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THE p-ADIC HULL OF ABELIAN GROUPS

BY

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ABSTRACT. In this paper we define "p-adic hull" for p-reduced groups K. The p-adic hull K^P of K is a module over the ring P of p-adic integers containing K and satisfying certain additional properties. The notion is investigated and then used to prove some known and some new theorems on $\mathrm{Ext}(K, T)$ and $\mathrm{Hom}(K, T)$ for K torsion-free and T a reduced p-group.

1. Introduction. The well-known method of "change of rings" put forth in Cartan-Eilenberg [2] permits the embedding of an abelian group K in a module over the ring P of p-adic integers provided only that the torsion subgroup of K is p-primary. The disadvantage of this p-adic embedding is that the module need not be p-reduced although K is p-reduced. Also, a group which is a p-adic module to start with may be properly enlarged. In §2 a "p-adic hull" K^P is introduced axiomatically. This hull is investigated and it is shown, among other things, that it has the properties mentioned above.

The concept of "p-adic hull" was suggested by investigations of the author [8] of the following two problems.

I. For which torsion-free groups K is $\mathrm{Ext}(K, T)[p] \neq 0$ for some p-group T?

II. Which torsion-free groups K possess unbounded reduced p-primary epimorphic images?

It is shown that the answer to both questions remains the same when K is replaced by its p-adic hull K^P. Now, the theory of torsion-free P-modules is much simpler than that of torsion-free groups. See Kaplansky [6, §§15 and 16]. In particular, a reduced countably generated torsion-free P-module is free, and a pure rank one submodule of any P-module is a direct summand. These facts are used in §3 to give new, simple proofs of results of Baer [1] and Mader [8]. In §4 the second fact is used to prove a theorem concerning Question II. In a final §5, we compare the two possible p-adic embeddings mentioned above.

We use the notation of Fuchs' book [3] which also contains most facts and concepts needed in this paper. We write maps on the right. If K is a P-module and S a subset, then PS denotes the submodule generated by S. P-modules K

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will have to be considered as modules and abelian groups simultaneously. Certain notions coincide whether K is considered a P-module or a $\mathbf{Z}$-module, among these are the following: divisible, reduced, p-height, $p^n K$, $K[p^n]$, direct sum, complete direct sum, maximal divisible subgroup (-module). Otherwise it will be made clear what is meant. If no mention of the ring of operators is made, we mean the $\mathbf{Z}$-module notions. For instance, "homomorphism" means group homomorphism.

2. **The p-adic hull.** The fact which makes things work in this paper is the standard embedding of the ring of rational integers $\mathbf{Z}$ in the ring P of p-adic integers. We have

$$(2.1) \qquad \mathbf{Z} \rightarrowtail P \twoheadrightarrow P/\mathbf{Z} \quad \text{(ex)} \quad \text{with } P/\mathbf{Z} \text{ divisible and } (P/\mathbf{Z})[p] = 0.$$

We derive some simple but useful consequences.

2.2 **Lemma.** *Let A, K be P-modules, and K reduced. Then*

(a) $\mathrm{Hom}(A, K) = \mathrm{Hom}_P(A, K)$. *In particular,* $\mathrm{Hom}(P, K) \cong K$.

(b) *K is in a unique way a (unitary) P-module.*

(c) *If L is a subgroup of K which is a P-module, then L is a submodule of K.*

Proof. (a) From (2.1) it follows that $\mathrm{Hom}(P, K) \rightarrowtail \mathrm{Hom}(\mathbf{Z}, K)$ is exact, i.e. every homomorphism $P \longrightarrow K$ is uniquely determined by its image at 1. Let $f \in \mathrm{Hom}(A, K)$, and $a \in A$. The map $P \longrightarrow K$: $\lambda \longrightarrow (\lambda a)f - \lambda(af)$ is homomorphic and has value 0 at 1. Hence $(\lambda a)f = \lambda(af)$ for all $\lambda \in P$. Since a was arbitrary, this proves that every homomorphism is P-linear. Since every P-homomorphism is additive, (a) is proven.

(b) If λx and $\lambda \cdot x$ are two scalar products, then $\lambda \longrightarrow \lambda x$ and $\lambda \longrightarrow \lambda \cdot x$ are two homomorphisms $P \longrightarrow K$ which coincide on 1. By (a) $\lambda x = \lambda \cdot x$ for all $\lambda \in P$.

