

DIRICHLET PROBLEM FOR DEGENERATE ELLIPTIC EQUATIONS⁽¹⁾

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ABSTRACT. Let L_0 be a degenerate second order elliptic operator with no zeroth order term in an m -dimensional domain G , and let $L = L_0 + c$. One divides the boundary of G into disjoint sets $\Sigma_1, \Sigma_2, \Sigma_3$; Σ_3 is the noncharacteristic part, and on Σ_2 the "drift" is outward. When c is negative, the following Dirichlet problem has been considered in the literature: $Lu = 0$ in G , u is prescribed on $\Sigma_2 \cup \Sigma_3$. In the present work it is assumed that $c \leq 0$. Assuming additional boundary conditions on a certain finite number of points of Σ_1 , a unique solution of the Dirichlet problem is established.

Introduction. Consider the second order degenerate elliptic operator with smooth coefficients

$$(0.1) \quad Lu = \frac{1}{2} \sum_{i,j=1}^m a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i(x) \frac{\partial u}{\partial x_i}$$

in a smoothly bounded domain G in R^m . The Dirichlet problem for the equation $Lu + c(x)u = 0$ in G has been treated by many authors (see [5] and the references contained therein). In all of these approaches, the boundary ∂G is decomposed as follows:

$$\Sigma_3 = \left\{ x \in \partial G; \sum_{i,j=1}^m a_{ij}(x) \nu_i \nu_j > 0 \right\} \quad ((\nu_i) = \text{outward normal}),$$

$$\Sigma_2 = \left\{ x \in \partial G \setminus \Sigma_3; \sum_{i=1}^m \left(b_i(x) - \frac{1}{2} \sum_{j=1}^m \frac{\partial a_{ij}}{\partial x_j}(x) \right) \nu_i > 0 \right\},$$

$$\Sigma_1 = \left\{ x \in \partial G \setminus \Sigma_3; \sum_{i=1}^m \left(b_i(x) - \frac{1}{2} \sum_{j=1}^m \frac{\partial a_{ij}}{\partial x_j}(x) \right) \nu_i \leq 0 \right\}.$$

A typical result of these theories asserts that the equation $Lu + c(x)u = 0$ has a unique solution in some function space when data are prescribed on

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$\Sigma_2 \cup \Sigma_3$ and when it is further required that $c(x) \leq c_0 < 0$ in G . In case $c_0 = 0$, Stroock and Varadhan [7] have shown that the Dirichlet problem in R^m has a unique solution with data prescribed on $\Sigma_2 \cup \Sigma_3$ provided that the paths of the associated diffusion process exit G "sufficiently fast." We intend to show that, if this last condition fails, we can still find a unique solution *provided that we assign data also on a certain portion of Σ_1* . This possibility was never considered before, to the best of our knowledge. Our method is entirely probabilistic, inspired by our previous work [3], [4] on the stability properties of stochastic differential equations.

The points of Σ_1 at which we must assign data are precisely the points to which the associated Markov process converges when $t \rightarrow \infty$. These "distinguished boundary points" are defined in terms of the normal *and* tangential behavior of the diffusion process, in contrast to $\Sigma_2 \cup \Sigma_3$ which only depends on the normal components of diffusion and drift. For technical reasons we will only consider cases where there exist a finite number of distinguished boundary points, together with an arbitrary configuration of $\Sigma_2 \cup \Sigma_3$. We denote by Σ_1^- the component of ∂G containing all of the distinguished boundary points.

In §§2-5 we consider the case $m = 2$, and in §6 we consider the case $m \geq 2$.

§§1 and 2 contain preliminary results on the boundary behavior of solutions $x(t)$ of the stochastic equations

$$dx_i = \sum_{r=1}^n \sigma_{ir}(x) dw^r + b_i(x)dt \quad (1 \leq i \leq 2)$$

in a special domain in the plane. For technical reasons we assume that when σ and b vanish simultaneously on Σ_1^- , they do not vanish faster than a linear function. In §3 we consider a general domain in the plane and show that either $x(t)$ attains $\Sigma_2 \cup \Sigma_3$ in finite time or else converges to some distinguished boundary point while remaining inside G for all $t < \infty$. In §4 we prove the differentiability (as a function of the starting point) of the probability that the process will converge to a given distinguished boundary point. Finally in §5 we consider the Dirichlet problem in a general domain in the plane, combining the results of the previous sections with known results [6] on the behavior of the diffusion process near $\Sigma_2 \cup \Sigma_3$.

The results of §§2-5 can be extended to $m \geq 3$; this is briefly discussed in §6. However, the main result of §6 is a theorem which even for $m = 2$ is not included in §§2-5.

1. **Boundary behavior of stochastic solutions in annular domains.** Consider a system of two stochastic differential equations

$$(1.1) \quad dx_i = \sum_{s=1}^n \sigma_{is}(x) dw^s + b_i(x)dt \quad (i = 1, 2)$$

where $w^1(t), \dots, w^n(t)$ are independent Brownian motions. Let $a = (a_{ij}) = \sigma\sigma^*$ where $\sigma = (\sigma_{is})$, $\sigma^* =$ transpose of σ . We assume

(A) The functions $\sigma_{is}(x)$, $b_i(x)$ and their first two derivatives are continuous and bounded in R^2 .

Let G be a bounded domain in R^2 . For simplicity we first take

$$G = \{x; 1 < |x| < 2\}.$$

Denote by ∂G the boundary of G . We shall assume

(B) On ∂G ,

$$(1.2) \quad \sum_{i,j=1}^2 a_{ij} \nu_i \nu_j = 0,$$

$$(1.3) \quad \sum_{i=1}^2 \left[b_i - \frac{1}{2} \sum_{j=1}^2 \frac{\partial a_{ij}}{\partial x_j} \right] \nu_i \leq 0$$

where ν is the outward normal to ∂G (with respect to G).

Let $R(x)$ be a positive C^2 function in G , which coincides with $\text{dist}(x, \partial G)$ when the latter is sufficiently small. Let

$$\begin{aligned} \mathcal{Q} &= \sum_{i,j=1}^2 a_{ij} \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j}, \\ \mathcal{B} &= \sum_{i=1}^2 b_i \frac{\partial R}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^2 a_{ij} \frac{\partial^2 R}{\partial x_i \partial x_j}, \quad \mathcal{Q} = \frac{1}{R} \left(\mathcal{B} - \frac{\mathcal{Q}}{2R} \right). \end{aligned}$$

(C) For some $\mu > 0$ sufficiently small,

$$Q(x) \leq -\theta_0 < 0 \quad \text{if } 1 < |x| \leq 1 + \mu \quad (\theta_0 \text{ constant}),$$

$$Q(x) \geq \theta_0 > 0 \quad \text{if } 2 - \mu \leq |x| < 2,$$

$$\mathcal{Q}(x) > 0 \quad \text{if } 1 + \mu \leq |x| \leq 2 - \mu, \nabla_x R(x) \neq 0,$$

$$\sum_{i,j=1}^2 a_{ij}(x) \frac{\partial^2 R}{\partial x_i \partial x_j} < 0 \quad \text{if } 1 + \mu \leq |x| \leq 2 - \mu, \nabla_x R(x) = 0.$$

By Theorem 1.1 of [4] and by slightly modifying the proof of Theorem 2.1 of [4] we get

Lemma 1.1. *If (A)–(C) hold then, for any solution $x(t)$ of (1.1) with $x(0) \in G$,*

$$P\{x(t) \in G \text{ for all } t > 0\} = 1, \quad P\{|x(t)| \rightarrow 1 \text{ as } t \rightarrow \infty\} = 1.$$

It is actually sufficient to assume that σ_{ij} are continuously differentiable in R^2 and twice continuously differentiable in a neighborhood of ∂G .

We shall now analyze the limit set of $x(t)$ on $|x| = 1$ (as $t \rightarrow \infty$). For this,

we introduce polar coordinates (r, ϕ) as in [4]. We find that

$$(1.4) \quad dr = \sum_{s=1}^n \tilde{\sigma}_s(r, \phi) dw^s + \tilde{b}(r, \phi) dt, \quad d\phi = \sum_{s=1}^n \tilde{\sigma}'_s(r, \phi) dw^s + \tilde{b}'(r, \phi) dt$$

where

$$\begin{aligned} \tilde{\sigma}_s(r, \phi) &= \sigma_{1s} \cos \phi + \sigma_{2s} \sin \phi, & \tilde{b}(r, \phi) &= b_1 \cos \phi + b_2 \sin \phi + \frac{1}{2r} \langle a(x) \lambda^\perp, \lambda^\perp \rangle, \\ \tilde{\sigma}'_s(r, \phi) &= -\frac{\sin \phi}{r} \sigma_{1s} + \frac{\cos \phi}{r} \sigma_{2s}, & \tilde{b}'(r, \phi) &= -\frac{\sin \phi}{r} b_1 + \frac{\cos \phi}{r} b_2 - \frac{1}{r^2} \langle a(x) \lambda, \lambda^\perp \rangle; \end{aligned}$$

here

$$\lambda = (\cos \phi, \sin \phi), \quad \lambda^\perp = (-\sin \phi, \cos \phi)$$

and

$$\langle a(x) \mu, \nu \rangle = \sum a_{ij}(x) \mu_i \nu_j \quad (\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2)).$$

Thus, if $(r(t), \phi(t))$ is a solution of (1.4) and if we define $x_1(t) = r(t) \cos \phi(t)$, $x_2(t) = r(t) \sin \phi(t)$, then $x(t) = (x_1(t), x_2(t))$ is a solution of (1.1).

The system (1.4) can also be written in the form (see [4])

$$(1.5) \quad \begin{aligned} dr &= r \left[\sum_{s=1}^n \tilde{\sigma}_s(\phi) dw^s + \tilde{b}(\phi) dt \right] + \left[\sum_{s=1}^n R_s dw^s + R_0 dt \right] \\ d\phi &= \left[\sum_{s=1}^n \tilde{\sigma}'_s(\phi) dw^s + \tilde{b}'(\phi) dt \right] + \left[\sum_{s=1}^n \Theta_s dw^s + \Theta_0 dt \right] \end{aligned}$$

where $R_s = o(r)$, $\Theta_s = o(1)$ ($0 \leq s \leq n$) as $r \rightarrow 1$, uniformly with respect to ϕ . It is useful to compare $\phi(t)$ with the solution of the single equation

$$(1.6) \quad d\phi = \sigma(\phi) dw + b(\phi) dt$$

where $\sigma(\phi) = \{ \sum_{s=1}^n [\tilde{\sigma}'_s(\phi)]^2 \}^{1/2}$, $b(\phi) = \tilde{b}'(\phi)$.

