoted to a characterization of the multiplier algebra of $l_1(S)$ for certain semigroups S. An example is furnished of a semigroup S for which $l_1(\Omega(S))$ is isomorphic to a proper subset of $\mathcal{M}(l_1(S))$.

2. **Preliminaries.** Let $(B,*)$ be a commutative Banach algebra under $\|\cdot\|$. Let $\Delta(B)$ denote the maximal ideal space of B, that is, the space of all continuous homomorphisms of B into the complex field C together with the weak-* (Gelfand) topology [17]. As usual for any $\alpha \in B$, define $\hat{\alpha}(\chi) = \chi(\alpha)$ for each $\chi \in \Delta(B)$, and let $\hat{B} = \{\hat{\alpha}: \alpha \in B\}$. A weak bounded approximate identity of norm R for B is a net $\{E_\rho\}_{\rho \in \mathcal{D}}$ of elements of B such that (a) $\|E_\rho\| \leq R$ for some natural number R and for all $\rho \in \mathcal{D}$, and (b) $(\alpha * E_\rho)^\wedge(\chi) \to \hat{\alpha}(\chi)$ for all $\chi \in \Delta(B)$ and for every $\alpha \in B$. A bounded approximate identity of norm R for B is a net $\{E_\rho\}$ of elements of B such that (a) $\|E_\rho\| \leq R$ for some natural number R and for all ρ, and (b) $\|\alpha * E_\rho - \alpha\| \to 0$ for all $\alpha \in B$; we sometimes use the

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terminology "bounded (norm) approximate identity of norm R" for the same concept.

If X is any normed linear vector space, the continuous linear dual of X is denoted by X^*; (α, f) represents the action of $f \in X^*$ on $\alpha \in X$; and if $Y \subseteq X$, $Z \subseteq X^*$, then let $w(Y, Z)$ be the weak topology on $Y \subseteq X$ induced by $Z \subseteq X^*$. The natural mapping of X into X^{**} is denoted by j: $X \to X^{**}$. Often, we simply denote $(f, j(\alpha))$ by (α, f), $\alpha \in X$, $f \in X^*$, in those circumstances where the meaning is clear.

A bounded linear operator T from B to B is called a multiplier of B if $T(\alpha * \beta) = \alpha * T(\beta)$ for all $\alpha, \beta \in B$. The set of multipliers of B, in turn, forms a Banach algebra of operators under operator norm $|||\cdot|||$; denote this algebra by $\mathscr{M}(B)$. $\mathscr{M}(B)$ contains an identity, and if B is semisimple, it contains an isomorphic copy of B as an ideal. If B has an identity E, then each $T \in \mathscr{M}(B)$ is given by multiplication by the fixed element $T(E)$ of B and so $\mathscr{M}(B)$ reduces to B. If B is semisimple, then $\mathscr{M}(B)$ is semisimple, and $\Delta(B)$ is homeomorphic to an open subset of $\Delta(\mathscr{M}(B))$, both in their Gelfand topologies. For a detailed review of results on multipliers, see [13]; also consult [21] and [2].

Now let S be a commutative locally compact Hausdorff semigroup with jointly continuous multiplication (sometimes herein referred to as a commutative locally compact topological semigroup), and let $M(S)$ denote the complex Banach algebra of all bounded regular Borel measures on S where the product $*$ is defined by convolution. For μ, $\nu \in M(S)$, F a Borel subset of S, $(\mu * \nu)(F) = \int_S \int_S \phi_F(xy)\, d\mu(x)\, d\nu(y)$, where ϕ_F denotes the characteristic function of F. The norm on $M(S)$ is the total variation norm, denoted $\|\cdot\|$. See Taylor [20]. A semicharacter χ on S is a nonzero continuous complex valued function on S of modulus less than or equal to one which satisfies $\chi(xy) = \chi(x)\chi(y)$ for all x, $y \in S$. The collection of semicharacters of S is denoted by $\hat{S}$. It is well known that $C_0(S)^* = M(S)$, where $\mu \in M(S)$ induces a linear functional on $C_0(S)$ by

$$(g, \mu) = \int_S g(x)\, d\mu(x) \quad \text{for all } g \in C_0(S).$$

The set of discrete measures in $M(S)$ forms a subalgebra of $M(S)$, denoted by $l_1(S)$. Of course, if S is discrete, then $l_1(S) = M(S)$. Hewitt and Zuckerman present a detailed study of $l_1(S)$ in [10]. They show that the existence of an identity in $l_1(S)$ is equivalent to the existence of a finite set of relative units in S, where U is defined to be a set of relative units for S, if for every $x \in S$, there exists $u \in U$ such that $xu = x$. Lardy [12] proves that the same conditions on S are necessary and sufficient for the existence of an identity in $M(S)$, and in fact an identity for $M(S)$ must lie in $l_1(S)$.

A portion of this paper is devoted to characterizing the multiplier algebras of certain semisimple convolution measure algebras. For a definition of convolution measure algebra see [20]; $M(S)$ is an example of a convolution measure algebra. Also, if G is a locally compact abelian topological group, and if $L_1(G)$ is the algebra of Haar integrable functions on G under convolution multiplication, then $L_1(G)$ is a convolution measure algebra.

Taylor proves in [20] that if $(A, *)$ is a commutative convolution measure algebra, we may identify the maximal ideal space of $(A,*)$ with $\hat{\Gamma}$, the set of all semicharacters on a compact topological semigroup Γ, which he labels the structure semigroup of $(A,*)$ (Γ will denote the structure semigroup of whatever convolution measure algebra is under discussion). There is a homomorphism $p: \mu \to \mu_p$ of A into $M(\Gamma)$ with the following pertinent properties: $p(A)$ is weak-* dense in $M(\Gamma)$, that is, dense in the $w(M(\Gamma), C(\Gamma))$ topology (where we have identified $C(\Gamma)$ with its natural image in $M(\Gamma)^*$); p is an isometry if and only if $(A,*)$ is semisimple. Γ also has the property that the uniformly closed linear span of $\hat{\Gamma}$ is $C(\Gamma)$. We make use of this fact in observing that $\hat{\Gamma} \subseteq \Delta(M(\Gamma))$ is enough to decide the semisimplicity of $M(\Gamma)$: that is, suppose $\mu, \nu \in M(\Gamma)$ and $\hat{\mu}(\chi) = \hat{\nu}(\chi)$ for all $\chi \in \hat{\Gamma}$; then because the linear span of $\hat{\Gamma}$ is uniformly dense in $C(\Gamma)$, the formula $(\chi, \mu) = (\chi, \nu)$ for all $\chi \in \hat{\Gamma}$ can be extended to all $f \in C(\Gamma)$; therefore, μ and ν agree as linear functionals on $C(\Gamma)$ and so as elements of $M(\Gamma)$.

Suppose $(A,*)$ is semisimple. Then it is proved in [11] that A has a weak bounded approximate identity if and only if Γ has a finite set of relative units. In fact, the existence of an identity in Γ is equivalent to the existence of a weak bounded approximate identity of norm one in A.

3. **Multipliers of convolution measure algebras.** Throughout this section $(A,*)$ is a semisimple convolution measure algebra, S is a locally compact abelian Hausdorff semigroup with jointly continuous multiplication, and $M(S)$ is assumed to be semisimple. The following is a representation theorem for $\mathcal{M}(A)$.

Theorem 3.1. *Let $\{E_\rho\}$ be a weak bounded approximate identity for $(A,*)$, $\|E_\rho\| \leq R$ for all ρ. Then if $T \in \mathcal{M}(A)$ there is a unique measure $\mu_T \in M(\Gamma)$ such that $T(\alpha) = \alpha * \mu_T$ for all $\alpha \in A$ and $\mathcal{M}(A)$ is isomorphic to $\{\mu \in M(\Gamma): \mu * \alpha \in A$ for all $\alpha \in A\}$. Furthermore, the correspondence $T \mapsto \mu_T$ is an isometry if and only if A has a weak bounded approximate identity of norm one*; *in any case* $|||T||| \leq \|\mu_T\| \leq R|||T|||$.

