

ADDENDUM TO "CONVERSE THEOREMS AND EXTENSIONS IN
 CHEBYSHEV RATIONAL APPROXIMATION TO
 CERTAIN ENTIRE FUNCTIONS IN $[0, +\infty)$ "

BY

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Recently (Trans. Amer. Math. Soc. 170 (1972), 171–185) we have proved among other results the following theorem. The notation and numbering are the same as in [1].

Theorem 7 [1, p. 183]. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of order $\rho = 0$ with $a_0 > 0$ and $a_k \geq 0$ for all $k \geq 1$ such that there exist finite numbers

$$1 < \rho_l = \Lambda + 1 \equiv \overline{\lim}_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log \log r},$$

and $0 < b_l \leq B_l$ such that

$$(3.11) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{(\log r)^{\Lambda+1}} = B_l, \quad \underline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{(\log r)^{\Lambda+1}} = b_l.$$

Then, there exists a sequence of real polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $P_n \in \pi_n$ for each $n \geq 0$ such that

$$(3.12) \quad \overline{\lim}_{n \rightarrow \infty} \left\{ \left\| \frac{1}{P_n(x)} - \frac{1}{f(x)} \right\|_{L_{\infty}[0, \infty)} \right\}^{1/n} = 0.$$

Our aim in writing this addendum is to present under the assumptions of Theorem 7, a much sharper result than (3.12), by modifying slightly the proof of Theorem 7.

Lemma [2, Lemma 7]. If $f(z)$ satisfies the assumptions of Theorem 7, then

$$\overline{\lim}_{n \rightarrow \infty} \frac{n^{\Lambda+1}}{[\log |1/a_n|]^{\Lambda}} = \frac{(\Lambda + 1)^{(\Lambda+1)}}{\Lambda^{\Lambda}} B_l$$

Remark. The notation used here is slightly different from the one used in [2].

Theorem 7*. Under the assumptions of Theorem 7, we get

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$$(3.12^*) \quad \overline{\lim}_{n \rightarrow \infty} \left\| \left\| \frac{1}{P_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[0, \infty)} \right\}^{1/n \log n} < e^{-1/\Lambda}.$$

Proof. We have from [1, p. 181],

$$(3.5') \quad \left| \frac{1}{f(x)} - \frac{1}{P_n(x; r)} \right| \leq \frac{2\delta_n(r)}{f^2(0)}, \quad x \in [0, r],$$

and

$$(3.6) \quad \left| \frac{1}{f(x)} - \frac{1}{P_n(x; r)} \right| \leq \frac{2}{f(r)}, \quad x \geq r.$$

From [1, (3.4)] we get

$$(3.4') \quad \delta_n(r) \leq \sum_{k=n+1}^{\infty} a_k r^k.$$

Hence, from (3.4') and (3.5'), we obtain

$$(3.16) \quad \left| \frac{1}{f(x)} - \frac{1}{P_n(x; r)} \right| \leq \frac{2 \sum_{k=n+1}^{\infty} a_k r^k}{a_0^2}.$$

Now from Lemma 1, we have for all $k \geq k_0(\epsilon)$

$$(3.17) \quad \begin{aligned} |a_k| r^k &\leq \left[\exp \left\{ \frac{-k\Lambda}{(\Lambda+1)} \left(\frac{k}{(\Lambda+1)(B_1 + \epsilon)} \right)^{1/\Lambda} \right\} \right] r^k, \\ &\leq \exp \left\{ \log r - \frac{\Lambda}{(\Lambda+1)} \left(\frac{k}{(\Lambda+1)(B_1 + \epsilon)} \right)^{1/\Lambda} \right\} k. \end{aligned}$$

Set

$$(3.18) \quad \log r = \frac{\Lambda}{(\Lambda+1)} \left(\frac{(n+1)}{(\Lambda+1)(B_1 + \epsilon)} \right)^{1/\Lambda} - \frac{\log(n+1)}{\Lambda}.$$