In case $\sigma(\phi) \neq 0$ for all ϕ and $\int_0^{2\pi} b(z) / \sigma^2(z) dz \neq 0$, it was proved in [4] that the algebraic angle $\phi(t)$ (i.e., the component $\phi(t)$ of the solution $(r(t), \phi(t))$ of (1.4)) satisfies

$$(1.7) \quad P \left\{ \lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = c \right\} = 1 \quad (c \text{ constant} \neq 0).$$

Suppose $\sigma(z)$ is degenerate, but it has only a finite number of zeros $\alpha_1, \dots, \alpha_k$ ($k \geq 1$) in the interval $[0, 2\pi)$. Then the conclusion (1.7) is still valid [4] provided $b(\alpha_j) > 0$ for all j , or $b(\alpha_j) < 0$ for all j , and provided the following condition holds:

(i) For some $\bar{\epsilon} > 0$,

$$\sum_{s=1}^n [\tilde{\sigma}'_s(r, \phi)]^2 = \sum_{s=1}^n [\tilde{\sigma}'_s(\phi)]^2 [1 + \eta(r, \phi)] \quad (1 \leq r \leq 1 + \bar{\epsilon})$$

where $\eta(r, \phi) \rightarrow 0$ if $r \rightarrow 1$, uniformly with respect to ϕ .

We shall now consider the degenerate case in situations where the $b(\alpha_j)$ may

vanish. The condition (i) will not be assumed in the sequel.

Our basic assumptions are:

(D) If $b(\alpha_j) = 0$ for some j ($1 \leq j \leq k$) then there is a simple C^3 curve Δ_{α_j} given by $r = r^*(t)$, $\phi = \phi^*(t)$ ($t_1 \leq t \leq t_2$) such that $(r^*(t_1), \phi^*(t_1)) = (1, \alpha_j)$, $(r^*(t_2), \phi^*(t_2))$ lies outside \bar{G} , and $(r^*(\tilde{t}), \phi^*(\tilde{t}))$, for some $t_1 < \tilde{t} < t_2$, is a point on ∂G different from $(1, \alpha_j)$, and such that

(i) a part $\{(r^*(t), \phi^*(t)); t_1 \leq t \leq t_1 + \epsilon_1\}$ coincides with the segment $1 \leq r \leq 1 + \bar{\epsilon}$, $\phi = \alpha_j$, and

(ii) the following relations hold at each point of Δ_{α_j} :

$$(1.8) \quad \sum_{i,j=1}^2 a_{ij} \nu_i \nu_j = 0, \quad \sum_{i=1}^2 \left[b_i - \frac{1}{2} \sum \frac{\partial a_{ij}}{\partial x_j} \right] \nu_i = 0$$

where (ν_1, ν_2) is the normal to Δ_{α_j} .

Finally, if $b(\alpha_j) = b(\alpha_b) = 0$ and $b(\alpha_i) \neq 0$ for all α_i between α_j and α_b , then the points (r, ϕ) with $r = 1 + \bar{\epsilon}$, ϕ in the interval (α_j, α_b) cannot be connected (in G) to points (r, ϕ) with $r = 1 + \bar{\epsilon}$, ϕ outside the interval (α_j, α_b) , without crossing either Δ_{α_j} or Δ_{α_b} .

Note that the conditions in (1.8) along the ray $1 \leq r \leq 1 + \epsilon$, $\phi = \alpha_j$ hold if and only if

$$\tilde{\sigma}_s(r, \alpha_j) = 0 \quad (1 \leq s \leq n), \quad \tilde{b}(r, \alpha_j) = 0 \quad \text{for } 1 \leq r \leq 1 + \bar{\epsilon}.$$

(E) If $b(\alpha_j) = 0$ for some j ($1 \leq j \leq k$), then $b(z)$, $\sigma(z)$ vanish at $z = \alpha_j$ to the first order only, and

$$Q_{\alpha_j} \equiv \lim_{z \rightarrow \alpha_j} \frac{2(z - \alpha_j)b(z)}{\sigma^2(z)} \neq 1.$$

Note that the limit exists since $b(z)$, $\sigma(z)$ are continuously differentiable.

We first consider the case where $b(z)$ vanishes at two consecutive points, say $\alpha = \alpha_b$, $\beta = \alpha_{b+1}$. Introduce straight segments

$$l_\alpha = \{(r, \alpha); 1 \leq r \leq 1 + \bar{\epsilon}\}, \quad l_\beta = \{(r, \beta); 1 \leq r \leq 1 + \bar{\epsilon}\}.$$

Denote by m_η ($\eta \geq 0$) the curve $\{(r, \phi); r = 1 + \eta, \alpha \leq \phi \leq \beta\}$, and by Ω_η ($\eta > 0$) the domain bounded by $m_0, m_\eta, l_\alpha, l_\beta$. Denote by $A_{\alpha\beta}$ the set of points in the probability space for which $\alpha \leq \phi(t) \leq \beta$ for a sequence of t 's converging to ∞ . By (D), and the proof of Theorem 1.1 (or rather Theorem 1.1) of [4], if $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$ then the solution $(r(t), \phi(t))$ never intersects $\Delta_\alpha \cup \Delta_\beta$. It follows (by the last part of (D)) that on the set $A_{\alpha\beta}$

$$(1.9) \quad \alpha < \phi(t) < \beta$$

for all t sufficiently large.

Lemma 1.2. Let (A)–(E) hold and let $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$. If $Q_\alpha > 1$, $Q_\beta < 1$ then, a.s. on the set $A_{\alpha\beta}$, $\phi(t) \rightarrow \beta$ if $t \rightarrow \infty$.

Proof. Consider the function

$$(1.10) \quad g(x) = \int_{\alpha^*}^x \frac{1}{\beta(y)} \left[\int_y^{\alpha^*} \frac{2\beta(z)}{\sigma^2(z)} dz \right] dy \quad \text{in } (\alpha + \epsilon_0, \beta - \epsilon_0)$$

for some small $\epsilon_0 > 0$, where

$$\beta(y) = \exp \left\{ \int_{\alpha^*}^y \frac{2b(z)}{\sigma^2(z)} dz \right\}.$$

It is easily seen that

$$\hat{L}g \equiv \frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) = -1 \quad \text{in } (\alpha + \epsilon_0, \beta - \epsilon_0).$$

Further,

$$(1.11) \quad g'(x) > 0 \quad \text{at } x = \alpha + \epsilon_0,$$

$$< 0 \quad \text{at } x = \beta - \epsilon_0$$

if $\alpha + \epsilon_0 < \alpha^* < \beta - \epsilon_0$;

$$(1.12) \quad g'(x) > 0 \quad \text{at } x = \alpha + \epsilon_0, x = \beta - \epsilon_0$$

if $\alpha < \alpha^* < \alpha + \epsilon_0$;

$$(1.13) \quad g'(x) < 0 \quad \text{at } x = \alpha + \epsilon_0, x = \beta - \epsilon_0$$

if $\beta - \epsilon_0 < \alpha^* < \beta$.

Set

$$f(x) = \begin{cases} -A_1 \log(x - \alpha) + B_1 & \text{in } (\alpha, \alpha + \epsilon_0), \\ A_2 \log(\beta - x) + B_2 & \text{in } (\beta - \epsilon_0, \beta), \\ g(x) & \text{in } [\alpha + \epsilon_0, \beta - \epsilon_0]. \end{cases}$$

If $A_1 > 0$, $A_2 > 0$ then, by using the assumptions $Q_\alpha > 1$, $Q_\beta < 1$ we find that $\hat{L}f(x) \leq -\nu < 0$ in $(\alpha, \alpha + \epsilon_0)$, $(\beta - \epsilon_0, \beta)$ where ν is a positive constant, provided ϵ_0 is sufficiently small. Choose α^* so that (1.13) holds, and determine the constants A_i, B_i in such a way that $f(x)$ is continuously differentiable at $x = \alpha + \epsilon_0$, $x = \beta - \epsilon_0$. We then find that $A_1 > 0$, $A_2 > 0$.

Let $\{F_m\}$ be a sequence of continuous functions which approximate f'' in the following manner:

$$F_m''(x) = f''(x) \quad \text{if } |x - (\alpha + \epsilon_0)| > 1/m, \quad |x - (\beta - \epsilon_0)| > 1/m,$$

and $F_m''(x)$ connects linearly $f''(\alpha + \epsilon_0 - 1/m)$ to $f''(\alpha + \epsilon_0 + 1/m)$ and $f''(\beta - \epsilon_0 - 1/m)$ to $f''(\beta - \epsilon_0 + 1/m)$. Let

$$f_m(x) = f(\alpha^*) + f'(\alpha^*)(x - \alpha^*) + \int_{\alpha^*}^x \int_{\alpha^*}^y F_m(z) dz dy.$$

Then $f_m(x) - f(x) \rightarrow 0, f'_m(x) - f'(x) \rightarrow 0$ uniformly in the interval (α, β) , and $f''_m(x) - f''(x) = 0$ outside the intervals with centers $\alpha + \epsilon_0, \beta - \epsilon_0$ and length $2/m$.