Proof. Assume A is embedded in $M(\Gamma)$. Suppose $T \in \mathcal{M}(A)$. Then $\{T(E_\rho)\}$ is a subset of the closed ball of $M(\Gamma)$ of radius $R|||T|||$. In the weak-* topology this ball is compact. Thus, there is a subnet $\{T(E_{\rho'})\}$ and $\mu_T \in M(\Gamma)$ such that μ_T is a weak-* limit of $\{T(E_{\rho'})\}$; hence, for all $f \in C(\Gamma)$, $(f, T(E_{\rho'})) \to (f, \mu_T)$ and in particular $(\chi, T(E_{\rho'})) \to (\chi, \mu_T)$ for all $\chi \in \hat{\Gamma}$. If $\alpha \in A$, $(\chi, T(E_{\rho'}))(\chi, \alpha) \to (\chi, \mu_T)(\chi, \alpha)$ and so $(\chi, \alpha * T(E_{\rho'})) \to (\chi, \alpha * \mu_T)$ for all $\chi \in \hat{\Gamma}$. On the other hand, since $\{E_\rho\}$ is a weak bounded approximate identity, $(\chi, \alpha * T(E_\rho)) = (\chi, T(\alpha) * E_\rho) \to (\chi, T(\alpha))$ for all $\chi \in \hat{\Gamma}$ and for all $\alpha \in A$. Thus, $(\chi, T(\alpha)) = (\chi, \alpha * \mu_T)$ for all $\chi \in \hat{\Gamma}$. Because the linear span of $\hat{\Gamma}$ is uniformly dense in $C(\Gamma)$, this formula can be extended to all $f \in C(\Gamma)$. Therefore, $T(\alpha)$ and $\mu_T * \alpha$ agree as linear functionals on $C(\Gamma)$ and so as elements of $M(\Gamma)$.

To see that μ_T is unique, suppose $\mu, \lambda \in M(\Gamma)$ and $\mu * \alpha = \lambda * \alpha$ for all $\alpha \in A$. If $\chi \in \hat{\Gamma}$, there is an $\alpha \in A$ such that $(\chi, \alpha) \neq 0$. Thus, $(\chi, \mu - \lambda)(\chi, \alpha) = 0$ implies that $(\chi, \mu - \lambda) = 0$. Since this is true for all $\chi \in \hat{\Gamma}$ and the linear span of $\hat{\Gamma}$ is dense in $C(\Gamma)$, $(f, \mu - \lambda) = 0$ for all $f \in C(\Gamma)$. Thus, $\mu - \lambda = 0$ or $\mu = \lambda$. These arguments show that $\mathscr{M}(A)$ is isomorphic to $\{\mu \in M(\Gamma): \mu * \alpha \in A$ for all $\alpha \in A\}$.

Now, since μ_T is a weak-* limit point of a net in the closed ball of radius $R|||T|||$, $\|\mu_T\| \leq R|||T|||$, so the mapping $T \mapsto \mu_T$ is continuous. Also, $T(\alpha) = \alpha * \mu_T$ for all $\alpha \in A$ implies that $\|T(\alpha)\| \leq \|\mu_T\| \, \|\alpha\|$ and hence $|||T||| \leq \|\mu_T\|$. Thus, $|||\cdot|||$ on $\mathscr{M}(A)$ is equivalent to $\|\cdot\|$ on $\{\mu \in M(\Gamma): \mu * \alpha \in A$ for all $\alpha \in A\}$. From the relationship $|||T||| \leq \|\mu_T\| \leq R|||T|||$ we see that if $R = 1$, $T \mapsto \mu_T$ is actually an isometry.

On the other hand, if the mapping $T \mapsto \mu_T$ is an isometry, let I denote the identity multiplier and μ_I the corresponding measure in $M(\Gamma)$ such that $\alpha = I(\alpha) = \alpha * \mu_I$ for all $\alpha \in A$. Then, because A is weak-* dense in $M(\Gamma)$, μ_I is the identity for $M(\Gamma)$: let $\mu \in M(\Gamma)$; then there exists a net $\{\mu_d\} \subset A$ such that $\mu_d \to \mu$ weak-*; of course, $\mu_d * \mu_I = \mu_d$ for all d, and hence

$$(\chi, \mu_d) = (\chi, \mu_d * \mu_I) = (\chi, \mu_d)(\chi, \mu_I) \to (\chi, \mu)(\chi, \mu_I) = (\chi, \mu * \mu_I)$$

for all $\chi \in \hat{\Gamma}$; thus, $(\chi, \mu) = (\chi, \mu * \mu_I)$ for all $\chi \in \hat{\Gamma}$ and so $\mu = \mu * \mu_I$. Therefore, $\mu_I \in l_1(\Gamma)$ [12], and in fact μ_I is the Hewitt and Zuckerman identity for $l_1(\Gamma)$ constructed from the finite set of relative units of Γ [12], [10]. If $\|\mu_I\| = |||I||| = 1$, it must be that μ_I is a unit point mass concentrated at, say, $e \in \Gamma$. Thus, e is the identity for Γ and so A must have a weak bounded approximate identity of norm one by Corollary 3.2 of [11]. This completes the proof.

In Theorem 3.1 of [11] it is proved that A has a weak bounded approximate identity if and only if $M(\Gamma)$ has an identity. This means that $\mathscr{M}(A)$ will not have a representation as a subalgebra of $M(\Gamma)$ unless A has a weak bounded approximate identity as in Theorem 3.1.

If G is a locally compact abelian topological group, the structure semigroup corresponding to $L_1(G)$ under convolution product $*$ is $\bar{G}$, the Bohr compactification of G [20]. Since $L_1(G)$ is a semisimple convolution measure algebra having a bounded approximate identity of norm one, it follows from Theorem 3.1 that if $T \in \mathscr{M}(L_1(G))$, there exists a measure $\mu_T \in M(\bar{G})$ such that $|||T||| = \|\mu_T\|$ and $T(\alpha) = \mu_T * \alpha$ for all $\alpha \in L_1(G)$. In the next theorem we demonstrate that μ_T is supported on G; that is, $\mu_T \in M(G)$. In this way Theorem 3.1 can be viewed as a generalization of Wendel's result [22] which identifies $\mathscr{M}(L_1(G))$, isomorphically and isometrically, as $M(G)$.

Theorem 3.2 (Wendel). *If G is a locally compact abelian topological group, $\mathscr{M}(L_1(G))$ is isometrically isomorphic to $M(G)$.*

Proof. Let $T \in \mathscr{M}(L_1(G))$. By Theorem 3.1 there is a unique measure $\mu_T \in M(\bar{G})$ such that $|||T||| = \|\mu_T\|$ and $T(\alpha) = \mu_T * \alpha$ for all $\alpha \in L_1(G)$, where

$\overline{G}$ is the Bohr compactification of G. To see that μ_T is concentrated on G, observe first that $\hat{\mu}_T$ is a continuous function on $\hat{G}$, i.e., it is $w(\hat{G}, L_1(G))$ continuous. Next note that since G is dense in $\overline{G}$ each $\chi \in \hat{G}$ extends to a character on $\overline{G}$; in like manner, if $f(x) = \sum_{i=1}^{n} c_i \chi_i(x)$, $\chi_i \in \hat{G}$, $c_i \in C$ for $i = 1, 2, \ldots, n$, is a trigonometric polynomial on G, f can be extended to a trigonometric polynomial on $\overline{G}$ without increase in norm and

$$\begin{aligned}\left|\sum_{i=1}^{n} c_i \hat{\mu}_T(\chi_i)\right| &= \left|\sum_{i=1}^{n} c_i (\mu_T, \chi_i)\right| \\ &= \left|\sum_{i=1}^{n} c_i \int_{\overline{G}} \chi_i(\overline{x})\, d\mu_T(\overline{x})\right| \\ &= \left|\int_{\overline{G}} f(\overline{x})\, d\mu_T(\overline{x})\right| \leq \|\mu_T\|\, \|f\|_\infty .\end{aligned}$$

Now, following Rudin's proof of Eberlein's theorem [18, p. 32], there is a measure $\mu \in M(\overline{G})$ such that

$$\hat{\mu}_T(\chi) = \int_{\overline{G}} \chi(\overline{x})\, d\mu(\overline{x}), \qquad \chi \in \hat{G},$$

and in fact μ is actually concentrated on G. $\hat{\mu}_T(\chi) = \hat{\mu}(\chi)$ for all $\chi \in \hat{G}$ implies $\mu_T \equiv \mu$, and hence μ_T is concentrated on G.

