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CHEBYSHEV CONSTANT AND CHEBYSHEV POINTS

BY

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ABSTRACT. Using λth power means in the case $\lambda \geq 1$, it is proven that the Chebyshev constant for any compact set in R_n, real Euclidean n-space, is equal to the radius of the spanning sphere. When $\lambda > 1$, the Chebyshev points of order m for all $m \geq 1$ are unique and coincide with the center of the spanning sphere. For the case $\lambda = 1$, it is established that Chebyshev points of order m for a compact set E in R_2 are unique if and only if the cardinality of the intersection of E with its spanning circle is greater than or equal to three.

1. **Introduction.** The transfinite diameter and the Chebyshev constant, two set functions originally defined in the complex plane by Fekete [3] using a geometric averaging process, can be generalized by choosing different averaging processes. This approach was used by Pólya and Szegö [7] who considered certain compact sets in R_1, R_2, and R_3 (R_n being real Euclidean n-space) and calculated their transfinite diameters and Chebyshev constants for λth power averages. In this paper we restrict ourselves to the consideration of calculating the Chebyshev constant in the $\lambda \geq 1$ averaging process, but we obtain results valid for *all* compact sets in R_n. Specifically the main theorem in §3 proves that the Chebyshev constant for any compact set in R_n is equal to the radius of the spanning sphere. Moreover it is established there that for the case $\lambda > 1$, the Chebyshev points of order m for all $m \geq 1$ are unique and coincide with the center of the spanning sphere. For the case $\lambda = 1$, the Chebyshev points of order m, $m \geq 1$, are unique in R_2 if and only if the cardinality of the intersection of the compact set and its spanning circle is greater than or equal to three.

To develop the techniques for the proof of the main theorem we are led to a consideration of two topics. One concerns some geometric properties of point sets, and the other deals with a problem known in the literature as Steiner's problem. These are treated in §2. Although the main theorem is proved for R_n, we shall discuss the aforementioned topics in R_2 in order to avoid unnecessary notational complications, noting the modifications necessary to extend the concepts to higher dimensions. Theorems denoted by letter are well known;

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theorems denoted by number are the work of the author.

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2. Geometric preliminaries and Steiner's problem.

I. Let E be a compact(1) set in R_2. The boundary of the disc of smallest radius which contains E will be called the *spanning circle of* E and will be denoted by C_E.

Theorem A. *The spanning circle of E is unique and it either* (1) *contains two boundary points of E which are at the ends of a diameter of the circle, or* (2) *contains three boundary points of E which form an acute triangle.*

Theorem A may be found as a problem (with solution) in Yaglom [8].

If R denotes the radius of C_E, we have

$$d(E) \leq 2R \tag{1}$$

where $d(E)$ denotes the diameter of E. Thus $d(E)/2$ provides a lower bound for R. An upper bound was given by H. W. E. Jung [5].

$$R \leq d(E)\sqrt{3}/3 \tag{2}$$

We note that we have equality in (1) whenever the cardinality of $E \cap C_E$ is two, and equality in (2) whenever E is an equilateral triangle.

The concept of a spanning circle can be generalized to n dimensions by consideration of the n-dimensional ball of smallest radius which contains a compact set E. The boundary of such a ball shall be called the *spanning sphere of* E, and denoted by S_E.

The analogue of Theorem A can be stated as follows:

Theorem 1. *Let E be a compact set in R_n. S_E is unique and its center $0'$ is in the convex hull of $E \cap S_E$.*

Uniqueness can be established without too much difficulty. The second assertion follows from showing that if S is a sphere which encloses E with H as its center, then if H is not in the convex hull of $E \cap S$, the elements of $E \cap S$ all lie in some open hemisphere of S. From this one can easily show that S cannot be the spanning sphere of E, for a small translation of S could "detach" the points of E from S and hence $E \cap S$ would be empty. Thus there would exist a sphere of smaller radius which encloses E.

(1) This definition, as well as Theorem A is valid for point sets which are merely bounded, but we shall always restrict our discussion to compact sets.