Denote by $f_m^\delta(x), f^\delta(x)$ any C^2 (2π) -periodic functions of $x \in R^1$ which coincide, respectively, with $f_m(x)$ and $f(x)$ in $(\alpha + \delta, \beta - \delta)$; δ is any positive number smaller than ϵ_0 . Denote by $R(r)$ any C^2 function satisfying

$$R(r) = 1 \quad \text{if } 1 < r < 1 + \eta_0, \quad R(r) = 0 \quad \text{if } 1 + \eta_1 < r < \infty,$$

where $0 < \eta_0 < \eta_1$ and $\eta_1 < \bar{\epsilon}$. Let

$$\tilde{\sigma}_s^\epsilon(r, \phi) = \{[\tilde{\sigma}_s(r, \phi)]^2 + \epsilon\}^{1/2} \quad (\epsilon > 0).$$

Denote by $(r^\epsilon, \phi^\epsilon)$ the solution of (1.4) when $\tilde{\sigma}_s$ is replaced by $\tilde{\sigma}_s^\epsilon$. Denote by L_ϵ the elliptic generator corresponding to the process $(r^\epsilon, \phi^\epsilon)$. Set

$$\Phi_m^\delta(r, \phi) = R(r)f_m^\delta(\phi), \quad \Phi^\delta(r, \phi) = R(r)f^\delta(\phi).$$

By Itô's formula,

$$\begin{aligned} & \Phi_m^\delta(r^\epsilon(t), \phi^\epsilon(t)) - \Phi_m^\delta(r^\epsilon(0), \phi^\epsilon(0)) \\ (1.14) \quad &= \int_0^t \nabla \Phi_m^\delta(r^\epsilon(r), \phi^\epsilon(r)) \cdot \sigma^\epsilon(r^\epsilon(r), \phi^\epsilon(r)) dw(r) + \int_0^t L_\epsilon \Phi_m^\delta(r^\epsilon(r), \phi^\epsilon(r)) dr \end{aligned}$$

where $\nabla \Phi$ is the gradient of Φ and $\sigma^\epsilon(r, \phi)$ is the matrix corresponding to $\tilde{\sigma}_s(r, \phi), \tilde{\sigma}_s^\epsilon(r, \phi)$. Since L_ϵ is uniformly elliptic, with bounded and uniformly Hölder continuous coefficients, the corresponding parabolic operator has a fundamental solution (see [2]). We can therefore go to the limit with $m \rightarrow \infty$ in (1.14) (cf. [4]) and conclude that

$$\begin{aligned} & R(r(t))f^\delta(\phi(t)) - R(r(0))f^\delta(\phi(0)) \\ (1.15) \quad &= \int_0^t \nabla \Phi^\delta(r^\epsilon(r), \phi^\epsilon(r)) \cdot \sigma^\epsilon(r^\epsilon(r), \phi^\epsilon(r)) dw(r) + \int_0^t L_\epsilon \Phi^\delta(r^\epsilon(r), \phi^\epsilon(r)) dr. \end{aligned}$$

We shall now consider the behavior of $(r^\epsilon(r), \phi^\epsilon(r))$ on the set $A_{\alpha\beta}$. Given $0 < \eta < \eta_0$, let T_η be the last time $(r(t), \phi(t))$ is outside $\Omega_{\eta/2}$. With t fixed, and a.s. in $A_{\alpha\beta}$,

$$(r^\epsilon(r), \phi^\epsilon(r)) \rightarrow (r(r), \phi(r)) \quad \text{uniformly in } r, \quad 0 \leq r \leq t,$$

for a subsequence $\epsilon = \epsilon' \searrow 0$. Hence, if $T_\eta \leq r \leq t, (r^\epsilon(r), \phi^\epsilon(r)) \in \Omega_\eta$ for all $\epsilon = \epsilon' \leq \epsilon^*(\omega)$. Given $\delta \leq \delta^*(\omega)$ sufficiently small, we have for any $\epsilon = \epsilon'$ sufficiently small and for any $\gamma > 0$,

$$\begin{aligned}
 \int_{T_\eta}^t L_\epsilon \Phi^\delta(r^\epsilon(r), \phi^\epsilon(r)) dr &= \int_{T_\eta}^t L_\epsilon \Phi(r^\epsilon(r), \phi^\epsilon(r)) dr \\
 &= \int_{T_\eta}^t (L_\epsilon f)(r^\epsilon(r), \phi^\epsilon(r)) dr \\
 &= \int_{T_\eta}^t \left\{ \left[\frac{1}{2} \sum_{s=1}^n [\tilde{\sigma}_s^\delta(r^\epsilon, \phi^\epsilon)]^2 + \epsilon \right] f''(\phi^\epsilon) + \tilde{b}(r^\epsilon, \phi^\epsilon) f'(\phi^\epsilon) \right\} dr \\
 &= \int_{T_\eta}^t \left\{ \left[\frac{1}{2} \sigma^2(\phi^\epsilon) + \epsilon \right] f''(\phi^\epsilon) + b(\phi^\epsilon) f'(\phi^\epsilon) + \theta \gamma \right\} dr \\
 &\leq -(t - T_\eta) \nu + \int_{T_\eta}^t \epsilon f''(\phi^\epsilon) dr + \bar{\theta} \gamma (t - T_\eta)
 \end{aligned}$$

where $|\theta| \leq 1$, $|\bar{\theta}| \leq 1$, provided $\eta \leq \eta^*(\gamma)$. Here ν is any positive number such that $Lf(x) \leq -\nu$ for all $x \neq \alpha + \epsilon_0, x \neq \beta - \epsilon_0$. It follows that

$$(1.16) \quad \overline{\lim}_{\epsilon=\epsilon' \searrow 0} \int_{T_\eta}^t L_\epsilon \Phi^\delta(r^\epsilon(r), \phi^\epsilon(r)) dr \leq -(t - T_\eta) \frac{\nu}{2}$$

if $\gamma < \nu/2$. Since $(r^\epsilon(r), \phi^\epsilon(r))$ does not intersect the set $I_\alpha \cup I_\beta$ for $r \geq 0$, $f''(\phi^\epsilon(r))$ ($0 \leq r \leq T_\eta$) remains bounded as $\epsilon = \epsilon' \searrow 0$. We conclude that

$$(1.17) \quad \overline{\lim}_{\epsilon=\epsilon' \searrow 0} \int_0^t L_\epsilon \Phi^\delta(r^\epsilon(r), \phi^\epsilon(r)) dr \leq C - \frac{\nu}{2} t$$

where C is a.s. finite valued random variable.

Consider next the stochastic integral in (1.15). If $T_\eta \leq r \leq t$, then the vector

$$(1.18) \quad b_\epsilon^\delta(r) \equiv \nabla \Phi^\delta(r^\epsilon(r), \phi^\epsilon(r)) \cdot \sigma^\epsilon(r^\epsilon(r), \phi^\epsilon(r))$$

has components $\{(d/d\phi) f^\delta(\phi^\epsilon(r))\} \{[\tilde{\sigma}_s^\delta(r^\epsilon(r), \phi^\epsilon(r))]^2 + \epsilon\}^{1/2}$. If we let $\epsilon = \epsilon' \searrow 0$ through an appropriate subsequence ϵ'' , then we obtain a.s. (cf. [4, §2])

$$\begin{aligned}
 (1.19) \quad \lim_{\epsilon=\epsilon'' \searrow 0} \int_{T_\eta}^t b_\epsilon^\delta(r) \cdot dw(r) &= \int_{T_\eta}^t b^\delta(r) \cdot dw(r) = \int_{T_\eta}^t b^0(r) \cdot dw(r) \\
 &= \sum_{s=1}^n \int_{T_\eta}^t f'(\phi(r)) \tilde{\sigma}_s^\delta(r, \phi(r)) dw^s(r)
 \end{aligned}$$

where $b^0(r)$ is defined by (1.18) with Φ^δ replaced by Φ and with $\epsilon = 0$.

If $0 \leq r \leq T_\eta$, then as $\epsilon \searrow 0$ through an appropriate subsequence of ϵ'' ,

$$\int_0^{T_\eta} b_\epsilon^\delta(r) \cdot dw(r) \rightarrow \int_0^{T_\eta} b^\delta(r) \cdot dw(r) = \int_0^{T_\eta} b^0(r) \cdot dw(r) \equiv \hat{C}$$

where $b^0(r)$ has a more complicated expression than in (1.19) (involving $R(r)$)

and its first derivative); \hat{C} is a.s. finite. We conclude from this and from (1.15), (1.17), (1.19) that, a.s. on $A_{\alpha\beta}$,

$$f(\phi(t)) - f(\phi(0)) \leq C - \frac{\nu}{2}t + \hat{C} + \sum_{s=1}^n \int_{T_\eta}^t f'(\phi(r)) \tilde{\sigma}_s(r(r), \phi(r)) dw^s(r).$$

By Lemma 1.3 of [3], the last integral is $o(t)$. Hence

$$\overline{\lim}_{t \rightarrow \infty} \frac{f(\phi(t))}{t} \leq -\frac{\nu}{2} < 0 \quad \text{a.s. in } A_{\alpha\beta}.$$

This implies that $\phi(t) \rightarrow \beta$ as $t \rightarrow \infty$, a.s. in $A_{\alpha\beta}$.

2. **Boundary behavior of stochastic solutions (continued).** Divide the zeros $\alpha_1, \dots, \alpha_k$ of $\sigma(z)$ in $[0, 2\pi)$ into blocks

$$B_j = \{\alpha_{j,1}, \dots, \alpha_{j,k_j}\} \quad (k_j > 1)$$

where $\alpha_{j,i} < \alpha_{j,i+1}$, $\alpha_{j,k_j} = \alpha_{j+1,1}$. (Here we agree that $\alpha_k < \alpha_1$.) For each block B_j , $b(\alpha_{j,1}) = 0$, $b(\alpha_{j,k_j}) = 0$, and $b(\alpha_{j,i}) \neq 0$ if $2 \leq i \leq k_j - 1$. Let

$$A_j = \{\omega; \alpha_{j,1} < \phi(t) < \alpha_{j,k_j} \text{ for all } t \text{ sufficiently large}\}.$$

In view of the Lemma 1.1 and the fact that $(r(t), \phi(t))$ never crosses the segments $\{(r, \alpha_{j,1}); 1 \leq r \leq 1 + \bar{\epsilon}\}$ we conclude that $\sum P(A_j) = 1$.

Consider now a block B_j , and set $\alpha = \alpha_{j,1}$, $\beta = \alpha_{j,k_j}$. Suppose

$$(2.1) \quad b(\alpha) = 0, \quad b(\beta) = 0, \quad b(\alpha_{j,i}) > 0 \quad (2 \leq i \leq k_j - 1),$$

$$(2.2) \quad Q_\alpha > 1, \quad Q_\beta < 1.$$

Lemma 2.1. *Let (A)–(E) and (2.1), (2.2) hold. If $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$, then a.s. in A_j , $\phi(t) \rightarrow \beta$ as $t \rightarrow \infty$.*

Proof. Let

$$f(x) = \begin{cases} -A_1 \log(x - \alpha) + B_1 & \text{in } (\alpha, \alpha + \epsilon_0), \\ A_2 \log(\beta - x) + B_2 & \text{in } (\beta - \epsilon_0, \beta), \\ g(x) & \text{in } [\alpha + \epsilon_0, \beta - \epsilon_0]; \end{cases}$$

the function $g(x)$ consists of three parts:

$$\begin{aligned} &A_3 g_1(x) + B_3 && \text{in } [\alpha + \epsilon_0, \alpha_{j,2} - \epsilon'], \\ &A_4 g_2(x) + B_4 && \text{in } [\alpha_{j,2} - \epsilon', \alpha_{j,k_j-1} + \epsilon'], \\ &g_3(x) && \text{in } (\alpha_{j,k_j-1} + \epsilon', \beta - \epsilon_0] \end{aligned}$$

where $\epsilon' > 0$ is sufficiently small. The function g_2 is constructed as the function f in the proof of Theorem 3.2 in [3]; thus $Lg_2 \leq -\nu < 0$ in $[\alpha_{j,2} - \epsilon', \alpha_{j,k_j-1} + \epsilon']$ and $g'(x) < 0$ at the endpoints. The function g_1 is defined as the function g in (1.10), (1.13) with β replaced by $\alpha_{j,2} - \epsilon'$. Finally, the function g_3 is defined as the function g in (1.10), (1.13) with α replaced by $\alpha_{j,k_j-1} + \epsilon'$. We can choose the constants A_i, B_i so that $f(x)$ is continuously differentiable; the A_i are all positive.