Conversely, it is well known that all elements of $M(G)$ are multipliers of $L_1(G)$. This completes the proof.

Theorem 3.1 gives a characterization of $\mathscr{M}(M(S))$ when $M(S)$ has a weak bounded approximate identity. Using the approach of Rennison [16] in applying Arens product [1] techniques to the task of describing the structure semigroup Γ corresponding to $(M(S), *)$, it is possible to better identify those measures in $M(\Gamma)$ that give rise to multipliers of $M(S)$. Whenever no ambiguity arises, we denote the uniformly closed subspace of $M(S)^*$ generated by $\Delta(M(S))$ by Λ, regardless of the particular semigroup S.

Definition 3.3. Let Γ be the structure semigroup of $(M(S), *)$. For each $\mu \in M(\Gamma)$, define $m_\mu \colon \Lambda \to \Lambda$ by defining m_μ on the linear span of $\Delta(M(S))$ as

$$m_\mu\left(\sum_{i=1}^{n} c_i \chi_i\right) = \sum_{i=1}^{n} c_i (\chi_i, \mu) \chi_i,$$

$$\chi_i \in \Delta(M(S)),\ c_i \in C \text{ for } i = 1, 2, \ldots, n.$$

That is, $m_\mu(f)$ is the Arens product of $\mu \in M(\Gamma) = \Lambda^*$ and $f \in \Lambda$, and $\|m_\mu\| \leq \|\mu\|$.

We use the standard notation that if $T\colon B \to B$ is a function defined on a linear vector space B, then the adjoint $T^*\colon B^* \to B^*$ is defined by $T^*(f) = f \circ T$.

Lemma 3.4. *Let $\mu, \nu \in M(\Gamma)$. Then $(m_\mu(f), \nu) = (f, \mu * \nu)$ for all $f \in \Lambda$. Hence, $m_\mu^*(\nu) = \mu * \nu$ for all $\nu \in M(\Gamma)$.*

Proof. For each $\chi \in \Delta(M(S))$,

$$(\chi, m_\mu^*(\nu)) = (m_\mu(\chi), \nu) = ((\chi, \mu)\chi, \nu) = (\chi, \mu)(\chi, \nu) = (\chi, \mu * \nu).$$

Hence, $(f, m_\mu^*(\nu)) = (f, \mu * \nu)$ for all $f \in \Lambda$ since the linear span of $\Delta(M(S))$ is dense in Λ. Thus, $m_\mu^*(\nu) = \mu * \nu$ for all $\nu \in M(\Gamma)$. This completes the proof.

If $M(S)$ has a weak bounded approximate identity and $T \in \mathscr{M}(M(S))$, the following calculations show that $T^*|_\Lambda = m_{\mu_T}$: if $\chi \in \Delta(M(S))$,

$$\begin{aligned}(T^*(\chi), \alpha) &= (\chi, T(\alpha)) = T(\alpha)^\wedge(\chi) \\ &= (\mu_T * \alpha)^\wedge(\chi) = \hat{\mu}_T(\chi)\hat{\alpha}(\chi) \\ &= (\hat{\mu}_T(\chi)\chi, \alpha) = (m_{\mu_T}(\chi), \alpha) \quad \text{for all } \alpha \in M(S);\end{aligned}$$

thus, $T^*(\chi) = m_{\mu_T}(\chi)$ for all $\chi \in \Delta(M(S))$. Also, T^* (and hence m_{μ_T}) is a continuous function on Λ in the $w(\Lambda, M(S))$ topology [19, Lemma 5.10]. We wish to investigate the converse question: that is, if $\mu \in M(\Gamma)$ is such that m_μ is continuous on Λ in the $w(\Lambda, M(S))$ topology, does μ determine a multiplier of $M(S)$? Indeed, we show that if $\mu \in M(\Gamma) = \Lambda^*$ is in addition continuous on Λ in the $w(\Lambda, M(S))$ topology, μ determines a multiplier of $M(S)$. In what immediately follows, we remove the restriction that $M(S)$ has a weak bounded approximate identity and assume only that $M(S)$ is semisimple. First, we proceed to establish some preliminary lemmas.

Recall that if B is a linear vector space, E is a total subspace of B^* if $g(x) = 0$ for all $g \in E$ implies that $x = 0$, $x \in B$.

Lemma 3.5. *If E is a weak-* dense subspace of B^*, then E is a total subspace of B^*.*

Proof. Suppose $x \in B$ and $(x, g) = 0$ for all $g \in E$. If $h \in B^*$, there exists a net $\{h_\rho\} \subset E$ such that $h_\rho \to h$ weak-*, that is $(y, h_\rho) \to (y, h)$ for all $y \in B$. Thus, $0 = (x, h_\rho) \to (x, h)$ implies that $(x, h) = 0$ for all $h \in B^*$. Therefore, $x \equiv 0$. This completes the proof.

Lemma 3.5 thus implies that $p(M(S))$ is a total subspace of Λ^*.

Lemma 3.6. *If $\mu \in M(\Gamma)$ is $w(\Lambda, M(S))$ continuous on Λ, then m_μ is also $w(\Lambda, M(S))$ continuous on Λ.*

Proof. Suppose $f \in \Lambda$, $\{f_\rho\} \subset \Lambda$ and $f_\rho \to f$ in the $w(\Lambda, M(S))$ topology. If $\nu \in M(S)$ is fixed, for all $\beta \in M(S)$ we have

$$(m_\nu(f_\rho), \beta) = (f_\rho, \nu * \beta) \to (f, \nu * \beta) = (m_\nu(f), \beta),$$

and hence $m_\nu(f_\rho) \to m_\nu(f)$ in the $w(\Lambda, M(S))$ topology for each $\nu \in M(S)$. The $w(\Lambda, M(S))$ continuity of μ on Λ implies that $(m_\mu(f_\rho), \nu) = (m_\nu(f_\rho), \mu) \to (m_\nu(f), \mu) = (m_\mu(f), \nu)$ for all $\nu \in M(S)$. Thus, $m_\mu(f_\rho) \to m_\mu(f)$ in the $w(\Lambda, M(S))$ topology on Λ.

Theorem 3.7. *If $\mu \in M(\Gamma)$ is such that m_μ is $w(\Lambda, M(S))$ continuous on Λ, then μ determines a multiplier of $M(S)$.*

Proof. For each $\beta \in M(S)$, the function $f \mapsto (m_\mu(f), \beta), f \in \Lambda$, is continuous in the $w(\Lambda, M(S))$ topology on Λ: $f_\rho \to f$ implies $m_\mu(f_\rho) \to m_\mu(f)$ implies $(m_\mu(f_\rho), \beta) \to (m_\mu(f), \beta)$ for all $\beta \in M(S)$, $\{f_\rho\} \subset \Lambda$, $f \in \Lambda$. By Lemma 3.5, $p(M(S))$ is a total subspace of Λ^*, so there exists an element $K\beta \in M(S)$ such that $(m_\mu(f), \beta) = (f, p(K\beta)) = f(K\beta)$ for all $f \in \Lambda$ [7, V, 3.9]. $p(K\beta)$ is linear on Λ and

$$\|K\beta\| = \|p(K\beta)\| = \sup_{f \in \Lambda; \|f\|_\infty \leq 1} |(f, p(K\beta))|$$
$$= \sup_{f \in \Lambda; \|f\|_\infty \leq 1} |(m_\mu(f), \beta)| \leq \|\mu\| \, \|\beta\|.$$