(c) Follows immediately from (b).

The next lemma justifies the definition of "p-adic hull" which will be given below.

2.3 **Lemma.** *Let K be a p-reduced group. Suppose K' is a group such that*

(a) $K' > K$,

(b) *K' is a reduced P-module,*

(c) $(K'/K)[p] = 0$,

(d) $K' = PK$. (*Hence K'/K is p-divisible.*)

Then

(A) *For every reduced P-module L, any homomorphism $K \longrightarrow L$ has a unique extension $K' \longrightarrow L$. The extension is a P-homomorphism.*

(B) *If K' and K'' satisfy* (a)–(d), *then there is a unique P-isomorphism $K' \longrightarrow K''$ which is the identity on K.*

(C) *For each p-reduced group K there is a group K' satisfying* (a)–(d).

Proof. (A) Let $L^* = \mathrm{Ext}(Z(p^\infty), L)$. With standard homological tools (see Harrison [4]) we find $L < L^*$, $(L^*/L)[p] = 0$, L^*/L is divisible, L^* is reduced, $\mathrm{Ext}(A, L^*) = 0$ for every group A with $A[p] = 0$, L^* is a P-module. By 2.2 the P-module structure of L^* is unique and L is a submodule. The exact sequence $K \rightarrowtail K' \twoheadrightarrow K'/K$ implies

$$\mathrm{Hom}(K'/K, L^*) = 0 \to \mathrm{Hom}(K', L^*) \to \mathrm{Hom}(K, L^*) \to \mathrm{Ext}(K'/K, L^*) = 0 \quad \text{(ex)}.$$

Hence every $\phi: K \to L \in \mathrm{Hom}(K, L^*)$ has a unique extension $\phi': K' \to L^*$. By 2.2(a) ϕ' is a P-homomorphism, and $K'\phi' = (PK)\phi' = P(K\phi') \subset PL = L$, thus $\phi' \in \mathrm{Hom}(K', L)$.

(B) Immediate consequence of (A).

(C) $K < K^* = \mathrm{Ext}(Z(p^\infty), K)$. Let $K' = PK \subset K^*$. Then $K' > K$, K' is reduced since K^* is reduced, K' is by construction a P-module and $K' = PK$, finally $(K'/K)[p] = 0$ since $K'/K < K^*/K$ and $(K^*/K)[p] = 0$.

2.4 **Definition.** Let K be a p-reduced group. Any group K' satisfying (a)–(d) of 2.3 will be called a p-adic hull or P-hull of K. We write $K' = K^P$.

The p-adic hull has the same degree of uniqueness as does the well-known divisible hull. The statement $K' = K^P$ reads "K' is a p-adic hull of K". As soon as a specific hull is chosen, it is meant by K^P and the ambiguity disappears. We next determine K^P in some cases, and note some of its properties.

2.5 **Proposition.** (a) *If K is a reduced P-module, then $K^P = K$.*

(b) *If K is a reduced p-group, then $K^P = K$.*

(c) *If K is p-reduced and $K[p] = 0$, then K^P is torsion-free.*

(d) *$(K^P)^P = K^P$ for every p-reduced group K.*

(e) *If $\{K_i\}$ is a family of p-reduced groups, then $(\bigoplus K_i)^P = \bigoplus K_i^P$.*

(f) *If K is a p-pure subgroup of P, then $K^P = P$.*

(g) *If K is a p-reduced torsion-free group and either K/pK is finite or K countable, then K^P is a free P-module of rank $\dim(K/pK)$.*

(h) *If K is free, then K^P is a free P-module. The converse does not hold.*

(i) *If L is a p-reduced group, $K < L$ and either $(L/K)[p] = 0$ or L/K is p-reduced, then the submodule PK of L^P generated by K is a p-adic hull of K.*

(j) *If $\{a_i \mid i \in I\}$ is a maximal p-independent subset of the torsion-free p-reduced group K, then $\{a_i \mid i \in I\}$ is a maximal p-independent subset of the module K^P.*

Proof. (a) K satisfies (a)–(d) of 2.3.

(b) Every p-group is a P-module hence (a) applies.