We can now proceed similarly to the proof of Lemma 1.2.

Suppose now that (2.1), (2.2) are replaced by

$$(2.3) \quad b(\alpha) = 0, \quad b(\beta) = 0, \quad b(\alpha_{j,i}) > 0 \quad (2 \leq i \leq k_j - 1),$$

$$(2.4) \quad Q_\alpha < 1, \quad Q_\beta > 1.$$

Lemma 2.2. *Let (A)–(E) and (2.3), (2.4) hold. If $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$, then a.s. in A_j , $\phi(t) \rightarrow \alpha$ as $t \rightarrow \infty$.*

The proof is similar to the proof of Lemma 2.1. Here one takes $f(x) = A_1 \log(x - \alpha) + B_1$ in $(\alpha, \alpha + \epsilon_0)$, $f(x) = -A_2 \log(\beta - x) + B_2$ in $(\beta - \epsilon_0, \beta)$.

Consider next the cases where

$$(2.5) \quad b(\alpha) = 0, \quad b(\beta) = 0, \quad Q_\alpha < 1, \quad Q_\beta < 1.$$

We further assume that one of the following three conditions holds:

$$(2.6) \quad b(\alpha_{j,i}) > 0 \quad (2 \leq i \leq k_j - 1),$$

$$(2.7) \quad b(\alpha_{j,i}) < 0 \quad (2 \leq i \leq k_j - 1),$$

$$(2.8) \quad b(\alpha_{j,i}) < 0 \quad (2 \leq i \leq i_0),$$

$$b(\alpha_{j,i}) > 0 \quad (i_0 + 1 \leq i \leq k_j - 1).$$

Lemma 2.3. *Let (A)–(E) and (2.5) hold, and let one of the conditions (2.6), (2.7), (2.8) hold. If $x(0) \notin (\Delta_\alpha \cup \Delta_\beta)$, then a.s. in A_j , either $\lim_{t \rightarrow \infty} \phi(t) = \alpha$ or $\lim_{t \rightarrow \infty} \phi(t) = \beta$.*

The proof is similar to the proof of Lemma 2.1. One takes $f(x) = A_1 \log(x - \alpha) + B_1$ in $(\alpha, \alpha + \epsilon_0)$, $f(x) = A_2 \log(\beta - x) + B_2$ in $(\beta - \epsilon_0, \beta)$. In case (2.8) holds one takes g_2 to be the function occurring in the proof of Theorem 4.2 in [3].

The case $b(\alpha) = 0, b(\beta) = 0, Q_\alpha > 1, Q_\beta < 1$ will not be considered in this paper. In this case $\phi(t)$ may oscillate between α and β without having a limit, as suggested by the case of linear equations [3].

3. Behavior of solutions in general domains. We shall now extend the results of §2 to a general bounded domain G . A point x_0 on the boundary ∂G of G is said to belong to Σ_3 if $\sum_{i,j} a_{ij}(x_0)\nu_i\nu_j > 0$. It belongs to Σ_2 if (1.2) and

$$\sum_i \left[b_i - \frac{1}{2} \sum_j \frac{\partial a_{ij}}{\partial x_j} \right] \nu_i > 0 \text{ at } x_0$$

hold. Finally, x_0 belongs to Σ_1 if (1.2), (1.3) hold at x_0 .

Denote by $R(x)$ a continuous function in \bar{G} , C^2 and positive in $G \cup \Sigma_2 \cup \Sigma_3$, that coincide with $\text{dist}(x, \Sigma_1)$ when the latter is sufficiently small. With $R(x)$ fixed from now on, we define $\mathfrak{A}, \mathfrak{B}, Q$ as in §1.

We shall need the following assumption:

(P) ∂G consists of a finite number of curves $\Gamma_1, \dots, \Gamma_q$. Each curve belongs entirely to either $\Sigma_2 \cup \Sigma_3$ or to Σ_1 . A curve Γ_j of $\Sigma_2 \cup \Sigma_3$ is in C^2 , and a curve Γ_j of Σ_1 is in C^3 . There is a positive constant μ such that if a curve Γ_i belongs to Σ_1 then either (i) $Q(x) \leq -\theta_0 < 0$ (θ_0 constant) for all $x \in G$ whose distance to Γ_i is $\leq \mu$ [we then say that Γ_i belongs to Σ_1^-], or (ii) $Q(x) \geq \theta_0 > 0$ (θ_0 constant) for all $x \in G$ whose distance to Γ_i is $\leq \mu$ [we then say that Γ_i belongs to Σ_1^+]. Finally, Σ_1^- is nonempty.

We shall maintain the assumptions (A), drop the assumption (B), and replace (C) by

$$(C^*) \mathfrak{A}(x) > 0 \text{ for all } x \in G \text{ with } \text{dist}(x, \Sigma_1) \geq \mu, \nabla_x R(x) \neq 0;$$

$$\sum_{i,j=1}^2 a_{ij}(x) \frac{\partial^2 R}{\partial x_i \partial x_j} < 0 \text{ for all } x \in G \text{ with } \text{dist}(x, \Sigma_1) \geq \mu,$$

$$\nabla_x R(x) = 0.$$

By slightly modifying the construction of $R(x)$ in the proof of Lemma 2.1 of [4], one can show that if the exterior boundary of G is not in Σ_1 then there actually exists a function $R(x)$ with $\nabla_x R(x) \neq 0$ everywhere in G .

The proof of Theorem 2.2 of [4] can be modified to yield the following extension of Lemma 1.1.

Theorem 3.1. *Let (A), (P), (C*) hold. Then, with probability 1, either (i) $x(t)$ exits G in finite time by crossing $\Sigma_2 \cup \Sigma_3$, or (ii) $x(t) \in G$ for all $t > 0$ and $\text{dist}(x(t), \Sigma_1^-) \rightarrow 0$ as $t \rightarrow \infty$.*

Suppose for definiteness that $\Gamma_1 \subset \Sigma_1^-$, and Γ_1 is not the outer boundary of G . If $x_1 = f(\tau)$, $x_2 = g(\tau)$ are parametric equations for Γ_1 ($\tau =$ length parameter), then we can introduce new variables

$$y_1 = (1 + \rho) \cos(2\pi\tau/L), \quad y_2 = (1 + \rho) \sin(2\pi\tau/L) \quad (L = \text{length of } \Gamma_1)$$

where the "polar coordinates" ρ, τ are defined by

$$x_1 = f(\tau) + \rho \dot{g}(\tau), \quad x_2 = g(\tau) - \rho \dot{f}(\tau).$$

As in [4] we can extend this mapping into a diffeomorphism from the exterior of Γ_1 onto the set $\{y: |y| \geq 1\}$. In the new coordinates

$$d\rho = \sum_{s=1}^n \tilde{\sigma}_s dw^s + \tilde{b} dt,$$

$$d\phi = \sum_{s=1}^n \tilde{\sigma}_s dw^s + \tilde{b} dt \quad \left(\phi = \frac{2\pi r}{L} \right),$$

and

$$\frac{L}{2\pi} \tilde{\sigma}_s(0, \phi) = f\sigma_{1s} + \dot{g}\sigma_{2s},$$

$$\frac{L}{2\pi} \tilde{b}(0, \phi) = (\dot{j}b_1 + \dot{g}b_2) - (\dot{g}, -\dot{j}) \begin{pmatrix} \Sigma\sigma_{1s}^2 & \Sigma\sigma_{1s}\sigma_{2s} \\ \Sigma\sigma_{1s}\sigma_{2s} & \Sigma\sigma_{2s}^2 \end{pmatrix} \begin{pmatrix} \dot{j} \\ \dot{g} \end{pmatrix}.$$

Set $\sigma(\phi) = \{\sum_{s=1}^n [\tilde{\sigma}_s(0, \phi)]^2\}^{1/2}$, $b(\phi) = \tilde{b}(0, \phi)$.

We now assume:

(D') The condition (D) holds with $r = 1 + \rho$. More precisely: $\sigma(z)$ vanishes at a finite number of points $\alpha_1, \dots, \alpha_k$ ($k \geq 1$). If $b(\alpha_j) = 0$ for some j , then there is a simple C^3 curve $\Delta_{\alpha_j}^1$ given by $x = x^*(t)$ ($t_1 \leq t \leq t_2$) such that $x^*(t_1) = (f(\alpha_j), g(\alpha_j))$, $x^*(t_2)$ lies outside \bar{G} , and $x^*(\tilde{t})$, for some $\tilde{t} \in (t_1, t_2)$, lies on ∂G and is different from $(f(\alpha_j), g(\alpha_j))$, and such that

(i) a part $\{x^*(t), t_1 < t \leq t_1 + \epsilon\}$ of $\Delta_{\alpha_j}^1$ lies in G and is nontangential to ∂G at $t = t_1$;

(ii) the relations (1.8) hold along $\Delta_{\alpha_j}^1$.

Finally, if $b(\alpha_j) = b(\alpha_b) = 0$ and $b(\alpha_i) \neq 0$ for all the α_i between α_j and α_b , then points of G corresponding to (ρ, ϕ) with $\rho = \epsilon$, ϕ in the interval (α_j, α_b) [$\epsilon > 0$ small], cannot be connected (in G) to points of G corresponding to (ρ, ϕ) with $\rho = \epsilon$, ϕ outside the interval (α_j, α_b) , without crossing either $\Delta_{\alpha_j}^1$ or $\Delta_{\alpha_b}^1$.

(E') The condition (E) holds. Furthermore, $b(\alpha_j) = 0$ for at least one value of j .