Now, K is a linear transformation on $M(S)$ and $|||K||| \leq \|\mu\|$. Thus, K is a bounded linear transformation on $M(S)$ with $m_\mu = K^* |_\Lambda$ since $(m_\mu(f), \beta) = f(K\beta)$ for all $\beta \in M(S)$ and for all $f \in \Lambda$. Moreover, if $\beta \in M(S)$,

$$\begin{aligned}
(\chi, p(K\beta)) &= (\chi, j(K\beta)) = (\chi, K^{**}j(\beta)) \\
&= (\chi, j(\beta) \circ K^*) = (m_\mu(\chi), j(\beta)) \\
&= (m_\mu(\chi), p(\beta)) = ((\chi, \mu)\chi, p(\beta)) \\
&= (\chi, \mu)(\chi, p(\beta)) = (\chi, \mu * p(\beta)) \quad \text{for all } \chi \in \Delta(M(S)).
\end{aligned}$$

Hence, for each $\beta \in M(S)$, $p(K\beta)$ and $\mu * p(\beta)$ are the same measure in $M(\Gamma)$, and so $\mu * p(\beta) \in p(M(S))$ for all $\beta \in M(S)$. Thus, if we now identify $M(S)$ with $p(M(S))$, $T_\mu(\beta) = \mu * \beta$, $\beta \in M(S)$, defines a multiplier of $M(S)$. This completes the proof.

We are now in a position to characterize those measures in $M(\Gamma)$ that give rise to multipliers of $M(S)$ when $M(S)$ has a weak bounded approximate identity.

Corollary 3.8. (a) *If $\mu \in M(\Gamma)$ is such that μ is $w(\Lambda, M(S))$ continuous on Λ, then μ determines a multiplier of $M(S)$.*

(b) *Suppose $M(S)$ has a weak bounded approximate identity. Then T is a multiplier of $M(S)$ if and only if there exists a unique measure $\mu_T \in M(\Gamma)$ such that m_{μ_T} is $w(\Lambda, M(S))$ continuous on Λ. Moreover, T and μ_T satisfy $T(\beta) = \mu_T * \beta$ for all $\beta \in M(S)$.*

Proof. (a) Lemma 3.6 and Theorem 3.7.

(b) The remarks prior to Lemma 3.5 show that if $T \in \mathcal{M}(M(S))$, then m_{μ_T} is $w(\Lambda, M(S))$ continuous on Λ, where μ_T is the unique measure in $M(\Gamma)$ that corresponds to T [Theorem 3.1]. On the other hand, if $\mu \in M(\Gamma)$ is such that m_μ is $w(\Lambda, M(S))$ continuous on Λ, Theorem 3.7 establishes that μ determines a multiplier of $M(S)$.

Corollary 3.9. *Assume that $M(S)$ has an identity. Suppose g is a complex valued $w(\Delta(M(S)), M(S))$ continuous function on $\Delta(M(S))$ that has a linear $w(\Lambda, M(S))$ continuous extension to Λ. Then $g = \hat{\mu}$ for some $\mu \in M(S)$ if and only if there exists a constant R such that $|\sum_{i=1}^{n} c_i g(\chi_i)| \leq R\|f\|_\infty$ for all $f = \sum_{i=1}^{n} c_i \chi_i$, $\chi_i \in \Delta(M(S))$, $c_i \in C$, $i = 1, 2, \ldots, n$.*

Proof. If $\hat{\mu} \in M(S)^\wedge$, the inequality holds, where $R = \|\mu\|$; clearly, $\hat{\mu}$ has a linear $w(\Lambda, M(S))$ continuous extension to Λ. On the other hand, if g has a linear extension $\bar{g}$ to Λ and satisfies the stated inequality, $\bar{g} \in \Lambda^*$. The fact that $\bar{g}$ is $w(\Lambda, M(S))$ continuous on Λ implies that $\bar{g} \in \mathcal{M}(M(S))$ by Corollary 3.8(a), and since $M(S)$ has an identity, $\bar{g} = \hat{\mu}$ for some $\mu \in M(S)$. This completes the proof.

If $T \in \mathcal{M}(M(S))$, then $\hat{T}$ (and hence μ_T, if $M(S)$ has a weak bounded approximate identity) is $w(\Delta(M(S)), M(S))$ continuous on $\Delta(M(S))$ since $\Delta(M(S))$ is homeomorphic to an open subset of $\Delta(\mathcal{M}(M(S)))$ [2] or [21]. Corollary 3.8(a) gives a partial converse to this result; however, we require that $\mu \in M(\Gamma)$ be $w(\Lambda, M(S))$ continuous on Λ and not simply on $\Delta(M(S)) \subset \Lambda$ in order that μ determine a multiplier of $M(S)$. From this viewpoint there is some similarity to Eberlein's characterization of bounded continuous functions on the dual of a locally compact abelian group G that are transforms of measures in $M(G)$ [8]. If G is a locally compact abelian topological group, the Bohr compactification $\overline{G}$ of G is the structure semigroup for $L_1(G)$, and $\mathcal{M}(L_1(G))$ is isometrically isomorphic to $M(G)$ by Theorem 3.2. Now if $\mu \in M(G)$, $\mu \in M(\overline{G})$ and μ(or $\hat{\mu}$) is $w(\hat{G}, L_1(G))$ continuous on $\hat{G}$; likewise, any measure $\bar{\mu} \in M(\overline{G})$ that is $w(\hat{G}, L_1(G))$ continuous on $\hat{G}$ is actually concentrated on G and hence belongs to $M(G)$ [18]. In keeping with this train of thought, Corollary 3.9 characterizes those $w(\Delta(M(S)), M(S))$ continuous functions on the maximal ideal space of $M(S)$ that have $w(\Lambda, M(S))$ continuous linear extensions to Λ, in the event that $M(S)$ has an identity.

It should be pointed out that Rennison's approach [16] in arriving at the structure semigroup Γ of $M(S)$ makes it clear that $M(\Gamma)$ is the Birtel extension [3] of $M(S)$. Birtel discusses in [3] the multiplier algebra of a Banach algebra B and states several results in which he assumes the topological embedding of either B or $\mathcal{M}(B)$ into what we are now referring to as the Birtel extension of B. Relating the work of Taylor [20] to that of Rennison [16], it turns out that the embedding of $M(S)$ into its Birtel extension $M(\Gamma)$ is an isometry if and only if $M(S)$ is semisimple [20]. Our main contribution with regard to Birtel's work is that Theorem 3.1 with $A = M(S)$ establishes the topological embedding $T \mapsto \mu_T$ of $\mathcal{M}(M(S))$ into $M(\Gamma)$ with $|||T||| \leq \|\mu_T\| \leq R|||T|||$; moreover, $T \mapsto \mu_T$ is an isometry if and only if the weak bounded approximate identity is of norm one.

For sake of completeness we note that Corollary 3.8(b) is analogous to Theorem 2 of [14], but the conditions are different.

4. Multipliers of $l_1(S)$. Assume that S is a commutative discrete semigroup. The semisimplicity of $l_1(S)$ is equivalent to the algebraic condition on S that $x^2 = y^2 = xy$ implies $x = y$, $x, y \in S$. If S satisfies this condition, we follow Petrich [15] in saying that S is separative. A function $\sigma: S \to S$ having the property that $\sigma(xy) = x\sigma(y)$ for all $x, y \in S$ is called a multiplier of S. The set of all multipliers of S is denoted $\Omega(S)$; under the operation $\circ$ of composition of functions, $\Omega(S)$ is a semigroup with identity $\bar{e}$ (though not necessarily commutative). Further, define S to be reductive if $yx = zx$ for all $x \in S$ implies $y = z$. There is a natural mapping $x \mapsto \sigma_x$ of S into $\Omega(S)$, where $\sigma_x(y) = xy$ for all $y \in S$. The natural mapping is one-to-one if and only if S is reductive and onto if and only if S has an identity. The set $\{\sigma_x : x \in S\} \subseteq \Omega(S)$ forms an ideal in $\Omega(S)$ and $\sigma_x \circ \sigma = \sigma \circ \sigma_x = \sigma_{\sigma(x)}$ for all $\sigma \in \Omega(S)$, $x \in S$. For more details on these and other results on semigroups, consult the survey article [15].