(c) Suppose $px = 0$ for $x \in K^P$. Since $(K^P/K)[p] = 0$, $x \in K[p] = 0$.

(d) Consequence of (a).

(e) and (f) Conditions (a)–(d) of 2.3 are easily checked.

(g) K^P is reduced. If K is countable, then K^P is countably generated and by Kaplansky [6, p. 46, Theorem 20], K^P is free. Note that always $K^P/pK^P = K + pK^P/pK^P \cong K/K \cap pK^P = K/pK$. If K/pK is finite, any basic submodule B of K^P is complete and by Kaplansky [6, p. 52, Theorem 23], B is a direct summand of K^P. Since K^P/B is divisible and K^P is reduced, we have $K^P = B$ and is free. In both cases the rank of K^P is $\dim(K^P/pK^P) = \dim(K/pK)$.

(h) Combine (e) and (f). That the converse does not hold is clear from (f) or (g).

(i) The submodule PK of L^P satisfies (a), (b), (d) of 2.3. Suppose $(L/K)[p] = 0$. If $x \in PK$ and $px \in K$, then $px \in L$ and hence $x \in L$. But $x \in L$ and $px \in K$ implies $x \in K$ since $(L/K)[p] = 0$. Now suppose that L/K is p-reduced. $PK \cap L/K < L/K$ so $PK \cap L/K$ is p-reduced. Further $PK = (\mathbf{Z} + pP)K = K + p(PK)$, and so $PK \cap L = (K + p(PK)) \cap L = K + [p(PK) \cap L] = K + p(PK \cap L)$ using the Dedekind identity and $(L^P/L)[p] = 0$. So $PK \cap L/K = (p(PK \cap L) + K)/K = p(PK \cap L/K)$. We now have that $PK \cap L/K$ is both p-reduced and p-divisible, so $PK \cap L = K$. Suppose $x \in PK[\subset L^P]$ and $px \in K[\subset L]$. Then $x \in L \cap PK = K$. This proves (c) of 2.3 also in the second case.

(j) Let $B = \bigoplus_{i \in I} \mathbf{Z}a_i$ be the p-basic subgroup of K generated by $\{a_i\}$. By (i) we may assume that $B^P \subset K^P$. Let $\hat{B} = \Pi_I P$. We shall utilize a representation of the whole set-up in $\hat{B}$. First of all $\phi: B \to \hat{B}: (\Sigma n_i a_i)\phi = (\cdots n_i \cdots)$ is clearly an embedding. Since $(\hat{B}/B\phi)[p] = 0$, $P(B\phi) = (B\phi)^P$, and clearly $(B\phi)^P = \bigoplus_I P$. Since $(K^P/B)[p] = 0$ and K^P/B is divisible, we conclude from $B \rightarrowtail K^P \twoheadrightarrow K^P/B$ (ex) that $0 \to \mathrm{Hom}(K^P, \hat{B}) \to \mathrm{Hom}(B, \hat{B}) \to \mathrm{Ext}(K^P/B, \hat{B}) = 0$ is exact. In particular the embedding $\phi: B \to \hat{B}$ has a unique extension $\phi: K^P \to \hat{B}$. We claim that ϕ is injective. In fact, suppose $x \in K^P$ and $x\phi = 0$. Since K^P/B is p-divisible, given n, we can write $x = b_n + p^n x_n$ for some $b_n \in B$ and some $x_n \in K^P$. Now $0 = x\phi = b_n\phi + p^n(x_n\phi)$ implies $b_n\phi \in B\phi \cap p^n\hat{B} = p^n(B\phi) = (p^nB)\phi$. Since ϕ is monomorphic on B, $b_n \in p^nB$ and $x \in p^nK^P$. So $x \in \bigcap_n p^nK^P = 0$. Thus $\phi: K^P \to \hat{B}$ is an embedding as claimed, and ϕ is also a P-homomorphism by 2.2. Clearly $(K^P)\phi = (PK)\phi = P(K\phi) = (K\phi)^P$, and $(B^P)\phi = (B\phi)^P$. The latter proves $B^P = \bigoplus_{i \in I} Pa_i$. Since obviously $(\hat{B}/(B\phi)^P)[p] = 0$ we have $((K\phi)^P/(B\phi)^P)[p] = 0$, and since $K^P/B^P \cong (K\phi)^P/(B\phi)^P$ we have $(K^P/B^P)[p] = 0$. Further $K^P = K + pK^P = B + pK + pK^P = B^P + pK^P$, so K^P/B^P is p-divisible. So B^P is a free, p-pure, dense submodule of K^P with free generators a_i, which shows that $\{a_i \mid i \in I\}$ is a maximal p-independent subset of K^P.