We shall be interested in the set $E \cap S_E$. In R_2, Theorem A tells us that this set contains either a pair of diametrically opposite points, or the vertices of an acute triangle. Theorem 1 does not allow us to focus in the same way on some finite subset of this set. However we state here the following theorem by Carathéodory.

Theorem B. *Let Q be a subset of R_n and $0'$ a point in the convex hull of Q. Then there exists a subset Q_1 of Q, containing at most $n+1$ points such that $0'$ is in the convex hull of Q_1.*

Thus substituting for Q the set $E \cap S_E$, we see that $0'$ lies in the convex hull of a subset of $E \cap S_E$ containing at most $n+1$ points.

II. Let A_1, A_2, and A_3 be points in the plane. The problem of determining a point Q in the plane such that $\sum_{i=1}^{3} |A_i - Q|$ shall be minimal, where $|A_i - Q|$ denotes the distance from the point A_i to the point Q, known as Steiner's problem, is discussed in Courant and Robbins [1]. The point Q is called the *Steiner point* of the triangle. Our work leads to a consideration of the following related problem:

If A_1, A_2, and A_3 are the vertices of an acute triangle, can we attach non-negative weights w_1, w_2, and w_3 to the points A_1, A_2, and A_3 to make any given point Q in the interior of triangle $A_1A_2A_3$ a "λ-Steiner point"; that is for $\lambda \geq 1$, the point $Q = P$ minimizes the function

$$\sum_{i=1}^{3} \mu_i |A_i - P|^\lambda \tag{3}$$

for all choices of P in the plane, when $u_i = w_i$.

This problem can be related to one in approximation theory as follows: Let E be a compact set in the complex plane $\mathbf{C}$, and let μ be a positive measure on $\mathbf{C}$. We denote the identity function by $I(z)$ and ask, "What is the best approximation in p-norm, $p \geq 1$, to $I(z)$ on E from the subspace consisting of the constant functions?" Thus we are asked to find some constant Q such that

$$\int |I(z) - P|^p \, d\mu \geq \int |I(z) - Q|^p \, d\mu \tag{4}$$

for all constants P.

Now, if we choose E to be the finite set of points $\{A_1, A_2, A_3\}$ and choose the measure μ to assign the weights u_i at A_i, $i = 1, 2, 3$, then we see that setting $p = \lambda$, the left-hand side of (4) is identical to the function (3).

Looking at the Steiner-type problem in terms of an approximation problem enables one to use the following theorem.

Theorem C.(2) *For $p \geq 1$, the condition*

$$(*) \qquad \sum_{i=1}^{3} u_i |A_i - Q|^{p-1} \overline{\operatorname{sgn} u_i(A_i - Q)} P = 0$$

for every P in $\mathbf{C}$ *implies* (4).

Theorem 2. *$u_i > 0$, $i = 1, 2, 3$, may be chosen so as to satisfy* (*).

Proof. Since Q is in triangle $A_1A_2A_3$ we may write Q using its barycentric coordinates relative to the points A_1, A_2, and A_3. Thus $Q = \Sigma_{i=1}^3 \lambda_i A_i$ with $\Sigma_{i=1}^3 \lambda_i = 1$ and since Q is in the interior of $\Delta A_1A_2A_3$, $\lambda_i > 0$, $i = 1, 2, 3$. Hence $\Sigma_{i=1}^3 \lambda_i (A_i - Q) = 0$ and also $\Sigma_{i=1}^3 \lambda_i \overline{(A_i - Q)} = 0$. Set

$$(5) \qquad u_i = w_i = \lambda_i |A_i - Q|^{2-p} \quad \text{for } i = 1, 2, 3.$$

Then

$$0 = \sum_{i=1}^{3} \lambda_i \overline{(A_i - Q)} = \sum_{i=1}^{3} w_i |A_i - Q|^{p-2} \overline{(A_i - Q)}$$

$$= \sum_{i=1}^{3} w_i |A_i - Q|^{p-1} \overline{\operatorname{sgn}} (A_i - Q).$$

Hence $\Sigma_{i=1}^3 w_i |A_i - Q|^{p-1} \overline{\operatorname{sgn}} (A_i - Q) P = 0$ for all $P \in \mathbf{C}$.