The next proposition summarizes some of the relationships between S and $\Omega(S)$ when given various conditions on S. Some of the statements in the proposition can be found in [15]; in any case, in the presence of commutativity, all are readily proved.

Proposition 4.1. (a) *If S has a set of relative units, then S is reductive.*

(b) *If S is reductive, then $\Omega(S)$ is commutative.*

(c) *If S is separative, then S is reductive and $\Omega(S)$ is separative.*

(d) *If S is cancellative, then $\Omega(S)$ is cancellative, and in addition, S and $\Omega(S)$ have isomorphic quotient groups.*

(e) *If S is idempotent, then $\Omega(S)$ is idempotent.*

(f) *If S is a union of groups, then $\Omega(S)$ is a union of groups.*

Thus, we have conditions on S that imply the commutativity of $\Omega(S)$, and we are able to relate the semisimplicity of $l_1(S)$ to that of $l_1(\Omega(S))$. Throughout the remainder of this section, let us assume that $\Omega(S)$ is a commutative semigroup. If we are interested in describing $\mathscr{M}(l_1(S))$, then elements of $l_1(\Omega(S))$ determine a class of multipliers of $l_1(S)$.

Proposition 4.2. *There is a norm-decreasing homomorphism $\tau \mapsto T_\tau$ of $l_1(\Omega(S))$ into $\mathscr{M}(l_1(S))$.*

Proof. Let $\tau = \sum_{\sigma\in\Omega(S)} \tau(\sigma)\delta_\sigma \in l_1(\Omega(S))$ and let P be the linear subspace of $l_1(S)$ spanned by the point masses $\{\delta_z\}_{z\in S}$. Define $T_\tau: P \to l_1(S)$ by $T_\tau(\delta_z) = \sum_{\sigma\in\Omega(S)} \tau(\sigma)\delta_{\sigma(z)}$, $z \in S$, and extend linearly. $\|T_\tau(\delta_z)\| \leq \sum_{\sigma\in\Omega(S)} |\tau(\sigma)| = \|\tau\|$ for all $z \in S$; if $\alpha = \sum_{1\leq i\leq n} \alpha(z_i)\delta_{z_i} \in P$, then

$$\|T_\tau(\alpha)\| \leq \sum_{1\leq i\leq n} |\alpha(z_i)| \, \|T_\tau(\delta_{z_i})\| \leq \|\tau\| \, \|\alpha\|.$$

Thus, for each $\tau \in l_1(\Omega(S))$, T_τ is a bounded operator from P into $l_1(S)$. The fact that P is dense in $l_1(S)$ allows us to extend T_τ to be a bounded linear operator on $l_1(S)$, and indeed $|||T_\tau||| \leq \|\tau\|$ (no confusion arises if we not rename T_τ). It is a routine verification to show that $T_\tau \in \mathscr{M}(l_1(S))$ for each $\tau \in l_1(\Omega(S))$. Now if S is reductive, S is identified with a subsemigroup of $\Omega(S)$, and T_τ is simply convolution multiplication by $\tau \in l_1(\Omega(S))$.

Proposition 4.3. *The natural homomorphism $g \mapsto \sigma_g$ of S into $\Omega(S)$ induces a homomorphism of $l_1(S)$ into $l_1(\Omega(S))$ which is* (a) *one-to-one if and only if $x \mapsto \sigma_x$ is one-to-one and* (b) *onto if and only if $x \mapsto \sigma_x$ is onto.*

Proof. (a) If $\alpha = \sum_{x \in S} \alpha(x)\delta_x \in l_1(S)$, the induced homomorphism of $l_1(S)$ into $l_1(\Omega(S))$ maps $\sum_{x\in S} \alpha(x)\delta_x \to \sum_{x \in S} \alpha(x)\delta_{\sigma_x}$. This map is one-to-one if and only if the set $\{\delta_{\sigma_x}\}$ is linearly independent, which is true if and only if $x \mapsto \sigma_x$ is one-to-one.

(b) If $x \mapsto \sigma_x$ is onto and $\sigma \in \Omega(S)$, then there exists $x \in S$ such that $\sigma = \sigma_x$, or in other words $\delta_\sigma = \delta_{\sigma_x}$. Conversely, if the homomorphism of $l_1(S)$ into $l_1(\Omega(S))$ is onto, then given $\sigma \in \Omega(S)$, there exists $\alpha \in l_1(S)$ such that $\delta_\sigma = \sum_{x \in S} \alpha(x)\delta_{\sigma_x}$. If $\sigma \neq \sigma_x$ for all $x \in S$, then $0 = \delta_\sigma(\sigma_x) = \alpha(x)$ for all $x \in S$ implies that $\delta_\sigma \equiv 0$. Thus, there exists $x \in S$ such that $\sigma = \sigma_x$ and $\sigma \mapsto \sigma_x$ is onto. This completes the proof.

Since the homomorphic image $\mathscr{I}$ of $l_1(S)$ in $l_1(\Omega(S))$ is an ideal in $l_1(\Omega(S))$, it is true that $\Delta(l_1(\Omega(S)))\backslash h(\mathscr{I})$ is homeomorphic to $\Delta(\mathscr{I})$ by Theorem 3.1.18 of [17], where $h(\mathscr{I})$ is the hull of $\mathscr{I}$. For instance, if S is separative, then we can consider S to be a subset of $\Omega(S)$; $\Omega(S)$ is commutative and separative. Moreover, $\Omega(S)^\wedge$ is homeomorphic to $\hat{S}$ in union with the hull of $l_1(S)$.

Proposition 4.4. *Suppose S is a semigroup having the following property: Given $\sigma \in \Omega(S)$ there exists $x_\sigma \in S$ such that $\sigma(x_\sigma) = \sigma'(x_\sigma)$ implies $\sigma \equiv \sigma'$, where $\sigma' \in \Omega(S)$. Then $\tau \mapsto T_\tau$ mapping $l_1(\Omega(S))$ into $\mathscr{M}(l_1(S))$ is one-to-one.*

Proof. Suppose $\tau = \sum_{\sigma \in \Omega(S)} \tau(\sigma)\delta_\sigma$ and $T_\tau(\delta_x) = 0$ for all $x \in S$, $\tau \neq 0$. Choose $\sigma \in \Omega(S)$ such that $\tau(\sigma) \neq 0$. Then there exists $x_\sigma \in S$ such that $\sigma(x_\sigma) = \sigma'(x_\sigma)$ implies $\sigma \equiv \sigma'$. Therefore,

$$0 = T_\tau(\delta_{x_\sigma}) = \tau(\sigma)\delta_{\sigma(x_\sigma)} + \sum_{\sigma' \in \Omega(S); \sigma' \neq \sigma} \tau(\sigma')\delta_{\sigma'(x_\sigma)}$$

implies that $\tau(\sigma) = 0$. Thus, $\tau \mapsto T_\tau$ is a one-to-one map.

Corollary 4.5. *If S is a cancellative semigroup, then $\tau \mapsto T_\tau$ is one-to-one.*

Proof. We verify the conditions of Proposition 4.4. For a given $\sigma \in \Omega(S)$ take x_σ to be any element in S. Since $\Omega(S)$ is cancellative, if $\sigma \circ \sigma_x = \sigma' \circ \sigma_x$, then it follows that $\sigma \equiv \sigma'$. This completes the proof.

If S is a semigroup with cancellation, then Proposition 4.1(d) establishes the existence of an isomorphic embedding of S and $\Omega(S)$ in a group G which has the property that $\Omega(S) = \{\sigma \in G: \sigma S \subset S\}$. We use this embedding and modify the arguments of Davis [6] in proving the next result.