We remark that K^P need not contain a p-adic hull for each of the subgroups of K. For example, let $\{a_i\}$ be a maximal independent subset of P and $A = \bigoplus_i \mathbf{Z}a_i$. Then $A^P \cong \bigoplus_2 \aleph_0 P$ which cannot be a submodule of $P^P = P$.

The next proposition shows that the process of forming p-adic hulls has great similarity with a functor.

2.6 **Proposition.** (a) *If K, L are p-reduced groups, K^P, L^P p-adic hulls of K, L and ϕ: $K \to L$ is a homomorphism, then there is a unique P-homomorphism ϕ^P: $K^P \to L^P$ extending ϕ.*

(b) *If K_i, $i = 1, 2, 3$, are p-reduced groups with p-adic hulls K_i^P, and if ϕ_1: $K_1 \to K_2$ and ϕ_2: $K_2 \to K_3$ are homomorphisms, then $(\phi_1\phi_2)^P = \phi_1^P\phi_2^P$.*

(c) *In the situation of* (a) *if ϕ is surjective so is ϕ^P. If $(L/K\phi)[p] = 0$ or $L/K\phi$ is p-reduced and ϕ is injective, so is ϕ^P.*

Proof. (a) The homomorphism ϕ: $K \to L^P$ has a unique extension ϕ^P: $K^P \to L^P$ by 2.3(A).

(b) Immediate consequence of (a).

(c) From $K\phi = L$ it follows that $K^P\phi^P = (PK)\phi^P = P(K\phi^P) = P(K\phi) = PL = L^P$. For the second part we first note that $(K\phi)^P = P(K\phi) \subset L^P$ by 2.5(i). Since ϕ: $K \to K\phi$ is an isomorphism so is ϕ^P: $K^P \to (K\phi)^P$. So ϕ^P: $K^P \to L^P$ is injective.

Since $K < L$ does not imply $K^P < L^P$ as remarked above it is also not true that ϕ injective implies ϕ^P injective in all cases.

2.7 **Remark.** The process described above is actually a functor on the category of p-reduced groups to a skeletal subcategory $\mathcal{C}$ of the category of reduced P-modules. Such a skeletal subcategory contains exactly one object from each isomorphism class of reduced P-modules. For each K, K^P is the unique object of $\mathcal{C}$ for which there is a monomorphism ϕ: $K \to K^P$ such that $K^P = (K\phi)^P$ in the sense of Definition 2.4. If $K \rightarrowtail M \twoheadrightarrow L$ is an exact sequence of p-reduced groups, then $0 \to K^P \to M^P \to L^P \to 0$ need not be exact. In order to see what happens we discuss the case where $L[p] = 0$ in some detail.

2.8 **Example.** Let K, L be p-reduced groups, $L[p] = 0$, $K^* = \mathrm{Ext}(Z(p^\infty), K)$ and $E = K^* \oplus L$. From $K \rightarrowtail K^* \twoheadrightarrow K^*/K$ (ex) it follows, using $L[p] = 0$, that $\mathrm{Hom}(L, K^*/K) \to \mathrm{Ext}(L, K) \to 0$ is exact. Thus every extension of K by L arises from a map of $\mathrm{Hom}(L, K^*/K)$. If $\xi \in \mathrm{Hom}(L, K^*/K)$ then $K \rightarrowtail M \overset{\Pi}{\twoheadrightarrow} L$ represents the image of ξ when $M < E$, $M = \{x + y \mid x \in K^*, y \in L$ and $y\xi = x + K\}$, and Π is the projection of E onto L. As in Mader [7] we easily calculate that $M \cap K^* = K$ and $M + K^* = E$. Since $E/M = (M + K^*)/M \cong K^*/K^* \cap M = K^*/K$ and $(K^*/K)[p] = 0$ we have $(E/M)[p] = 0$. So by 2.5(i) $M^P < E^P = K^* \oplus L^P$; also $K^P < K^*$.