Thus we have found a set of positive weights given by (5) such that $Q = P$ minimizes the function (3).

We now discuss how the aforegoing ideas are realized in R_n. In n-dimensional space the Steiner-type problem we are interested in becomes:

If $A_1, A_2, \cdots, A_k$, $2 \leq k \leq n + 1$, are k points in R_n, can we attach non-negative weights $w_1, w_2, \cdots, w_k$ to the points $A_1, A_2, \cdots, A_k$ so as to make any given point Q in the convex hull of $\{A_1, A_2, \cdots, A_k\}$ a "λ-Steiner point," i.e., for $\lambda \geq 1$, $Q = P$ minimizes a function of the form

$$(6) \qquad \sum_{i=1}^{k} u_i |A_i - P|^{\lambda}$$

for all choices of P in R_n when $u_i = w_i$, where $|A_i - P|$ denotes the Euclidean distance from the point A_i to P?

The approximation theory analogue can be stated as follows:

Let E be a compact set in R_n and let μ be a positive measure on R_n. Consider the space of continuous functions whose domain is E and whose range lies

(2) The condition stated is both necessary and sufficient for the best approximation in the case $p > 1$ (see Dunford and Schwartz [2]). For the case $p = 1$, see Kripke and Rivlin [6].

in some inner product space H. This space, which we denote by $C(E; H)$ can be made into a normed linear space in various ways. We shall define a norm as follows: If $f \in C(E; H)$, then by $\|f\|_p$, $p \geq 1$, we shall mean $(\int_E |f(x)|^p \, d\mu)^{1/p}$ where $|f(x)|$ denotes the inner-product space norm of the range value $f(x)$.

If V is a finite-dimensional subspace of $C(E; H)$ we may pose an approximation problem by asking: Given $f \in C(E; H)$. What is a best approximation Q to f in p-norm out of V? I.e., find a $Q \in V$ such that $\|f - X\|_p \geq \|f - Q\|_p$ or equivalently

$$\int_E |f(x) - P(x)|^p \, d\mu \geq \int_E |f(x) - Q(x)|^p \, d\mu \tag{7}$$

for all $P \in V$.

A sufficient condition for best approximation is

$$\int_E |f(x) - Q(x)|^{p-1} (P(x), \operatorname{sgn}(f(x) - Q(x))) \, d\mu = 0 \tag{8}$$

for all $P \in V$. Its proof follows along the lines of the proof of Theorem C with the use of the Schwarz inequality on the inner product. (We note here that we extend the definition of sgn to vectors by: If $s \in R_n$, we define $\operatorname{sgn} s = s/\|s\|$ for $s \neq 0$ and $\operatorname{sgn} s = 0$ for $s = 0$.)

Now we take E to be the finite set of points $\{A_1, A_2, \cdots, A_k\}$ and choose the measure μ to assign the weights u_i at A_i, $i = 1, 2, \cdots, k$. Let H be k-dimensional complex space with the inner-product defined by: If $z = (z_1, z_2, \cdots, z_k) \in H$ and $u = (u_1, u_2, \cdots, u_k) \in H$, then, $(z, u) = \sum_{i=1}^k z_i \bar{u}_i$. Let f be the identity function on E and let the subspace V consist of the constant functions. Then, setting $p = \lambda$, the left-hand side of (7) becomes $\sum_{i=1}^k |A_i - P|^\lambda u_i$ which is identical to the function (6). The sufficiency condition (8) becomes

$$\sum_{i=1}^{k} u_i |A_i - Q|^{p-1} (P, \operatorname{sgn}(A_i - Q)) = 0 \tag{9}$$

for all constants P.