Suppose (A), (P), (C*) and (D'), (E') hold. Denote by A^1 the set where $x(t) \in G$ for all $t > 0$ and $\text{dist}(x(t), \Gamma_1) \rightarrow 0$. Let A_j^1 be the subset of A^1 for which $\alpha_{j,1} < \phi(t) < \alpha_{j,k_j}$ holds for all t sufficiently large. Suppose a portion of each $\Delta_{\alpha_j}^1$ initiating at $(f(\alpha_j), g(\alpha_j))$ coincides with the normal to ∂G at that point. Then the proof of Lemmas 2.1–2.3 remain valid (in the y -coordinates). Here we use the fact that the diffeomorphism $x \rightarrow y$ given above does not affect the condition (D'), i.e., the conditions in (1.8) are invariant under a diffeomorphism. If $\Delta_{\alpha_j}^1$ does not contain the normal, then we perform a different local diffeomorphism from the x -space onto the y -space, such that Γ_1 is mapped onto the unit circle

and such that the image of a portion of $\Delta_{\alpha_j}^1$ does coincide with the normal to this circle. The new diffeomorphism does not affect the tangential stochastic equation, i.e., the functions $\sigma(\phi), b(\phi)$ remain the same.

We conclude: If $x \notin \bigcup_j \Delta_{\alpha_j,1}$ then almost surely on A_j^1 , either

- (i) $\phi(t) \rightarrow \beta$ if (2.1), (2.2) hold; or
- (ii) $\phi(t) \rightarrow \alpha$ if (2.3), (2.4) hold; or
- (iii) $\phi(t) \rightarrow \alpha$ or $\phi(t) \rightarrow \beta$ if (2.5) and one of the conditions (2.6), (2.7), (2.8) hold.

In what follows we assume:

(Q) For each block B_j , either (2.1), (2.2) or (2.3), (2.4) or (2.5) and one of the conditions (2.6), (2.7), (2.8) hold.

The segment $\{(\rho, \phi); \phi = \alpha_{j,1}, 0 \leq \rho \leq \bar{\epsilon}\}$ in the y -space is mapped onto an arc l_j in the x -space. l_j initiates at a point γ_{j1} on Γ_1 , is nontangential to Γ_1 at γ_{j1} , it is contained in $\Delta_{\alpha_{j,1}}^1$, and it lies in the interior of G (with the exception of its endpoint γ_{j1}). It divides a small G -neighborhood N_j of γ_{j1} into domains: N_{j1}^+ and N_{j1}^- .

Definition. The point γ_{j1} is called a *distinguished boundary point* of G if at the corresponding point $\alpha_{j,1}, Q_{\alpha_{j,1}} < 1$.

If $\Gamma_1 \subset \Sigma_1^-$ and the interior of Γ_1 contains G , then the above considerations remain valid with trivial changes; the assertions (i)–(iii) are unchanged.

Consider now the general case. We index the Γ_i so that

$$\begin{aligned} \Sigma_1^- &= \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_p, \\ \Sigma_1^+ &= \Gamma_{p+1} \cup \dots \cup \Gamma_{p+b}, \\ \Sigma_2 \cup \Sigma_3 &= \Gamma_{p+b+1} \cup \dots \cup \Gamma_q. \end{aligned}$$

We assume

- (D*) The condition (D') holds for each $\Gamma_j, 1 \leq j \leq p$.
- (E*) The condition (E') holds for each $\Gamma_j, 1 \leq j \leq p$.
- (Q*) The condition (Q) holds for each $\Gamma_j, 1 \leq j \leq p$.

We define distinguished boundary points on $\Gamma_2, \dots, \Gamma_p$ in the same way as for Γ_1 . Denote by $\zeta_j (1 \leq j \leq k)$ the set of all distinguished boundary points on Σ_1^- . With each ζ_i we associate two "half" G -neighborhoods N_j^+, N_j^- of ζ_j , in the same way that we have associated N_{j1}^+, N_{j1}^- with γ_{j1} .

In the condition (D*) there appear curves $\Delta_\alpha^l (1 \leq l \leq p)$ defined analogously to the curves Δ_α^1 . Denote these curves by $\Delta_j (1 \leq j \leq l)$ and set $\Lambda_j = \Delta_j \cap \bar{G}$. Each ζ_i is an endpoint of some Λ_j .

We sum up the previous considerations in the following theorem.

Theorem 3.2. *Let the conditions (A), (P), (C*) and (D*), (E*), (Q*) hold. If $x \notin \bigcup_{j=1}^l \Lambda_j$, then the probability space is a finite disjoint union*

$$(3.1) \quad \Omega_0 \cup \left(\bigcup_{j=1}^k \Omega_j^+ \right) \cup \left(\bigcup_{j=1}^k \Omega_j^- \right),$$

such that the following holds almost surely: if $\omega \in \Omega_0$, $x(t)$ exits from G in finite time by crossing $\Sigma_2 \cup \Sigma_3$; if $\omega \in \Omega_j^+$ then $x(t) \in G$ for all $t > 0$, and $x(t) \in N_j^+$, $x(t) \rightarrow \zeta_j$ as $t \rightarrow \infty$; if $\omega \in \Omega_j^-$ then $x(t) \in G$ for all $t > 0$, and $x(t) \in N_j^-$, $x(t) \rightarrow \zeta_j$ as $t \rightarrow \infty$. The decomposition (3.1) depends on $x(0)$.

Definition. If $x(t) \in G$ for all $t > 0$, and $x(t) \in N_i^+$, $x(t) \rightarrow \zeta_i$ as $t \rightarrow \infty$, then we shall write: $x(t) \rightarrow \zeta_i^+$ as $t \rightarrow \infty$. Similarly we define the concept: $x(t) \rightarrow \zeta_i^-$ as $t \rightarrow \infty$. We denote by $p_i^+(x)$ ($p_i^-(x)$) the probability that $x(t) \rightarrow \zeta_i^+$ ($x(t) \rightarrow \zeta_i^-$) as $t \rightarrow \infty$, given $x(0) = x \in G$.

Clearly $p_i^+(x) \geq 0$, $p_i^-(x) \geq 0$, $\sum_{i=1}^k p_i^+(x) + \sum_{i=1}^k p_i^-(x) \leq 1$. If $\Sigma_2 \cup \Sigma_3$ is empty, then the last sum is equal to 1 (by Theorem 3.2) if $x \notin \bigcup_{j=1}^l \Lambda_j$.

Definition. Denote by $q_i(x)$ ($1 \leq i \leq p$) the probability that $x(t) \in G$ for all $t > 0$ and $\text{dist}(x(t), \Gamma_i) \rightarrow 0$ as $t \rightarrow \infty$, given $x(0) = x \in G$.

Theorem 3.3. *Let the conditions (A), (P), (C*) hold. Then $q_i(x) \rightarrow 1$ if $\text{dist}(x, \Gamma_i) \rightarrow 0$ ($1 \leq i \leq p$).*

Proof. For any $\lambda > 0$ sufficiently small, let $\Gamma_{i\lambda}$ be the curve in G parallel to Γ_i at a distant λ . Denote by G_λ the domain bounded by Γ_i , $\Gamma_{i\lambda}$. Denote by L the elliptic operator corresponding to the diffusion process (1.1). Then

$$L[R(x)]^\epsilon = [\epsilon^2 \mathcal{Q}/2R^2 + \epsilon Q][R(x)]^\epsilon \quad (\epsilon > 0).$$

Since $\mathcal{Q} = O(R^2)$ in G_λ , $LR \leq 0$ in G_λ provided λ and ϵ are sufficiently small. Denote by τ_λ the exit time from G_λ . Then, by Itô's formula,

$$E[R(x(\tau_\lambda))]^\epsilon - [R(x)]^\epsilon = E \int_0^{\tau_\lambda} L[R(x(r))]^\epsilon dr \leq 0.$$

Since $x(\tau_\lambda) \in \Gamma_{i\lambda}$, $R(x(\tau_\lambda)) = \lambda$. Hence

$$[1 - q_i(x)]\lambda^\epsilon \leq [R(x)]^\epsilon = [\text{dist}(x, \Gamma_i)]^\epsilon,$$

and the assertion follows.

The above proof is valid also in any number of dimensions.

4. **Regularity of the functions $p_i^\pm(x)$.** Let

$$(4.1) \quad Lu \equiv \frac{1}{2} \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^2 b_i(x) \frac{\partial u}{\partial x_i}.$$

Let $\Lambda'_1, \dots, \Lambda'_l$ be disjoint C^3 curves (the endpoints are included) lying in G , and set

$$\Lambda = \bigcup_{i=1}^l \Lambda_i, \quad \Lambda' = \bigcup_{i=1}^{l'} \Lambda'_i.$$

The elliptic operator L will be allowed to degenerate in G only at the points of $\Lambda \cup \Lambda'$. We shall call Λ_i a "boundary spoke" and Λ'_j an "interior spoke."

Consider a parametric representation for Λ'_j :

$$(4.2) \quad x_1 = f(s), \quad x_2 = g(s) \quad (0 \leq s \leq L_j)$$

where s is the length parameter. Let

$$(4.3) \quad b_j^*(s) = (\dot{j}b_2 - \dot{g}b_1) - \frac{1}{2}(\dot{j}\bar{g} - \dot{g}\bar{j}) \sum_{r=1}^n (\dot{j}\sigma_{1r} + \dot{g}\sigma_{2r})^2$$

where the argument of b_j, σ_{ir} is (x_1, x_2) given by (4.2).

We shall need the following assumption regarding the degeneracy of L in G :

(G) On each $\Lambda'_j, b_j^*(s) \neq 0$ for $0 \leq s \leq L_j$. The elliptic operator L may degenerate on each arc Λ_i , and in a sufficiently small δ_0 -neighborhood of each Λ'_j ; at all the remaining points of G, L is nondegenerate.

The number δ_0 is positive and depends only on upper bounds on the first derivatives of σ_{ik}, b_i , and on a positive lower bound on the $|b_j^*(s)|$. Its precise nature will emerge from the proof of Theorem 4.1 below. Denote by Λ'_{δ_0} the δ_0 -neighborhood of Λ' .

Theorem 4.1. *Let the conditions (A), (P), (C*) and (G) hold. Then $p_i^\pm(x)$ ($1 \leq i \leq p$) are Lipschitz continuous functions in $G - \Lambda$, and C^2 solutions of $Lu = 0$ in $G - (\Lambda \cup \Lambda'_{\delta_0})$.*

Proof. We shall combine classical regularity theorems with the method of Freidlin [1]. Denote by $p_i(x)$ any one of the functions $p_i^\pm(x)$. Consider first a point $x_0 \in G$ where L is nondegenerate. Let N be a small disc with center x_0 such that L is nondegenerate in \bar{N} . By the strong Markov property, for any $x \in N$,

$$(4.4) \quad p_i(x) = E_x \{ p_i(x_{\tau_N}) \} = \int_{\partial N} p_i(y) P_x(x_{\tau_N} \in dS_y)$$

where τ_N is the exit time from N , and dS_y is the length element on ∂N . Note, by a standard argument, that $p_i(y)$ is a Borel function on ∂N .