(a) Π^P: $M^P \to L^P$ is surjective and $\mathrm{Ker}\,\Pi^P = M^P \cap K^*$.

Proof. Since Π is surjective, so is Π^P by 2.6(c). Further it is clear that Π^P is the projection of E^P onto L^P since this projection obviously extends Π. Therefore, $\mathrm{Ker}\,\Pi^P = M^P \cap K^*$.

(b) $K^P \rightarrowtail M^P \twoheadrightarrow L^P$ (ex) if and only if $K^P = M^P \cap K^*$.

This is immediate from (a).

(c) $M^P \cap K^*/K^P = p^\omega(M^P/K^P) =$ maximal divisible submodule of M^P/K^P.

Proof. Since $M^P/M^P \cap K^* \cong L^P$ is reduced and torsion-free we have $p^\omega(M^P/K^P) \subset M^P \cap K^*/K^P$. We are finished if we show that $M^P \cap K^*/K^P$ is divisible. Since $(E^P/E)[p] = 0$ and $(E/M)[p] = 0$ we have $(E^P/M)[p] = 0$. Since $(E^P/M)[p] = 0$ and M^P/M is p-divisible it follows easily that $(E^P/M^P)[p] = 0$. Now $E^P/M^P = PE/M^P = P(K^* + M)/M^P = (K^* + PM)/M^P = (K^* + M^P)/M^P \cong K^*/M^P \cap K^* \cong (K^*/K^P)/(M^P \cap K^*/K^P)$, so $M^P \cap K^*/K^P$ is pure in the divisible module K^*/K^P and so is itself divisible.

(d) Suppose $K^*/K^P \neq 0$. Then $M^P \cap K^* = K^P$ for every M only if every subset of L which is $\mathbf{Z}$-independent is P-independent in L^P.

Proof. (1) Let us first note that $(K^*/K)[p] = 0$ and K^P/K p-divisible together imply that $(K^*/K^P)[p] = 0$. Since K^*/K^P is a P-module this means that K^*/K^P is torsion-free (i.e. $\lambda x = 0$ implies $x = 0$).

(2) Given $\xi \in \operatorname{Hom}(L, K^*/K)$ and a corresponding extension M of K by L contained in E (i.e. $x + y \in M$ $(x \in K^*,\ y \in L)$ if and only if $y\xi = x + K$). We have $u + v \in M^P$ $(u \in K^*,\ v \in L^P)$ if and only if $u + v = \Sigma\lambda_i(x_i + y_i)$ where $x_i + y_i \in M$ $(x_i \in K^*,\ y_i \in L,\ y_i\xi = x_i + K)$. Further $u + v \in M^P \cap K^*$ if and only if $\Sigma\lambda_i y_i = 0$. Hence $M^P \cap K^* = K^P$ if and only if $\Sigma\lambda_i x_i \in K^P$ whenever $\Sigma\lambda_i y_i = 0$ for $x_i + y_i \in M$.

(3) Suppose $\{y_i\}$ is a (finite) $\mathbf{Z}$-independent subset of L but $\Sigma\lambda_i y_i = 0$ in L^P for $\lambda_i \in P$, not all 0. Let $x \in K^*, x \notin K^P$. If $\Sigma\lambda_i \neq 0$, choose $\xi \in \operatorname{Hom}(L, K^*/K)$ such that $y_i\xi = x$. Such a ξ exists since $\{y_i\}$ is $\mathbf{Z}$-independent and K^*/K is divisible. Then $\Sigma\lambda_i(x + y_i) = (\Sigma\lambda_i)x \in M^P \cap K^*$ but $(\Sigma\lambda_i)x \notin K^P$ by (1). Should it happen that $\Sigma\lambda_i = 0$ then $p\lambda_j + \Sigma_{i \neq j}\lambda_j \neq 0$ for some j. Now choose ξ such that $y_i\xi = x$ $(i \neq j)$ and $y_j\xi = px$. Then it follows exactly as before that $M^P \cap K^* \neq K^P$.