Theorem 2 is valid in R_n by choosing

$$u_i = w_i = \lambda_i |A_i - Q|^{2-p} \quad \text{for } i = 1, 2, \cdots, k \tag{10}$$

where the λ_i's are the barycentric coordinates of Q relative to the points A_1, $A_2, \cdots, A_k$, and $|A_i - Q|$ denotes Euclidean distance.

3. Chebyshev constant and Chebyshev points. Let E be a compact set in R_n. Let $X_1, X_2, \cdots, X_m$ be m points $(m \geq 1)$ of R_n. Consider the function:

$$r_m^{(\lambda)}(E) = \min_{X_i \in R_n} \max_{X \in E} \left[\frac{1}{m} \sum_{i=1}^{m} |X - X_i|^\lambda \right]^{1/\lambda}, \quad \lambda \neq 0,$$

where $|X - X_i|$ denotes the Euclidean distance between X and X_i. It can be shown (see Hille [4]) that $\lim_{m\to\infty} r_m^{(\lambda)}(E)$ exists, and this limit is called the *λth power average Chebyshev constant for* E, and denoted by $\chi^{(\lambda)}(E)$.

The case when $\lambda = 0$ corresponds to the geometric averaging process. In that case we have

$$r_m^{(0)}(E) = \min_{X_i \in R_n} \max_{X \in E} \left(\prod_{i=1}^{m} |X - X_i| \right)^{1/m}.$$

One can also show that $\lim_{m\to\infty} r_m^{(0)}(E)$ exists. Choosing the points $X_1, X_2, \cdots, X_m$ to lie in the complex plane rather than in R_n, Fekete [3] in 1923, originally defined the Chebyshev constant of a compact set E by this limit. One notes the value $r_m^{(0)}(E)$ is the mth root of the norm of the Chebyshev polynomial of degree m for E—that is, the monic polynomial of degree m of minimal maximum norm on E. Hence the name "Chebyshev constant" for this limit.

Points $X_1^*, X_2^*, \cdots, X_m^*$ such that

$$r_m^{(\lambda)}(E) = \max_{X \in E} \left[\frac{1}{m} \sum_{i=1}^{m} |X - X_i^*|^\lambda \right]^{1/\lambda}$$

are called *Chebyshev points of order m for E.*

The calculation of $\chi^{(\lambda)}(E)$ where E is a set consisting of two points, an interval, a circle, a disc, a sphere, and a ball can be found in Pólya and Szegö [7]. We look only at the case when $\lambda \geq 1$, but will now prove two theorems valid for all compact sets in R_n.

The main theorem may be stated as follows:

Theorem 3. *Let E be a compact set in R_n. Let S_E be the spanning sphere of E. Then $\chi^{(\lambda)}(E) = R$ for $\lambda \geq 1$, where R is the radius of S_E.*

Proof. Let $0'$ be the center of S_E. Placing $X = 0'$ for $i = 1, 2, \cdots, m$ gives

$$r_m^{(\lambda)}(E) \leq \max_{X \in E} \left[\frac{1}{m} \sum_{i=1}^{m} |X - 0'|^\lambda \right]^{1/\lambda} = \left(\frac{1}{m} m R^\lambda \right)^{1/\lambda} = R.$$

Our excursion into the geometric preliminaries and the Steiner problem helps to establish the reverse inequality.

We have noted that the center $0'$ lies in the convex hull of at most $n + 1$ points of $E \cap S_E$. Let this set be denoted by $\{A_1, A_2, \cdots, A_k\}$ where $2 \leq k \leq n + 1$. Now suppose X_i^*, $i = 1, 2, \cdots, m$, is a set of Chebyshev points of order m for E. Then

$$r_m^{(\lambda)}(E) = \max_{X \in E} \left(\frac{1}{m} \sum_{i=1}^{m} |X - X_i^*|^\lambda \right)^{1/\lambda}$$
$$(11) \qquad \geq \max_{X \in \{A_1, A_2, \cdots, A_k\}} \left(\frac{1}{m} \sum_{i=1}^{m} |X - X_i^*|^\lambda \right)^{1/\lambda}.$$