Theorem 4.6. *If S is a semigroup with cancellation, then $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$.*

Proof. $l_1(S)$ and $l_1(\Omega(S))$ are the subalgebras of $l_1(G)$ whose elements are supported on S and $\Omega(S)$ respectively. Since S is cancellative, the homomorphism $\tau \mapsto T_\tau$ of $l_1(\Omega(S))$ into $\mathcal{M}(l_1(S))$ is one-to-one by Corollary 4.5. We propose to show that it is onto. Let $T \in \mathcal{M}(l_1(S))$ and let $\alpha \in l_1(S)$. If $x \in S$, then $T(\alpha) * \delta_x = \alpha * T(\delta_x)$; thus, $T(\alpha) = \alpha * T(\delta_x) * \delta_{x^{-1}}$ for all $\alpha \in l_1(S)$. The interesting fact is that $T(\delta_x) * \delta_{x^{-1}} \in l_1(\Omega(S))$. For suppose there exists $y \in G, z \in S$ such that $yz \notin S$ and $[T(\delta_x) * \delta_{x^{-1}}](y) \neq 0$. Then $[\delta_z * T(\delta_x) * \delta_{x^{-1}}](yz) = [T(\delta_x) * \delta_{x^{-1}}](y) \neq 0$ and hence $\delta_z * T(\delta_x) * \delta_{x^{-1}} \notin l_1(S)$. This contradicts the fact that $\alpha * T(\delta_x) * \delta_{x^{-1}} \in l_1(S)$ for all $\alpha \in l_1(S)$. Thus, given $T \in \mathcal{M}(l_1(S))$ there exists $\tau \in l_1(\Omega(S))$ such that $T(\alpha) = \alpha * \tau = T_\tau(\alpha)$ for all $\alpha \in l_1(S)$ and hence $\tau \mapsto T_\tau$ is an onto map. From Proposition 4.2 we know that $|||T_\tau||| \leq \|\tau\|$ for all $\tau \in l_1(\Omega(S))$. But

$$|||T_\tau||| = \sup_{\alpha \in l_1(S); \|\alpha\| = 1} \|\tau * \alpha\| > \|\tau * \delta_x\| = \|\tau\|,$$

and hence $|||T_\tau||| = \|\tau\|$. Therefore, we have proved that $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$. This completes the proof.

For the remainder of this section we assume that $l_1(S)$ is semisimple, and that $l_1(S)$ has a weak bounded approximate identity of norm one. This is equivalent to the existence of an identity in the structure semigroup Γ of $(l_1(S), *)$ by Corollary 3.2 of [11]. In Proposition 4.1 of [11] S is realized as a dense subsemigroup of Γ by the isomorphism i_s: $S \to \Gamma$. Let us consider the question of embedding $\Omega(S)$ in Γ. If Γ does not have an identity, then clearly there can be no isomorphic embedding of $\Omega(S)$ in Γ which extends the natural map i_s of S into Γ, since an identity for $i_s(S)$ is an identity for Γ. As the next theorem shows, the presence of an identity in Γ is sufficient to guarantee the embedding of $\Omega(S)$ in Γ.

Theorem 4.7. *If S is separative, the natural isomorphism i_s of S into Γ has an extension to an isomorphism $i_{\Omega(S)}$ of $\Omega(S)$ into Γ if and only if Γ has an identity.*

Proof. If $i_{\Omega(S)}$ is an isomorphism of $\Omega(S)$ into Γ and extends the natural isomorphism i_s, Γ has an identity since $i_s(S)$ is $w(\Gamma, C(\Gamma))$-dense in Γ [11, Proposition 4.1] and $\Omega(S)$ has identity $\bar{e}$.

On the other hand, assume Γ has an identity e, and let $i_s(x) = \tilde{x}$ for each $x \in S$. By Proposition 4.1 of [11], there is a net $\{\tilde{u}_d\} \subset i_s(S)$ such that $\tilde{u}_d \to e$. Fix $\sigma \in \Omega(S)$ and consider $\{\sigma(u_d)^{\sim}\} \subset \Gamma$. Since Γ is compact, there exists a subnet $\{\sigma(u_{d'})^{\sim}\}$ and $\tilde{\sigma} \in \Gamma$ such that $\sigma(u_{d'})^{\sim} \to \tilde{\sigma}$; thus, $\tilde{\sigma} \cdot \tilde{x} = \lim \tilde{x}\sigma(u_{d'})^{\sim} = \lim \sigma(x)^{\sim} \tilde{u}_{d'} = \sigma(x)^{\sim}$ for all $x \in S$. Now, let $\{\tilde{x}_\rho\} \subset i_s(S)$ be any net such that $\tilde{x}_\rho \to e$ and suppose that in accordance with the compactness of Γ there is a subnet $\sigma(x_{\rho'})^{\sim} \to \tilde{\tau} \in \Gamma$; then $\tilde{\sigma} \cdot \tilde{x} = \sigma(x)^{\sim} = \tilde{\tau} \cdot \tilde{x}$ for all $x \in S$ implies that for each $\chi \in \hat{\Gamma}$, $\chi(\tilde{\sigma}) = \chi(\tilde{\tau})$. Indeed, given $\chi \in \hat{\Gamma}$, choose $z \in S$ such that $\chi(\tilde{z}) \neq 0$, in which case we have that

$$\chi(\tilde{\sigma})\chi(\tilde{z}) = \chi(\tilde{\sigma}\cdot\tilde{z}) = \chi(\tilde{\tau}\cdot\tilde{z}) = \chi(\tilde{\tau})\chi(\tilde{z});$$

since $\hat{\Gamma}$ separates points of Γ [20], it must be that $\tilde{\sigma} = \tilde{\tau}$. Thus, multiplication by $\tilde{\sigma} \in \Gamma$ determines an element of $\Omega(S)$. We may now define $i_{\Omega(S)}: \Omega(S) \to \Gamma$ by $i_{\Omega(S)}(\sigma) = \tilde{\sigma}$ for each $\sigma \in \Omega(S)$. $i_{\Omega(S)}$ is one-to-one because $i_{\Omega(S)}(\sigma) = \tilde{\sigma} = i_{\Omega(S)}(\tau)$, $\sigma, \tau \in \Omega(S)$ implies that $\sigma(x)^{\sim} = \tilde{\sigma}\cdot\tilde{x} = \tau(x)^{\sim}$ for all $x \in S$ and so $\sigma \equiv \tau$. This completes the proof.

Now that we have an embedding of $\Omega(S)$ in Γ, $l_1(\Omega(S))$ can be identified with a subalgebra of $l_1(\Gamma) \subseteq M(\Gamma)$. Note that this means that $\hat{\tau}(\chi) = 0$ for all $\chi \in \hat{S}$ implies $\tau \equiv 0$, $\tau \in l_1(\Omega(S))$. Also, Theorem 3.1 implies that the embedding $T \mapsto \mu_T$ of $\mathcal{M}(l_1(S))$ in $M(\Gamma)$ is an isometry, in which case we have $|||T_\tau||| = \|\tau\|$ for all $\tau \in l_1(\Omega(S))$. Thus, $\{T_\tau : \tau \in l_1(\Omega(S))\}$ is an operator-norm closed subalgebra of $\mathcal{M}(l_1(S))$. This leads to the conjecture that $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$. We propose to use the representation theory for multipliers of semisimple convolution measure algebras with weak bounded approximate identities developed in §3 in order to show that this is true. The measure $\mu \in M(\Gamma)$ is a multiplier of $l_1(S)$ if and only if m_μ is $w(\Lambda, l_1(S))$ continuous on Λ by Corollary 3.8(b). Our aim is to show that if $\mu \in M(\Gamma)$ is such that m_μ is $w(\Lambda, l_1(S))$ continuous on Λ, then indeed $\mu \in l_1(\Omega(S))$.