As a rule some $\mathbf{Z}$-independent subsets of L will become P-independent in L^P, and clearly $K^*/K^P \neq 0$ can be achieved. Thus $K^P \neq M^P \cap K^*$ can occur and $K^P \rightarrowtail M^P \twoheadrightarrow L^P$ need not be exact.

The last lemma of this section settles a technical matter which is needed in §4.

2.9 **Lemma.** (a) *Let M be an unbounded group with $p^\omega M = 0$. Then $M^P/p^\omega M^P$ is unbounded.*

(b) *Let L be a p-reduced group, $K < L$ such that L/K is unbounded and $p^\omega(L/K) = 0$. Then $K^P < L^P$ by 2.5(i). Let $K^0 < L^P$ be such that $K^0/K^P = p^\omega(L^P/K^P)$. Then K^0 is a submodule, $p^\omega(L^P/K^0) = 0$ and L^P/K^0 is unbounded.*

Proof. (a) Suppose first that $M/T(M)$ is not p-divisible. Then we have $K \rightarrowtail M \twoheadrightarrow L(\mathrm{ex})$ where $K/T(M) = p^\omega(M/T(M))$ and $L \cong M/K \cong (M/T(M))/(K/T(M)) = (M/T(M))/p^\omega(M/T(M))$. Hence L is $\neq 0$, torsion-free and p-reduced, and therefore $p^\omega L^P = 0$. Since $M^P \twoheadrightarrow L^P$ it follows that $M^P/p^\omega M^P \twoheadrightarrow L^P$ showing that

$M^P/p^\omega M^P$ is unbounded. Secondly suppose that M/T is p-divisible where $T = T(M)$. Let $T^* = \mathrm{Ext}(Z(p^\infty), T)$. Then T^*/T is torsion-free divisible. Since M is p-reduced and M/T is torsion-free and p-divisible we may assume that $T < M < T^*$. It is easily checked that $(T^*/M)[p] = 0$, and therefore $M^P = PM < T^*$. Now $p^\omega M^P \cap T \subset p^\omega T^* \cap T = p^\omega T \subset p^\omega M = 0$. Therefore $T \cong T + p^\omega M^P/p^\omega M^P < M^P/p^\omega M^P$. We are finished if T is unbounded. But if T is bounded, then $M \cong T \oplus M/T$ and since $p^\omega M = 0$. This means M is bounded which is not so.

(b) Put $L/K = M$. Since $L \twoheadrightarrow M$ we have $L^P \twoheadrightarrow M^P$. The composite map $L^P \twoheadrightarrow M^P/p^\omega M^P$ maps K^P and hence K^0 onto 0. So we have an induced map $L^P/K^0 \twoheadrightarrow M^P/p^\omega M^P$. By (a) $M^P/p^\omega M^P$ is unbounded hence so is L^P/K^0. By definition K^0 is a submodule and $p^\omega(L^P/K^0) = 0$.

3. **Applications to Hom and Ext.** We are concerned with the groups $\mathrm{Hom}(K, T)$, $\mathrm{Ext}(K, T)$ for T a reduced p-group and K a p-reduced group. For the results of this section we only need 2.3 and parts of 2.5 of our previous results.

3.1 **Theorem.** *Let T be a reduced p-group and K a p-reduced group. Then the following hold.*

(a) *The restriction map* $\mathrm{Hom}(K^P, T) \to \mathrm{Hom}(K, T)$ *is an isomorphism.*

(b) $\mathrm{Ext}(K^P/K, T) \rightarrowtail \mathrm{Ext}(K^P, T) \twoheadrightarrow \mathrm{Ext}(K, T)$ *is exact.*

(c) $\mathrm{Ext}(K^P, T) \cong \mathrm{Ext}(K^P/K, T) \oplus \mathrm{Ext}(K, T)$. *Let* $T^* = \mathrm{Ext}(Z(p^\infty), T)$. *Then* $\mathrm{Ext}(K^P/K, T) \cong \mathrm{Hom}(K^P/K, T^*/T)$ *and both groups are torsion-free divisible.*

(d) $\mathrm{Ext}(K^P, T)[p] \cong \mathrm{Ext}(K, T)[p]$.