The maximum of a set of numbers is greater than or equal to *any* average function of them. We shall use a weighted λth power average.(3) Thus

$$(12) \qquad r_m^{(\lambda)}(E) \geq \left[\frac{u_1(m^{-1} \Sigma_{i=1}^m |A_1 - X_i^*|^\lambda) + u_2(m^{-1} \Sigma_{i=1}^m |A_2 - X_i^*|^\lambda) + \cdots + u_k(m^{-1} \Sigma_{i=1}^m |A_k - X_i^*|^\lambda)}{\Sigma_{j=1}^k u_j} \right]^{1/\lambda}.$$

Rearranging terms, the right-hand side of (12) becomes

$$(13) \qquad \left[\frac{m^{-1} \Sigma_{j=1}^k u_j |A_j - X_1^*|^\lambda + m^{-1} \Sigma_{j=1}^k u_j |A_j - X_2^*|^\lambda + \cdots + m^{-1} \Sigma_{j=1}^k u_j |A_j - X_m^*|^\lambda}{\Sigma_{j=1}^k u_j} \right]^{1/\lambda}.$$

We choose $u_j = w_j$, $j = 1, 2, \cdots, k$, such that the point $0'$ is the "λ-Steiner point", as in accordance with (10) with $Q = 0'$. The last expression (13) thus is greater than or equal to

$$\left[\frac{m^{-1} \Sigma_{j=1}^k w_j |A_j - 0'|^\lambda + m^{-1} \Sigma_{j=1}^k w_j |A_j - 0'|^\lambda + \cdots + m^{-1} \Sigma_{j=1}^k w_j |A_j - 0'|^\lambda}{\Sigma_{j=1}^k w_j} \right]^{1/\lambda}$$
$$= \left[\frac{m^{-1} m \Sigma_{j=1}^k w_j |A_j - 0'|^\lambda}{\Sigma_{j=1}^k w_j} \right]^{1/\lambda} = R.$$

Hence $r_m^{(\lambda)}(E) = R$ for every m and $\chi^{(\lambda)}(E) = R$. The bounds for R in the n-dimensional case are given by Jung [5]:

$$(14) \qquad d(E)/2 \leq R \leq (n/2(n+1))^{1/2} d(E).$$

Having established the value of the Chebyshev constant, we now look at the

(3) Let $u_i > 0$, $i = 1, 2, \cdots, n$. The function $A_{(\lambda)}(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_n) = [(\Sigma_{i=1}^n u_i x_i^\lambda)/\Sigma_{i=1}^n u_i]^{1/\lambda}$, $\lambda \neq 0$, is called a weighted λth-power average of $x_1, x_2, \cdots, x_n$ with respect to the weights $u_1, u_2, \cdots, u_n$. (If for some i, $x_i = 0$ and $\lambda < 0$ then we set $A_{(\lambda)}(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_n) = 0$.)

question of uniqueness of Chebyshev points. We have the following result:

Theorem 4. *For* $\lambda > 1$, *the Chebyshev points of order* m *for* E *are unique and are* $X_i^* = 0'$ *for* $i = 1, 2, \cdots, m$.

Proof. We note from Theorem 3 that the set $X_i^* = 0'$ for $i = 1, 2, \cdots, m$ is indeed a set of Chebyshev points of order m for E. The approximation problem, being a problem in a finite dimensional space with p-norm, $p = \lambda > 1$ admits a unique solution. Thus there exist weights $w_1, w_2, \cdots, w_k$ such that

$$\sum_{j=1}^{k} u_j |A_j - X|^\lambda > \sum_{j=1}^{k} u_j |A_j - 0'|^\lambda$$

for $X \neq 0'$, when $u_j = w_j$. Hence for arbitrary X_i, $i = 1, 2, \cdots, m$, $X_i \neq 0'$ for every i, the expression (13) (with X_i written in place of X_i^*) is strictly greater than

$$\left[\frac{m^{-1}\sum_{j=1}^{k} w_j|A_j - 0'|^\lambda + m^{-1}\sum_{j=1}^{k} w_j|A_j - 0'|^\lambda + \cdots + m^{-1}\sum_{j=1}^{k} w_j|A_j - 0'|^\lambda}{\sum_{j=1}^{k} w_j}\right]^{1/\lambda} = R. \tag{15}$$