Let A be an interval on ∂N . Denote by I_A the characteristic function of A , and by η, ζ the endpoints of A . By classical theorems [2], there exists a unique solution u of

$$\begin{aligned}
 Lu &= 0 \text{ in } N, \quad u \in C^2(N), \\
 u(x) &\rightarrow I_A(y) \text{ if } x \rightarrow y \in \partial N, \quad y \neq \eta, \quad y \neq \zeta, \\
 u(x) &\text{ remains bounded as } x \rightarrow \eta, \text{ or } x \rightarrow \zeta.
 \end{aligned}$$

We can write u in terms of Green's function [2]

$$(4.5) \quad u(x) = \int_A \frac{\partial G(x, y)}{\partial \nu_y} dS_y$$

where ν_y is the inward normal. Denote by N_ϵ the disc with center x_0 and radius $= r_0 - \epsilon$, where r_0 is the radius of N . Let τ_{N_ϵ} be the exit time from N_ϵ . By Itô's formula,

$$(4.6) \quad u(x) = E_x \{ u(x_{\tau_{N_\epsilon}}) \} \quad (\epsilon > 0).$$

Since L is nondegenerate in \bar{N} , $x(t)$ exists N at ζ or at η with probability 0. Hence, taking $\epsilon \rightarrow 0$ in (4.6), we arrive at the formula

$$u(x) = E_x \{ u(x_{\tau_N}) \} = E_x \{ I_A(x_{\tau_N}) \} = P_x(x_{\tau_N} \in A).$$

Comparing this with (4.5), we conclude that

$$P_x(x_{\tau_N} \in A) = \int_A \frac{\partial G(x, y)}{\partial \nu_y} dS_y.$$

This implies that

$$P_x(x_{\tau_N} \in dS_y) = \frac{\partial G(x, y)}{\partial \nu_y} dS_y.$$

Hence (4.4) gives

$$(4.7) \quad p_i(x) = \int_{\partial N} p_i(y) \frac{\partial G(x, y)}{\partial \nu_y} dS_y.$$

This shows that $p_i(x)$ is continuous in N . By decreasing N we may assume that $p_i(x)$ is continuous in \bar{N} .

The solution v of $Lv = 0$ in N , $v = p_i$ on ∂N is also given by the right-hand side of (4.7). Hence $v = p_i$ in N . Since v belongs to $C^2(N)$, the same is true of p_i . This completes the proof of the second assertion of Theorem 4.1.

To prove the first assertion, consider first the case of an interior spoke Λ'_j having the form $\phi = \phi_0, r_0 \leq r \leq r_1$. Let B_δ be the domain $|\phi - \phi_0| < \delta, r_0 < r < r_1$. The condition $b_j^*(s) \neq 0$ reduces to $\tilde{b}(r, \phi) \neq 0$ where $\tilde{b}(r, \phi)$ is defined as in §1. Suppose, for definiteness, that $\tilde{b}(r, \phi) \geq \beta > 0$ inside B_δ . By Itô's formula we have

$$\phi(t) = \phi_0 + \sum_{s=1}^n \int_0^{t \wedge \tau_{B_\delta}} \tilde{\sigma}_s(r, \phi) dw^s + \int_0^{t \wedge \tau_{B_\delta}} \tilde{b}(r, \phi) dr,$$

and hence

$$\phi(t \wedge \tau_{B_\delta}) \geq \phi_0 + \sum_{s=1}^n \int_0^{t \wedge \tau_{B_\delta}} \tilde{\sigma}_s(r, \phi) dw^s + \beta(t \wedge \tau_{B_\delta}).$$

Thus

$$\beta E_x(t \wedge \tau_{B_\delta}) \leq \sup_{\phi \in \tilde{B}_\delta} |\phi - \phi_0| = \delta.$$

It follows that

$$(4.8) \quad E_x(\tau_{B_\delta}) \leq \delta/\beta = C \quad (x \in B_\delta).$$

By a standard iteration argument it follows that $P_x(\tau_{B_\delta} \geq nt_0) \leq (C/t_0)^n$ ($n = 1, 2, \dots$). Consequently,

$$P_x(\tau_{B_\delta} \geq t) \leq e^{-\alpha t}, \quad \alpha = -(1/t_0) \log(C/t_0).$$

Taking $t_0 = e$ we get

$$(4.9) \quad P_x(\tau_{B_\delta} \geq t) \leq e^{-\alpha t}, \quad \alpha = (1/e) \log(\beta e/\delta).$$

We may choose $\delta > 0$ sufficiently small to apply the following result of Freidlin [1, p. 1349] (which we state only in R^2):

Theorem. Suppose σ_{ij}, b_i are continuously differentiable in R^2 with

$$\max_{i,j,k} \left\{ \left| \frac{\partial \sigma_{ij}}{\partial x_k} \right|, \left| \frac{\partial b_i}{\partial x_k} \right| \right\} = K.$$

Let $\alpha_1 = 8K^2 + 4K$. Suppose the boundary is uniformly normally regular, and the boundary function ψ is the restriction to ∂B_δ of a C^2 function in a neighborhood of ∂B_δ . Then the function $E_x\{\psi(x_{\tau_{B_\delta}})\}$ is Lipschitz continuous in B_δ , provided $\alpha > \alpha_1$.

By choosing δ_0 (in the condition (G)) sufficiently small we can ensure that, for some $\delta > \delta_0$, L is nondegenerate on the boundary of B_δ , and $\alpha > \alpha_1$. The uniform normal regularity of ∂B_δ means that $E_x(\tau_{B_\delta}) \leq C_0|x - x_0|$ for all $x_0 \in \partial B_\delta, x \in B_\delta$ where C_0 is a constant. This property is guaranteed by the nonvanishing of the normal diffusion on ∂B_δ (see [1]). Further, since L is nondegenerate on $\partial B_\delta, p_i(x)$ is C^2 in a neighborhood of ∂B_δ . Hence we can apply Freidlin's theorem to deduce (upon recalling the first equation of (4.4), which holds for $N = B_\delta$) that $p_i(x)$ is Lipschitz continuous in B_δ .

To handle the case of a general spoke Λ'_j , we introduce new coordinates (ρ, s) by the equations

$$x_1 = f(s) + \rho \dot{g}(s), \quad x_2 = g(s) - \rho \dot{f}(s)$$

where $-\rho_0 < \rho < \rho_0$ (ρ_0 is positive and sufficiently small) and f, g are as in (4.2). The stochastic differentials $ds, d\rho$ are given by the formulas

$$d\rho = \sum_{r=1}^n \tilde{\sigma}_r dw^r + \tilde{b} dt,$$

$$d\phi = \sum_{r=1}^n \tilde{\sigma}_r^* dw^r + \tilde{b}^* dt \quad (\phi = 2\pi s/L_j).$$

Explicit calculation gives (cf. [4]) $\tilde{b}(0, \phi) = 2\pi b_j^*(s)/L_j$, where $b_j^*(s)$ is defined in (4.3). Since $b_j^*(s) \neq 0$, we can repeat the argument given in the previous special case.

Remark. Suppose σ_{ij}, b_i belong to $C^m(R^2)$. Using Theorem 3 of Freidlin [1] (instead of the above quoted theorem of [1]) we conclude that the $p_i^\pm(x)$ have $m-1$ Lipschitz continuous derivatives in $G - \Lambda$. Here the constant δ_0 occurring in the condition (G) depends also on m .

Definition. If $x \rightarrow \zeta_i^+$, $x \in N_i^+$ then we write $x \rightarrow \zeta_i^+$. Similarly we write $x \rightarrow \zeta_i^-$ if $x \rightarrow \zeta_i^-$, $x \in N_i^-$.

Theorem 4.2. Let the conditions (A), (P), (C*) and (D*), (E*), (Q*) hold. Then $p_i^+(x) \rightarrow 1$ if $x \rightarrow \zeta_i^+$, and $p_i^-(x) \rightarrow 1$ if $x \rightarrow \zeta_i^-$.

This theorem is of the same type as Theorem 3.3. The method of proof is also the same as for Theorem 3.3.

Proof. It suffices to prove the assertion for $p_i^+(x)$. Consider first the special case where the distinguished boundary point ζ_i lies in some Γ_j , say Γ_1 , which is the circle $r=1$, and G lies in the exterior of Γ_1 . Let N be "half G -neighborhood" of ζ_i given by $\zeta_i < \phi < \phi_1$, $1 < r < 1 + \delta$. Consider the function

$$f(R, \phi) = R^\epsilon + (\phi - \zeta_i)^\epsilon \quad \text{in } N$$

where $R = r - 1$, and $\epsilon > 0$ is sufficiently small. It is easily seen that $Lf \leq 0$ if δ and ϵ are sufficiently small. Let τ be the exit time from N . By Itô's formula

$$(4.10) \quad E\{f(R(t \wedge \tau), \phi(t \wedge \tau))\} \leq f(R(0), \phi(0)) \quad (t > 0).$$

Now, $x(t)$ cannot leave N in finite time by crossing either $R=0$ or $\phi = \zeta_i$. On the other hand, on the remaining boundary of N , $f(R, \phi)$ is bounded below by some constant $\gamma > 0$ (γ depends on δ, ϵ). Hence, taking $t \rightarrow \infty$ in (4.10), we obtain the inequality

$$\gamma P_x(\tau < \infty) \leq f(R(0), \phi(0)) \quad (x = x(0)).$$

Since $f(R(0), \phi(0)) \rightarrow 0$ if $x \rightarrow \zeta_i^+$, we conclude that

$$P_x(\tau < \infty) \rightarrow 0 \text{ if } x \rightarrow \zeta_i^+.$$

Since, by Theorem 3.2, $p_i^+(x) = 1 - P_x(\tau < \infty)$, the proof is complete.

Remark. By Theorem 4.2, the $p_i^\pm(x)$ are discontinuous at the points of the boundary spoke initiating at ζ_i , which are in some small neighborhood of ζ_i .