The fact that $\Omega(S)$ can be considered a subset of Γ affords us a new way of relating the semicharacters of S to their extensions in $\Omega(S)^{\wedge}$. Each $\chi \in \hat{S}$ has a unique extension $\tilde{\chi} \in \Omega(S)^{\wedge}$. Now Λ is isometrically isomorphic to $C(\Gamma)$ with the mapping denoted $g \mapsto g'$; therefore, it must be that for each $\chi \in \hat{S}$, $\chi' |_{\Omega(S)} = \tilde{\chi}$. Also, there is a one-to-one correspondence between the linear span of $\hat{S}$ in $l_1(S)^* = l_\infty(S)$ and the linear span L of $\{\tilde{\chi} \in \Omega(S)^{\wedge} : \chi \in \hat{S}\} \subset l_1(\Omega(S))^* = l_\infty(\Omega(S))$ by mapping

$$\sum_{i=1}^{n} c_i\chi_i \mapsto \sum_{i=1}^{n} c_i\chi_i' |_{\Omega(S)} .$$

The next proposition relates Λ to the closure $\bar{L}$ of L in $l_\infty(\Omega(S))$.

Proposition 4.8. *There is a one-to-one correspondence between Λ and the uniform closure of L in $l_1(\Omega(S))^*$.*

Proof. We have the following situation: $S \subset \Omega(S) \subset \Gamma$. Now if a sequence $\{\tilde{f}_n\} \subset L$ converges to $\tilde{f} \in \bar{L}$ uniformly on $\Omega(S)$, then certainly the sequence $\{\tilde{f}_n |_S\}$ converges to $\tilde{f} |_S \in \Lambda$ uniformly on S. The interesting point to recognize is that since uniform convergence on S and on Γ are equivalent, then uniform convergence of a sequence $\{f_n\}$ of elements derived from the linear span of $\hat{S}$ implies uniform convergence in $l_\infty(\Omega(S))$ of the sequence whose elements are extensions of f_n to $\tilde{f}_n \in L$ for all n.

Theorem 4.9. *Suppose $l_1(S)$ has a weak bounded approximate identity of norm one. If $\mu \in M(\Gamma) = \Lambda^*$ is such that m_μ is $w(\Lambda, l_1(S))$ continuous on Λ, then* *convolution multiplication by μ determines a multiplier of $l_1(\Omega(S))$. Hence, $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$.*

Proof. We may extend the isometry $p\colon l_1(S) \to M(\Gamma)$ in a natural way to $\bar{p}\colon l_1(\Omega(S)) \to M(\Gamma)$. The mapping $\bar{p}$ is induced by the embedding of $\Omega(S)$ in Γ, and indeed, $\bar{p}$ is an isometry. In what follows we may identify Λ with a closed subset of $l_1(\Omega(S))^*$ by Proposition 4.8; we also recognize that $\Lambda = C(\Gamma)$. Viewing Λ as a subset of $l_1(\Omega(S))^*$, note that $(f, \bar{p}(\tau)) = f(\tau)$ for all $\tau \in l_1(\Omega(S))$, $f \in \Lambda$. Let $\mu \in M(\Gamma)$ be such that m_μ is $w(\Lambda, l_1(S))$ continuous on Λ. Corollary 3.8(b) implies that μ determines a multiplier of $l_1(S)$. Now, m_μ is $w(\Lambda, l_1(\Omega(S)))$ continuous on Λ. Thus, for each $\tau \in l_1(\Omega(S))$, the function $f \mapsto (m_\mu(f), \tau)$, $f \in \Lambda$, is continuous on Λ in the $w(\Lambda, l_1(\Omega(S)))$ topology, and is also linear. Since $p(l_1(S)) \subset \bar{p}(l_1(\Omega(S)))$, by Lemma 3.5, $\bar{p}(l_1(\Omega(S)))$ is a total subspace of Λ^*; so there exists $K\tau \in l_1(\Omega(S))$ such that $(m_\mu(f), \tau) = (f, \bar{p}(K\tau)) = f(K\tau)$ for all $f \in \Lambda$ [7, V, 3.9]. Now $\bar{p}(K\tau)$ is linear on Λ and

$$
\begin{aligned}
\|K\tau\| = \|\bar{p}(K\tau)\| &= \sup_{f \in \Lambda; \|f\| \le 1} |(f, \bar{p}(K\tau))| \\
&= \sup_{f \in \Lambda; \|f\| \le 1} |(m_\mu(f), \tau)| \le \|\mu\|\,\|\tau\|.
\end{aligned}
$$

Also, K is a linear transformation on $l_1(\Omega(S))$ and $|||K||| \le \|\mu\|$. Since $(m_\mu(f), \tau) = f(K\tau)$ for all $\tau \in l_1(\Omega(S))$ and for all $f \in \Lambda$, $m_\mu = K^* |_\Lambda$. Moreover, if $\tau \in l_1(\Omega(S))$,

$$
\begin{aligned}
(\chi, \bar{p}(K\tau)) &= (\chi, j(K\tau)) = (\chi, K^{**}j(\tau)) \\
&= (\chi, j(\tau) \circ K^*) = (m_\mu(\chi), j(\tau)) \\
&= (m_\mu(\chi), \bar{p}(\tau)) = ((\chi, \mu)\chi, \bar{p}(\tau)) \\
&= (\chi, \mu)(\chi, \bar{p}(\tau)) = (\chi, \mu * \bar{p}(\tau))
\end{aligned}
$$

for all $\chi \in \hat{S}$. Hence, for each $\tau \in l_1(\Omega(S))$, $\bar{p}(K\tau)$ and $\mu * \bar{p}(\tau)$ are the same measure in $M(\Gamma)$. Therefore, $T_\mu(\tau) = \mu * \tau$, for all $\tau \in l_1(\Omega(S))$, defines a multiplier of $l_1(\Omega(S))$, where we have identified $l_1(\Omega(S))$ with $\bar{p}(l_1(\Omega(S)))$. Because $l_1(\Omega(S))$ has an identity, then $\mu \in l_1(\Omega(S))$. Finally, we have proved that every multiplier μ of $l_1(S)$ is actually an element of $l_1(\Omega(S))$. Thus, $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$. This completes the proof.

We conclude this paper with an example. At this point one might conjecture that every multiplier of $l_1(S)$ is of the form T_τ for some $\tau \in l_1(\Omega(S))$. However, this is not always true, as Example 4.10 shows.

Example 4.10. Let $S = \{0, 1, 2, \ldots, n, \ldots\}$ under the operation $n^2 = n$ for all n and $nm = 0$ for $n \ne m$. It can be shown that $\Omega(S) = \{F\colon F \text{ is a subset of } S\}$ under the semigroup operation of union of sets, where each $F \subset S$ gives rise to a multiplier of S by mapping

$$n \mapsto 0, \quad \text{if } n \in F \text{ or } n = 0,$$
$$n \mapsto n, \quad \text{if } n \notin F.$$

It is also a routine matter (as indicated in [10]) to verify that there is a one-to-one correspondence between the filters of $\{F\}_{F \subset S}$ and the semicharacters of $\Omega(S)$, each filter being a set where a semicharacter assumes the value 1. The semicharacters χ_n of S correspond to all the filters composed of singletons $\{n\}$, $n \neq 0$ (together with the function 1). We propose to determine Λ, and exhibit the structure semigroup Γ corresponding to $l_1(S)$. We also show that the operator norm of a multiplier T on $l_1(S)$ is equivalent to the supremum norm of the corresponding continuous function g_T on $\hat{S}$.

If we observe that $\sum_{i=0}^{n} c_i \chi_i$ corresponds to the sequence c_0, $c_0 + c_1$, $c_0 + c_2$, $\ldots$, $c_0 + c_n$, $c_0, \ldots, c_0, \ldots$, where $c_i \in C$, $i = 0, 1, 2, \ldots, n$, then it is fairly clear that the closed linear span of $\hat{S}$ in $l_\infty(S)$ is the space of all convergent sequences in $l_\infty(S)$.