A simple calculation based on (3.17) and (3.18) gives us for all $n \geq n_0$

$$(3.19) \quad \begin{aligned} \sum_{k=n+1}^{\infty} a_k r^k &\leq \left[\exp \left(\frac{-(n+1) \log(n+1)}{\Lambda} \right) \right] \left(\sum_{k=0}^{\infty} \frac{1}{n^{k/\Lambda}} \right) \\ &\leq \left[\exp \left[\frac{-(n+1) \log(n+1)}{\Lambda} \right] \right] \left(\frac{n^{1/\Lambda}}{n^{1/\Lambda} - 1} \right). \end{aligned}$$

On the other hand we get from (3.6) and (3.11) for all $r \geq r_0(\epsilon_1)$

$$(3.20) \quad M_f(r) \geq \exp [(\log r)^{\Lambda+1} (b_l - \epsilon_1)].$$

(3.18) and (3.20) give us for all $n \geq n_0$

$$(3.21) \quad M_f(r) \geq \exp \left\{ \left[\frac{\Lambda}{(\Lambda+1)} \left(\frac{(n+1)}{(\Lambda+1)(B_l + \epsilon)} \right)^{1/\Lambda} - \frac{\log(n+1)}{\Lambda} \right]^{(\Lambda+1)} (b_l - \epsilon_1) \right\}$$

$$\geq \exp \left[c_1 n^{(\Lambda+1)/\Lambda} \left(1 - c_2 \frac{\log(n+1)}{\Lambda(n+1)^{1/\Lambda}} \right)^{\Lambda+1} \right]$$

where

$$c_1 = \left(\frac{\Lambda}{(\Lambda+1)} \right)^{\Lambda+1} (b_l - \epsilon_1) \left(\frac{1}{(\Lambda+1)(B_l + \epsilon)} \right)^{(\Lambda+1)/\Lambda}, \quad c_2 = \frac{(\Lambda+1)(\Lambda+1)(B_l + \epsilon)^{1/\Lambda}}{\Lambda}.$$

Now we get from (3.16) and (3.19) for $x \in [0, r]$

$$(3.22) \quad \left\| \frac{1}{f(x)} - \frac{1}{P_n(x; r)} \right\|_{[0, r]}^{1/n \log n} \leq \exp \left(-\frac{1}{\Lambda} \right).$$

On the other hand from (3.6), (3.20) and (3.21) we get

$$(3.23) \quad \left\| \frac{1}{f(x)} - \frac{1}{P_n(x; r)} \right\|^{1/n \log n} = 0, \quad \text{for all } x \geq r.$$

Hence from (3.22) and (3.23) by setting (cf. [1, p. 182]) $P_n(x) \equiv P_n(x; r(n))$, we get (3.12*), i.e.

$$\overline{\lim}_{n \rightarrow \infty} \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0, \infty)}^{1/n \log n} \leq \exp \left(-\frac{1}{\Lambda} \right).$$

Remarks. There exist transcendental entire functions of order zero which fail to satisfy the assumptions of the Theorem 7* but satisfy

$$(3.24) \quad \overline{\lim}_{n \rightarrow \infty} \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|^{1/n \log n} < 1.$$

For example the function

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{2^{\log 2} 3^{\log 3} 4^{\log 4} \dots n^{\log n}}$$

is of order zero with $\Lambda = \infty$ and satisfies (3.24).

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By choosing

$$\log r = \frac{\Lambda}{2(\Lambda + 1)} \left(\frac{(n+1)}{(\Lambda + 1)(B_l + \epsilon)} \right)^{1/\Lambda}$$

instead of (3.18) and adopting the same technique we get instead of (3.12*) the following

$$(3.25) \quad \overline{\lim}_{n \rightarrow \infty} \left(\left\| \frac{1}{P_n(x)} - \frac{1}{f(x)} \right\|_{L_\infty[0, \infty)} \right)^{1/n^{1+1/\Lambda}} < 1.$$

The result (3.25) is the best possible in the sense that Λ cannot be replaced by $(\Lambda - \epsilon)$ for any $\epsilon > 0$ for all entire functions satisfying the assumptions of Theorem 7*. The details of this will appear separately. In view of (3.25) the following example may be of some interest.

Let $f(x) = \sum_0^\infty x^n / e^{2^n}$. For this function $\Lambda = 0$ and

$$(3.26) \quad \lim_{n \rightarrow \infty} (\lambda_{0,n})^{1/2^{(n+1)}} = \frac{1}{e}.$$

The proof of this is very lengthy. The details of this also will appear separately.

REFERENCES

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