5. The Dirichlet problem.

Lemma 5.1. Let $x(t) = (x_1(t), \dots, x_l(t))$ be a solution of a system of l stochastic equations of the form (1.1), with uniformly Lipschitz continuous coefficients σ_{ij}, b_i . Let τ be any finite valued random variable. Suppose the range of $x(t), 0 \leq t \leq \tau$, is contained in an open set $D \subset R^l$. Let $f(x)$ be a C^2 function in D . Then Itô's formula holds:

$$f(x(\tau)) - f(x(0)) = \sum_{i=1}^l \sum_{j=1}^n \int_0^\tau f_{x_i}(x(s)) \sigma_{ij}(x(s)) dw^j + \int_0^\tau Lf(x(s)) ds$$

where

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^l a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^l b_i(x) \frac{\partial f}{\partial x_i} \quad [(a_{ij}) = \sigma \sigma^*].$$

Proof. For any $\delta > 0$, modify f into a function $f^\delta(x)$ in $C^2(R^l)$, coinciding with $f(x)$ if $\text{dist}(x, R^l - D) > \delta$. Apply Itô's formula to $f^\delta(x(t))$, substitute $t = \tau$, and take $\delta \searrow 0$.

Now let the assumptions of Theorem 4.2 hold. Consider the Dirichlet problem

$$(5.1) \quad Lu = 0 \quad \text{in } G - \Lambda,$$

$$(5.2) \quad u = g \quad \text{on } \Sigma_2 \cup \Sigma_3,$$

$$(5.3) \quad \begin{cases} u(x) \rightarrow f_i^+ & \text{if } x \rightarrow \zeta_i^+, \\ u(x) \rightarrow f_i^- & \text{if } x \rightarrow \zeta_i^- \quad (1 \leq i \leq k). \end{cases}$$

Here L is defined by (4.1), g is a given continuous function on $\Sigma_2 \cup \Sigma_3$, and f_i^\pm are given numbers.

If u is continuous in $(G \cup \Sigma_2 \cup \Sigma_3) - \Lambda$ and is in $C^2(G - \Lambda)$, and if it satisfies (5.1)–(5.3), then we call it a *classical solution* of the Dirichlet problem (5.1)–(5.3). Notice that u is not required to be continuous on $\Sigma_1^- \cup \Sigma_1^+ \cup \Lambda$. Since $u(x) \rightarrow f_i^\pm$ as $x \rightarrow \zeta_i^\pm$, u cannot be continuous at the points of Λ near ζ_i if $f_i^+ \neq f_i^-$.

We shall prove in this section the existence and uniqueness of a classical solution.

Theorem 5.2. *Let (A), (P), (C*) and (D*), (E*), (Q*) hold. Then there exists at most one classical solution of the Dirichlet problem.*

Proof. Let $G_\epsilon = \{x \in G, \text{dist}(x, \partial G) > \epsilon\}$, $\epsilon > 0$. Denote by τ the exit time from G , and denote by τ_ϵ the exit time from G_ϵ . Let u be a classical solution. Since $x(t)$ (with $x(0) \in G - \Lambda$) remains in $G - \Lambda$ for $0 \leq t \leq T \wedge \tau_\epsilon$, where $T < \infty$, $\epsilon > 0$, we can apply Lemma 5.1. This gives, after taking the expectation,

$$u(x) = E_x\{u(x(T \wedge \tau_\epsilon))\}.$$

Taking $\epsilon \searrow 0$, $T \nearrow \infty$ and using the continuity of u at $\Sigma_2 \cup \Sigma_3$ and Theorem 3.2, we get

$$(5.4) \quad u(x) = E_x\{g(x(\tau))I_{(\tau < \infty)}\} + \sum_{i=1}^k f_i^+ p_i^+(x) + \sum_{i=1}^k f_i^- p_i^-(x)$$

where I_A is the indicator function of A . This implies that $u(x)$ is uniquely determined (in $G - \Lambda$).

We shall now prove the existence of a solution.

Theorem 5.3. *Let (A), (P), (C*), (G), (D*), (E*), (Q*) hold, and let g be continuous on $\Sigma_2 \cup \Sigma_3$. Then the function $u(x)$ given by (5.4) is continuous in $(G \cup \Sigma_2 \cup \Sigma_3) - \Lambda$, Lipschitz continuous in $G - \Lambda$, and C^2 in $G - (\Lambda \cup \Lambda'_{\delta_0})$, and it satisfies (5.1) in $G - (\Lambda \cup \Lambda'_{\delta_0})$ and (5.2), (5.3).*

Proof. In Theorem 4.1 we proved that $p_i^\pm(x)$ is Lipschitz continuous in $G - \Lambda$, and is a C^2 solution of (5.1) in $G - (\Lambda \cup \Lambda'_{\delta_0})$. The same proof works also for the function $E_x\{g(x(\tau))I_{(\tau < \infty)}\}$. Hence, the function u , given by (5.4), is Lipschitz continuous in $G - \Lambda$ and is a C^2 solution of (5.1) in $G - (\Lambda \cup \Lambda'_{\delta_0})$. The assertion $u(x) \rightarrow f_i^\pm$ as $x \rightarrow \zeta_i^\pm$ follows from Theorem 4.2 and the fact that

$$\sum_{j=1}^k p_j^+(x) + \sum_{j=1}^k p_j^-(x) + E_x\{I_{(\tau < \infty)}\} = 1.$$

(This is the assertion of Theorem 3.2.) Finally, the assertion that $u(x)$ is continuous at the points of $\Sigma_2 \cup \Sigma_3$ and it satisfies (5.2) follows from Theorem 2 of Pinsky [6].

Remark. The function $u(x)$ is a weak solution of (5.1) in G , in the sense that

$$u(x) = \int_{\partial N} u(y) P_x(x, \tau_N \in dS_y)$$

where N is a disc in G , $x \in N$, and τ_N is the exit time from N . The proof is the same as for (4.4).

We shall now strengthen the assumptions of Theorem 5.3 in order to achieve a classical solution.

(G*) The condition (G) holds and σ_{ij}, b_i are in $C^2(\Lambda'_{\delta_0})$. The positive constant δ_0 occurring in the condition (G) will now be smaller; it will be as in the remark following the proof of Theorem 4.1, with $m = 2$.

Theorem 5.4. *Let (A), (P), (C*), (G*), (D*), (E*), (Q*) hold, and let g be a continuous function on $\Sigma_2 \cup \Sigma_3$. Then (5.4) is the unique classical solution of the Dirichlet problem (5.1)–(5.3).*

Indeed, we only have to verify that u is in $C^2(G - \Lambda)$ and $Lu = 0$ in $G - \Lambda$. For $p_i^\pm(x)$ this follows from the remark following the proof of Theorem 4.1. For $E_x\{g(x(\tau))I_{(\tau < \infty)}\}$ the proof is the same.

Remark. Theorems 5.2–5.4 extend to the Dirichlet problem consisting of (5.5)

$$Lu + c(x)u = 0 \quad \text{in } G - \Lambda$$

and (5.2), (5.3), provided $c(x) \leq 0$ in G . Instead of (5.4) we have

$$(5.6) \quad \begin{aligned} u(x) = & E_x \left\{ g(x(\tau)) \exp \left[\int_0^\tau c(x(s)) ds \right] I_{(\tau < \infty)} \right\} \\ & + \sum_{i=1}^k f_i^+ E_x \left\{ \exp \left[\int_0^\infty c(x(s)) ds \right] I_{[p_i^+(x) > 0]} \right\} \\ & + \sum_{i=1}^k f_i^- E_x \left\{ \exp \left[\int_0^\infty c(x(s)) ds \right] I_{[p_i^-(x) > 0]} \right\}. \end{aligned}$$

If $c(x) \leq -c_0 < 0$ then the last two sums vanish, so that no boundary conditions on Σ_1 need to be given. This is in agreement with the treatment in [5], [7] (and the references given there) where c_0 is assumed to be positive.

6. The Dirichlet problem in m -dimensional domains. In subsection 6.1 we prove a theorem for $m \geq 2$ which even when $m = 2$ is not contained in §§2–5. In subsections 6.2, 6.3 we discuss the generalizations of the results of §§2–5 to $m \geq 2$.

6.1. Consider a system of m stochastic equations

$$(6.1) \quad dx_i = \sum_{r=1}^n \sigma_{ir}(x) dw^r + b_i(x) dt \quad (1 \leq i \leq m)$$

and let L , given by (0.1), be the corresponding elliptic operator, i.e., $\sigma\sigma^* = (a_{ij})$. We shall denote the analogs of the conditions (A), (P), (C*) for $m \geq 2$ by (A_m) , (P_m) , (C_m^*) respectively. Assuming that these conditions hold, the assertion of Theorem 3.1 remains valid.

With G a bounded m -dimensional domain, and $\Gamma_1, \dots, \Gamma_q$ its boundary hypersurfaces, we index the Γ_i as in §3. Thus, Σ_1^- is made up of $\Gamma_1, \dots, \Gamma_p$. Denote by Γ_i^ϵ ($\epsilon > 0$) the intersection of G with ϵ -neighborhood of Γ_i . We assume

(R) On each Γ_l ($1 \leq l \leq p$) there is a finite number of points ξ_{lj} such that $\sigma_{ir}(\xi_{lj}) = 0$, $b_i(\xi_{lj}) = 0$ for $1 \leq i \leq m$, $1 \leq r \leq n$. Let $R_{lj}(x) = |x - \xi_{lj}|$ if $|x - \xi_{lj}| < \epsilon'$ (for some $\epsilon' > 0$), and define $Q_{lj}(x)$ as $Q(x)$ in §1 when $R(x)$ is replaced by $R_{lj}(x)$. Then

$$Q_{lj}(x) \leq -\theta_0 < 0 \quad \text{if } |x - \xi_{lj}| < \epsilon', \quad x \in G.$$

For any $1 \leq l \leq p$, let $R_l^*(x)$ be a positive C^2 function for $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $x \neq \xi_{lj}$ (for some $\epsilon_0 > 0$) such that $R_l^*(x) = R_{lj}(x)$ if $|x - \xi_{lj}| < \epsilon'$. We shall assume

(S) For all $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $\min_j |x - \xi_{lj}| > \epsilon'$ ($1 \leq l \leq p$),

$$(6.2) \quad \sum_{i,j=1}^m a_{ij}(x) \frac{\partial R_l^*}{\partial x_i} \frac{\partial R_l^*}{\partial x_j} > 0 \quad \text{if } \nabla_x R_l^*(x) \neq 0;$$

$$(6.3) \quad \sum_{i,j=1}^m a_{ij}(x) \frac{\partial^2 R_l^*}{\partial x_i \partial x_j} < 0 \quad \text{if } \nabla_x R_l^*(x) = 0.$$

Notice that $R_l^*(x)$ can be constructed such that $\nabla_x R_l^*(x)$ is nonzero if $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $x \neq \xi_{lj}$, and $\nabla_x R_l^*(x)$ is not normal to Γ_l if $x \in \Gamma_l$, $x \neq \xi_{lj}$. Hence, if L is nondegenerate in $\Gamma_l^{\epsilon_0}$ and if the stochastic equations induced by (6.1) on Γ_l have a nondegenerate diffusion matrix [i.e., if the elliptic operator induced by L on Γ_l is nondegenerate] at each $x \neq \xi_{lj}$, then $\nabla_x R_l^*(x) \neq 0$ if $x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l$, $x \neq \xi_{lj}$ and (6.2) holds.