Let Γ denote the semigroup S topologized in the following manner: $S\backslash\{0\}$ retains the discrete topology of S; neighborhoods of 0 consist of 0 in union with all but a finite number of points of S. Then Γ is compact, and an application of the Stone-Weierstrass theorem shows that $\Lambda = C(\Gamma)$. Note that no new points have been added to S in order to obtain Γ. Clearly, $\hat{S} = \hat{\Gamma}$. Finally, Γ is Taylor's structure semigroup associated with $l_1(S)$ since topologically the maximal ideal space of $\Lambda = C(\Gamma)$ is Γ, and there is only one semigroup product on Γ that makes the Gelfand transforms of elements of $\hat{S}$ into semicharacters on Γ [16].

Let us turn now to a discussion of $\mathscr{M}(l_1(S))$. If $T \in \mathscr{M}(l_1(S))$ and $j \in S$,

$$T(\delta_0) = T(\delta_0) * \delta_j = \delta_0 * T(\delta_j) = t_{00}\delta_0$$

for some $t_{00} \in C$. Likewise, $T(\delta_j) = \delta_j * T(\delta_j)$ implies there exists $t_{jj} \in C$ such that

$$T(\delta_j) = (t_{00} - t_{jj})\delta_0 + t_{jj}\delta_j \quad \text{for all } j \in S.$$

Since $l_1(S)$ is semisimple, T is a bounded linear transformation on $l_1(S)$. Hence, $\|T(\delta_j)\| = |t_{00} - t_{jj}| + |t_{jj}|$ implies that there exists a natural number R such that $|t_{jj}| \leq R$ for all $j = 0, 1, 2, \ldots$. Thus, $T \in \mathscr{M}(l_1(S))$ implies that T has the matrix representation

$$T \leftrightarrow \begin{bmatrix} t_{00} & t_{00} - t_{11} & t_{00} - t_{22} & \cdots \\ 0 & t_{11} & 0 & \cdots \\ 0 & 0 & t_{22} & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix} = [t_{ij}]_{i,j=0}^{\infty}$$

where $|t_{ij}| \leq R$ for all j.

It is also fairly easy to see that every bounded linear transformation T: $l_1(S) \to l_1(S)$ which has such a matrix representation, with $|t_{ii}| \leq R$ for all i and some natural number R, must be a multiplier of $l_1(S)$. In this way a one-to-one correspondence is established between $\mathcal{M}(l_1(S))$ and $l_\infty(S)$. We now want to show that this correspondence is also a topological one.

If $T \in \mathcal{M}(l_1(S))$, then let g_T be the continuous function on $\hat{S}$ corresponding to T[21]. It is proved in [21] that $\|g_T\|_\infty \leq |||T|||$. Let $\alpha = \sum_{i=0}^\infty \alpha(i)\delta_i$, $\|\alpha\| \leq 1$. Then,

$$\begin{aligned}\|T(\alpha)\| &\leq \sum_{i=0}^\infty |\alpha(i)|\,\|T(\delta_i)\| \leq \sup_i \|T(\delta_i)\| \\ &= \sup_i \{|t_{00} - t_{ii}| + |t_{ii}|\}.\end{aligned}$$

Also, $g_T(\chi_0) = T(\delta_0)\hat{}\,(\chi_0) = (t_{00}\delta_0)\hat{}\,(\chi_0) = t_{00}$, while for all $j \neq 0$,

$$\begin{aligned}g_T(\chi_j) &= T(\delta_j)\hat{}\,(\chi_j)/\hat{\delta}_j(\chi_j) = T(\delta_j)\hat{}\,(\chi_j) \\ &= [(t_{00} - t_{jj})\delta_0 + t_{jj}\delta_j]\hat{}\,(\chi_j) = t_{jj}.\end{aligned}$$

Thus, $|t_{00} - t_{ii}| + |t_{ii}| \leq |t_{00}| + 2|t_{ii}| \leq 3\|g_T\|_\infty$ for all i. Therefore,

$$\|T(\alpha)\| \leq \sup_i \{|t_{00} - t_{ii}| + |t_{ii}|\} \leq 3\|g_T\|_\infty$$

for all α, $\|\alpha\| \leq 1$. Combining inequalities we find that

$$\|g_T\|_\infty \leq |||T||| \leq 3\|g_T\|_\infty.$$

This shows that $|||\cdot|||$ on $\mathcal{M}(l_1(S))$ is equivalent to $\|\cdot\|_\infty$ on $\{g_T \in C(\hat{S}): T \in \mathcal{M}(l_1(S))\}$. Thus, we have now proved that $\mathcal{M}(l_1(S))$ is topologically isomorphic to $l_\infty(S)$ by noting that there is a one-to-one correspondence between S and $\hat{S}$.

Our next step is to exhibit an element of $\mathcal{M}(l_1(S))$ that is not in $l_1(\Omega(S))$. Let $\sum_{n=1}^\infty a_n$ be a nonabsolutely convergent series of real numbers with $|a_n| \leq 1$ for all n; let T be given by the matrix representation

$$\begin{bmatrix} \sum_{n=1}^\infty a_n & \sum_{n\neq 1} a_n & \sum_{n\neq 2} a_n & \cdots \\ 0 & a_1 & 0 & \cdots \\ 0 & 0 & a_2 & \cdots \\ \cdot & \cdot & 0 & \cdots \end{bmatrix}$$

Then $T \in \mathcal{M}(l_1(S))$. We wish to show that $g_T \neq \hat{\tau}$ for any $\tau \in l_1(\Omega(S))$. If g_T is the transform of an element of $l_1(\Omega(S))$, then Corollary 3.9 implies that there exists a constant R such that

$$\left|\sum_{i=0}^{N} c_i g_T(\chi_i)\right| \le R\|f\|_\infty \quad \text{for all } f = \sum_{i=0}^{N} c_i \chi_i,\ \chi_i \in \hat{S},\ c_i \in C.$$

Let $R' > R$; there exists a rearrangement $\sum a_{m_n}$ with partial sums S'_n such that $\lim_n S'_n = R'$. Hence, there exists N such that $|S'_N - R'| < |R - R'|$; set $f = \sum_{n=1}^{N} \chi_{m_n}$, with $g_T(\chi_{m_n}) = a_{m_n}$. Hence, $\|f\|_\infty = 1$ and $|\sum_{n=1}^{N} g_T(\chi_{m_n})| = |\sum_{n=1}^{N} a_{m_n}| > R\|f\|_\infty$, a contradiction. Thus, $g_T \notin l_1(\Omega(S))^\wedge$.

However, although $l_1(\Omega(S))^\wedge \neq \mathcal{M}(l_1(S))$, it is of interest that $l_1(\Omega(S))^\wedge$ is dense in $l_\infty(\hat{S})$. First we recall [9] that if B is a Banach algebra with identity with the property that $\mathrm{Re}(\hat{B})$ is sup norm dense in $C_R(\Delta(B))$, then $\hat{B}$ contains the characteristic function of every open-closed subset of $\Delta(B)$. Thus, if S is a union of groups, then $l_1(\Omega(S))^\wedge$ contains the characteristic function of every open-closed subset of $\Omega(S)^\wedge$ [5].

Proposition 4.11. *Suppose S is a union of groups. If $\hat{S}$ has the discrete topology, then $l_1(\Omega(S))^\wedge|_{\hat{S}}$ is uniformly dense in $l_\infty(\hat{S})$.*

Proof. Since $\hat{S}$ is homeomorphic to a subset of $\Omega(S)^\wedge$ and $\hat{S}$ is discrete, the remarks in the above paragraph imply that $l_1(\Omega(S))^\wedge$ contains the characteristic function of every subset of $\hat{S}$. This fact is enough to conclude that $l_1(\Omega(S))^\wedge|_{\hat{S}}$ is uniformly dense in $l_\infty(\hat{S})$. This completes the proof.

Now, in Example 4.10 a routine verification proves that $\hat{S}$ has the discrete topology, and $|||\cdot|||$ on $\mathcal{M}(l_1(S))$ is equivalent to $\|\cdot\|_\infty$. Thus, as an operator subalgebra of $\mathcal{M}(l_1(S))$, $l_1(\Omega(S))$ is operator-norm dense in $\mathcal{M}(l_1(S))$.

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