Proof. The exact sequence $K \rightarrowtail K^P \twoheadrightarrow K^P/K$ implies the exact sequence $0 \to \mathrm{Hom}(K^P, T) \to \mathrm{Hom}(K, T) \to \mathrm{Ext}(K^P/K, T) \to \mathrm{Ext}(K^P, T) \to \mathrm{Ext}(K, T) \to 0$. By 2.3(A) $\mathrm{Hom}(K^P, T) \to \mathrm{Hom}(K, T)$ is surjective. This proves both (a) and (b). To prove (c) consider $T \rightarrowtail T^* \twoheadrightarrow T^*/T$ (ex). We obtain $0 \to \mathrm{Hom}(K^P/K, T^*/T) \to \mathrm{Ext}(K^P/K, T) \to \mathrm{Ext}(K^P/K, T^*) = 0$. Thus $\mathrm{Hom}(K^P/K, T^*/T) \cong \mathrm{Ext}(K^P/K, T)$. Since T^*/T is torsion-free divisible so is $\mathrm{Hom}(K^P/K, T^*/T)$. It follows that (b) splits and all of (c) is proved.

(d) Immediate consequence of (c).

There are immediate consequences when K^P is a free module.

3.2 **Corollary.** *If K is a torsion-free p-reduced group such that either K/pK is finite or K countable, and if T is a reduced p-group, then*

(a) $\mathrm{Hom}(K, T) \cong \prod_{\dim(K/pK)} T$.

(b) $\mathrm{Ext}(K, T)[p] = 0$.

Proof. By 2.5(g) $K^P = \bigoplus_d P$ where $d = \dim(K/pK)$. Hence $\mathrm{Hom}(K, T) \cong \mathrm{Hom}(K^P, T) = \mathrm{Hom}_P(K^P, T) = \prod_d T$. Further $\mathrm{Ext}(K^P, T) \cong \prod_d \mathrm{Ext}(P, T)$, and by

3.1(c) and 2.5(f) $\mathrm{Ext}(P, T) = \mathrm{Ext}(\mathbf{Z}^P, T) \cong \mathrm{Ext}(P/\mathbf{Z}, T) \oplus \mathrm{Ext}(\mathbf{Z}, T) = \mathrm{Ext}(P/\mathbf{Z}, T)$ which is torsion-free. Hence $\mathrm{Ext}(K^P, T)$ is torsion-free. By 3.1(d) the proposition follows.

These results were first proved by Baer [1, p. 229], and later differently by Mader [8].

4. **Reduced p-primary quotient groups.** Groups may have large ranks and no elements of infinite p-height but no reduced unbounded p-primary epimorphic images. See Baer [1, p. 231, 4.1], and Howard [5, p. 324, 2.2, and p. 325, 2.9]. We shall give a necessary and sufficient condition for the existence of reduced unbounded p-primary epimorphic images. The theorem is motivated by the results of Howard [5] and the one very obvious part of the theorem which we will do first.

4.1 **Proposition.** *If the group K has a reduced unbounded p-primary epimorphic image, then K is the union of an ascending sequence of subgroups $K_1 < \cdots < K_i < K_{i+1} < \cdots$ such that $p^{\omega}(K/K_i) = 0$ and K/K_i is unbounded for all i.*

Proof. Since every p-group can be mapped epimorphically onto any of its basic subgroups by Szele's theorem (Fuchs [3, p. 152, 36.1]) we may assume that K has the epimorphic image $B = \bigoplus_{j=1}^{\infty} B_j$ where each B_j is a direct sum of cyclic groups of order p^j and infinitely many B_j are not zero. Let K_i be the preimage of $\bigoplus_{1 \le j \le i} B_j$. Then $\{K_i\}$ obviously is as claimed.

Our main result is the converse of this proposition, i.e. we prove

4.2 **Theorem.** *A group K has a reduced unbounded p-primary epimorphic image if and only if K is the union of an ascending sequence of subgroups $K_1 < K_2 < \cdots < K_i < K_{i+1} < \cdots$ such that $p^{\omega}(K/K_i) = 0$ and K/K_i is unbounded.*

The theorem is proved by reducing it to the easier case of P-modules by means of the p-adic hull.