So far, assuming (A_m) , (P_m) and (C_m^*) , we already know that $x(t)$ does not intersect Σ_1 in finite time, and $\text{dist}(x(t), \Sigma_1^-) \rightarrow 0$ as $t \rightarrow \infty$, provided $x(t) \in G$ for all $t > 0$. We now employ the assumptions (R), (S) to construct a function

$$f(x) = \Phi(R_l^*(x)) \quad (x \in \Gamma_l^{\epsilon_0} \cup \Gamma_l, \quad x \neq \xi_{lj})$$

such that $Lf \leq -\nu$ (as in Theorem 2.2 of [4]), and then use it to deduce that, on the set where $\text{dist}(x(t), \Gamma_l) \rightarrow 0$, $\min_j |x(t) - \xi_{lj}| \rightarrow 0$.

We shall denote the set of all the points ξ_{lj} by ξ_1, \dots, ξ_k , and call them *distinguished boundary points*. We pose the Dirichlet problem

$$(6.4) \quad Lu = 0 \quad \text{in } G,$$

$$(6.5) \quad u = g \quad \text{on } \Sigma_2 \cup \Sigma_3,$$

$$(6.6) \quad u = f_i \quad \text{at } \zeta_i \quad (1 \leq i \leq k)$$

where g is a given continuous function on $\Sigma_2 \cup \Sigma_3$ and the f_i are given numbers.

Theorem 6.1. *Let the conditions (A_m) , (P_m) , (C_m^*) and (R), (S) hold, and let L be nondegenerate in G . Then there exists a unique solution of the Dirichlet problem (6.4)–(6.6).*

In fact, the solution is given by

$$(6.7) \quad u(x) = E_x \{f(x(\tau))I_{(\tau < \infty)}\} + \sum_{i=1}^k f_i p_i(x)$$

where τ is the exit time from G , and $p_i(x)$ is the probability that $x(t) \in G$ for all $t > 0$ and $x(t) \rightarrow \zeta_i$ as $t \rightarrow \infty$, given $x(0) = x$. The regularity of the terms on the right-hand side of (6.7) can be proved as in the case $m = 2$ (in §4).

Remark 1. Theorem 6.1 can be extended to the case where L may degenerate in a small neighborhood of a finite number of "interior spokes," as in the case $m = 2$. This can be proved by the same method as for $m = 2$.

Remark 2. Theorem 6.1 extends to the Dirichlet problem in which (6.4) is replaced by $Lu + c(x)u = 0$ in G , and $c(x) \leq 0$; cf. the remark at the end of §5.

6.2. Let the conditions (A_m) , (P_m) , (C_m^*) hold and consider the Dirichlet problem

$$(6.8) \quad \begin{cases} Lu = 0 & \text{in } G, \\ u = g & \text{on } \Sigma_2 \cup \Sigma_3, \\ u = f_i & \text{on } \Gamma_i \quad (1 \leq i \leq p) \end{cases}$$

where the f_i are constants; the Γ_i ($1 \leq i \leq p$) constitute the Σ_1^- boundary of G . If L is nondegenerate in G , then the function

$$(6.9) \quad u(x) = E_x \{g(x(\tau))I_{(\tau < \infty)}\} + \sum_{i=1}^p f_i q_i(x)$$

is the unique classical solution of the Dirichlet problem (6.8). The proof of uniqueness is the same as the proof of Theorem 5.2. The proof that $u(x)$ is in $C^2(G)$ is the same as the corresponding proof for $p_i^\pm(x)$. The assertion that $u(x) \rightarrow f_i$ as $\text{dist}(x, \Gamma_i) \rightarrow 0$ is a consequence of Theorem 3.3 (which holds in any number of dimensions). Finally the assertion that $u(x) \rightarrow g(y)$ if $x \rightarrow y$, $y \in \Sigma_2 \cup \Sigma_3$ follows from [6, Theorem 2].

6.3. All of the results of §§2-5 can be generalized to the case $m \geq 3$. The conditions needed, however, take a more complicated form. In order to clarify the procedure, we shall describe only a special case, namely, $m = 3$ and G is a shell given by $1 < r < 2$. We further assume that the conditions of Lemma 1.1 hold for $m = 3$ so that the assertion of Lemma 1.1 is valid, with respect to the system in polar coordinates

$$(6.10) \quad \begin{aligned} dr &= \sum_{j=1}^n \tilde{\alpha}_{1j} dw^j + \tilde{b}_1 dt, \\ d\theta &= \sum_{j=1}^n \tilde{\alpha}_{2j} dw^j + \tilde{b}_2 dt, \\ d\phi &= \sum_{j=1}^n \tilde{\alpha}_{3j} dw^j + \tilde{b}_3 dt. \end{aligned}$$

On $\tau = 1$, this system reduces to

$$(6.11) \quad \begin{aligned} d\theta &= \sum_{j=1}^n \tilde{\sigma}_{2j}(1, \theta, \phi) dw^j + \tilde{b}_2(1, \theta, \phi) dt, \\ d\phi &= \sum_{j=1}^n \tilde{\sigma}_{3j}(1, \theta, \phi) dw^j + \tilde{b}_3(1, \theta, \phi) dt. \end{aligned}$$

We shall assume

(T₁) Along the closed curve $\Gamma: (\theta = \theta_0, 0 \leq \phi \leq 2\pi)$ we have

$$\sum_{i,j=2}^3 \tilde{a}_{ij} \nu_i \nu_j = 0, \quad \sum_{i=2}^3 \left[\hat{b}_i - \frac{1}{2} \sum_{j=1}^2 \frac{\partial \tilde{a}_{ij}}{\partial \theta_j} \right] \nu_i = 0$$

when $\theta_1 = \theta, \theta_2 = \phi, (\nu_2, \nu_3)$ is normal to Γ , and $\tilde{a} = \tilde{\sigma} \tilde{\sigma}^*$.

(T₂) The condition (C*) holds with x replaced by (θ_1, θ_2) and $R(x)$ replaced by a positive function $R^*(\theta_1, \theta_2)$ coinciding with the distance function from Γ when the latter is sufficiently small.

(T₃) Define $Q(\theta_1, \theta_2)$ with respect to (6.11) and $R(\theta_1, \theta_2)$ in the same way that $Q(x)$ was defined with respect to the system (1.2) with respect to $R(x)$. Then $Q(\theta_1, \theta_2) \leq -\nu < 0$ ($0 \leq \phi \leq 2\pi, |\theta - \theta_0| < \epsilon'$) for some $\epsilon' > 0$.

(T₄) No solution of (6.10) crosses the conical surface $S: \theta = \theta_0$. This is the case if and only if $\sum_{i,j=1}^3 a_{ij} \nu_i \nu_j = 0, \sum_{j=1}^3 [b_i - \frac{1}{2} \sum_{j=1}^3 \partial a_{ij} / \partial x_j] \nu_i = 0$ on $\theta = \theta_0$, where (ν_i) is the normal to S .

Using the condition (T₁) we can prove (as in [4]) that the solution $(\theta_0(t), \phi_0(t))$ of (6.11) never crosses Γ . Using also the conditions (T₂), (T₃) we can construct a function $V(\theta, \phi)$ for $\theta \neq \theta_0$ such that $LV \leq -\nu < 0$ and $V \rightarrow -\infty$ if $\theta \rightarrow \theta_0$. If we apply Itô's formula to $V(\theta(t), \phi(t))$, where $(\nu(t), \theta(t), \phi(t))$ is a solution of the system (6.10) with $\theta(0) \neq \theta_0$, then we conclude (as in [4]) that

$$(6.12) \quad r(t) \rightarrow 1, \quad \theta(t) \rightarrow \theta_0 \quad (t \rightarrow \infty).$$

Next we consider the restriction of (6.11) to $\theta = \theta_0$, i.e.

$$d\phi = \sum_{j=1}^n \tilde{\sigma}_{3j}(1, \theta_0, \phi) dw^j + \tilde{b}_3(1, \theta_0, \phi) dt$$

and assume

(T₅) Conditions analogous to (D), (E) and (Q) hold for (6.11) with respect to Γ .

Thus, through each point where $\tilde{\sigma}_{3j}(1, \theta_0, \phi) = 0$ ($1 \leq j \leq n$) and $\tilde{b}_3(1, \theta_0, \phi) = 0$ there passes a "boundary spoke" Λ_j lying in the sphere. Λ_j connects a pair of adjacent α_j 's. With the aid of the condition (T₅), we construct a function $f(\phi)$ with $Lf \leq -\nu < 0$ as in §§1 and 2. Applying Itô's formula to $f(\phi(t))$ where $(\nu(t), \theta(t), \phi(t))$ is a solution of (6.10), we can then show that $\phi(t) \rightarrow \alpha_i^\pm$

($1 \leq i \leq k$) with probability $p_i^\pm(x)$ ($\sum_{i=1}^k [p_i^+(x) + p_i^-(x)] = 1$).

The points $\zeta_i = (1, \theta_0, \alpha_i)$ are called *distinguished boundary points*. We can now pose the Dirichlet problem

$$(6.13) \quad \begin{aligned} Lu &= 0 && \text{in } G, \\ u(x) &\rightarrow f_i^\pm, && x \rightarrow \zeta_i^\pm \quad (1 \leq i \leq k). \end{aligned}$$

Theorem 6.2. *Let the assumptions ($T_1 - T_5$) hold and let L be nondegenerate in $G - S$. Then there exists a unique solution of the Dirichlet problem (6.13). It is given by the formula*

$$u(x) = \sum_{i=1}^k f_i^+ p_i^+(x) + \sum_{i=1}^k f_i^- p_i^-(x) \quad (x \in S).$$

This theorem extends easily to general domains G with a $\Sigma_2 \cup \Sigma_3$ boundary component.

A subsequent treatment by one of us (M.P.) shows that condition (D) is superfluous. The details will appear in a forthcoming publication.

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