Therefore

$$\max_{X \in E}\left[\frac{1}{m}\sum_{i=1}^{m} |X - X_i|^\lambda\right]^{1/\lambda} > R$$

and thus the unique Chebyshev points are $X_i^* = 0'$ for $i = 1, 2, \cdots, m$.

For $\lambda = 1$, the best approximation problem does not necessarily yield a unique solution, so we must examine that case separately. The uniqueness question in R_2 (or $\mathbf{C}$) is characterized by the following theorem.

Theorem 5. *In* R_2, *for* $\lambda = 1$, *the Chebyshev points of order* m *for* E *are unique if and only if the cardinality of* $E \cap C_E$ *is greater than or equal to three.*

Proof. Let $E \cap C_E = \{A_1, A_2, A_3\}$. We refer to Theorem C and look at a best approximation in the $p = \lambda = 1$ case. The theorem states that in order that

$$\sum_{i=1}^{3} u_i|A_i - Q| \leq \sum_{i=1}^{3} u_i|A_i - P|$$

for all $P \in \mathbf{C}$, it is sufficient that

$$\sum_{i=1}^{3} u_i P\, \overline{\mathrm{sgn}}(A_i - Q) = 0 \tag{16}$$

for all P in $\mathbf{C}$. (*Note.* $\overline{\mathrm{sgn}}\, u_i(A_i - Q) = \overline{\mathrm{sgn}}(A_i - Q)$.)

In the proof of Theorem C one develops the conditions:

$$\sum_{i=1}^{3} u_i|A_i - Q| = \sum_{i=1}^{3} u_i(A_i - P)\,\overline{\mathrm{sgn}}\,(A_i - Q). \tag{17}$$

Since the left-hand side is real and positive, the right-hand side must also be. Hence

$$\arg\left[\sum_{i=1}^{3} (A_i - P)\,\overline{\mathrm{sgn}}\,(A_i - Q)\right] = 0. \tag{18}$$

Now

$$\begin{aligned}\sum_{i=1}^{3} u_i(A_i - P)\,\overline{\mathrm{sgn}}\,(A_i - Q) &\leq \left|\sum_{i=1}^{3} u_i(A_i - P)\,\overline{\mathrm{sgn}}\,(A_i - Q)\right| \\ &\leq \sum_{i=1}^{3} u_i|A_i - P|.\end{aligned}$$

The last inequality on the right is strict unless either (1) $A_i - P = 0$ for all i, or (2) $\arg[(A_i - P)\,\overline{\mathrm{sgn}}\,(A_i - Q)]$ is the same for all i. Possibility (1) can never happen since this would imply that $A = (A_1, A_2, A_3)$ is the same vector as $\hat{P} = (P, P, P)$ which is not the case since the A_i's are distinct points. In view of (18) possibility (2) states that $\arg[(A_i - P)\,\overline{\mathrm{sgn}}\,(A_i - Q)] = 0$ for all i. Thus,

$$\arg(A_i - P) + \arg\overline{\mathrm{sgn}}\,(A_i - Q) = 0 \quad \text{or} \quad \arg(A_i - P) - \arg(A_i - Q) = 0.$$

Thus there exists $\mu \neq 0$ such that $(A_i - P) = \mu(A_i - Q)$. Substituting in (17) we have

$$\sum_{i=1}^{3} u_i|A_i - Q| = \sum_{i=1}^{3} u_i\mu(A_i - Q)\,\overline{\mathrm{sgn}}\,(A_i - Q)$$

or

$$\sum_{i=1}^{3} u_i|A_i - Q| = \mu\sum_{i=1}^{3} u_i|A_i - Q|.$$

Hence $\mu = 1$ and $A_i - Q = A_i - P$ or $P = Q$. Using the fact that $0'$ is in the convex hull of $\Delta A_1A_2A_3$ we can attach weights w_1, w_2, and w_3 to A_1, A_2, and A_3 such that

$$\sum_{i=1}^{3} u_i|A_i - X| > \sum_{i=1}^{3} u_i|A_i - 0'| \tag{19}$$

when $u_i = w_i$. (If $\Delta A_1A_2A_3$ is a right triangle we attach the weight zero to the vertex of the right angle.) If one repeats the proof of Theorem 4 with $\lambda = 1$ condition (19) shows that the inequality from (13) to (15) remains strict. Thus the Chebyshev points of order m for those sets described in the hypothesis are $z_i = 0'$, for $i = 1, 2, \cdots, m$, and this set is unique.

We now turn our attention to the case where the cardinality of $E \cap C_E$ is equal to two. Let $E \cap C_E = \{P, P'\}$.

Consider the set $\mathfrak{E}$ of all ellipses with major axis PP'. Let $M_E \in \mathfrak{E}$ be the ellipse of largest eccentricity which encloses E. We shall call M_E *the spanning ellipse of* E. If we denote the eccentricity of M_E by ϵ, we note that if the cardinality of $E \cap C_E$ is equal to two, then $\epsilon > 0$. (For those sets such that the cardinality of $E \cap C_E$ is greater than or equal to three we have $\epsilon = 0$ and M_E coincides with C_E.) From the remark following (2), we have $d(E) = 2R$. Thus if we designate the foci of M_E by F and F' and its center by $0'$, then the length of the segments $0'F$ and $0'F'$ is $c = R\epsilon$. If m is even we let

$$z_i = \begin{cases} 0' + d, & i = 1, 2, \cdots, m/2, \\ 0' - d, & i = m/2 + 1, \cdots, m, \end{cases} \tag{20}$$

where the points $0' \pm d$ are translations of the point $0'$ along the major axis PP'. Then

$$\frac{1}{m} \max_{z \in E} \sum_{i=1}^{m} |z - z_i| = \frac{1}{m} \max_{z \in E} \left[\frac{m}{2}|z - (0' + d)| + \frac{m}{2}|z - (0' - d)|\right].$$

Now the ellipse with foci at $0' + d$ and $0' - d$ for $d \le c$ has eccentricity smaller than or at most equal to (in the case $d = c$) the eccentricity of M_E. Thus if $d < c$, then for every z in E, $z \ne P, P'$, we have

$$|z - (0' + d)| + |z - (0' - d)| < 2R \quad \text{and} \quad |P - (0' + d)| + |P - (0' - d)| = 2R.$$

If $d = c$ then for every z in E we have

$$|z - (0' + c)| + |z - (0' - c)| \le 2R$$

with equality for $z = P, P'$. Hence

$$\frac{1}{m} \max_{z \in E} \left[\frac{m}{2}|z - (0' + d)| + \frac{m}{2}|z - (0' - d)|\right] = \frac{1}{2}(2R) = R.$$

Thus the set (20) for any $d \le c = R\epsilon$ is a set of Chebyshev points of even order m for E.

One can show that the set

$$z_i = \begin{cases} 0' + d, & \text{for } i = 1, 2, \cdots, (m-1)/2, \\ 0', & \text{for } i = (m+1)/2, \\ 0' - d, & \text{for } i = (m+3)/2, \cdots, m, \end{cases} \tag{21}$$

for any $d \le c = R\epsilon$ is a set of Chebyshev points of odd order m for E.

We note finally that Theorem 5 includes the case of R_1 since for any compact set E in R_1, $E \cap C_E = \{a, b\}$ where $b = \max_{x \in E} x$ and $a = \min_{x \in E} x$. Thus Chebyshev points of order m, $m > 1$ for any set E in R_1 are not unique.

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