4.3 **Reduction.** *If $K = \bigcup K_i$ as in 4.2, then K^P is the union of an ascending sequence of submodules $L_1 < \cdots < L_i < L_{i+1} < \cdots$ such that K^P/L_i is unbounded and $p^{\omega}(K^P/L_i) = 0$.*

Proof. Since $p^{\omega}(K/K_i) = 0$ we have $K_i^P < K^P$ by 2.5(i). As we have seen in 2.8 K^P/K_i^P need not be reduced. Therefore let L_i be the submodule of K^P with $L_i/K_i^P = p^{\omega}(K^P/K_i^P)$. By 2.9 K^P/L_i is unbounded and $p^{\omega}(K^P/L_i) = 0$. It is obvious that $L_i < L_{i+1}$ for all i, and $K^P = PK = P(\bigcup K_i) \subset P(\bigcup L_i) = \bigcup L_i$.

Since $\mathrm{Hom}(K^P, T) \cong \mathrm{Hom}(K, T)$ for any reduced p-group T (3.1(a)) it remains to prove 4.2 for P-modules.

4.4 **Theorem.** *Let K be a P-module and $K_1 < K_2 < \cdots < K_i < K_{i+1} < \cdots$ an ascending sequence of submodules such that $p^{\omega}(K/K_i) = 0$, K/K_i is not* *bounded and* $\bigcup K_i = K$. *Then there exists a submodule* M *such that* K/M *is a reduced unbounded p-primary module. The converse also holds.*

Proof. Let $K_0 = p^\omega K$. Since $p^\omega(K/K_1) = 0$ we have $K_0 < K_1$. M will be obtained inductively as the union of a chain of submodules

$$M_0 < M_1 < \cdots < M_i < M_{i+1} < \cdots$$

satisfying

(1) $p^\omega(K/M_i) = 0$;

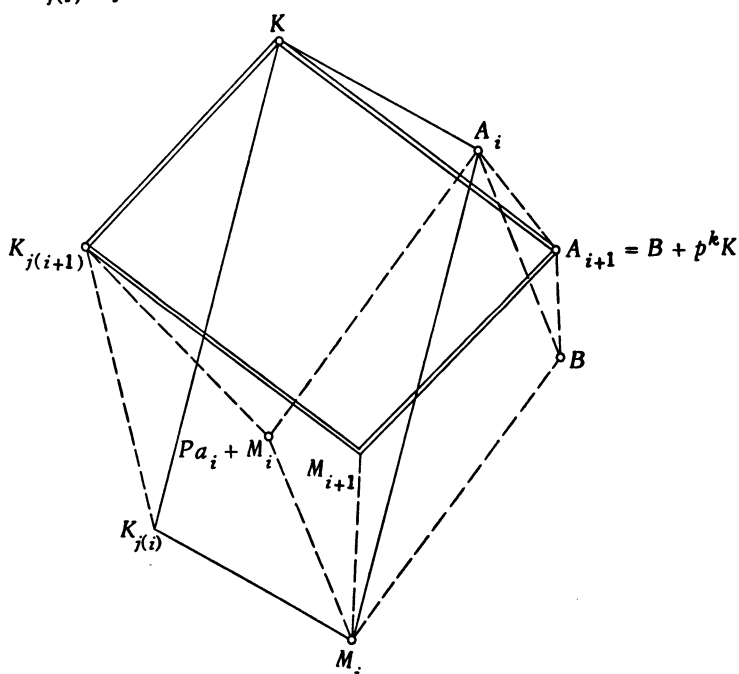
(2) there are integers $j(i)$ such that $0 < j(0) < j(1) < \cdots < j(i) < j(i+1) < \cdots$ and $K_{j(i)} > M_i$ for all i;

(3) there are submodules A_i of K such that $K/M_i = K_{j(i)}/M_i \oplus A_i/M_i$;

(4) for $i \geq 1$, $K_{j(i)}/M_i = (K_{j(i-1)} + M_i)/M_i \oplus C_i$ where $C_i = P(a_{i-1} + M_i)$ for some $a_{i-1} \in K$ and $\infty > \exp C_i \geq i$;

(5) for $i \geq 1$, $K_{j(i-1)} \cap M_i = M_{i-1}$, hence $(K_{j(i-1)} + M_i)/M_i \cong K_{j(i-1)}/M_{i-1}$;

(6) $K_{j(i)}/M_i$ is finitely generated and p-primary.



M_i, $j(i)$, A_i, a_{i-1}, C_i will be constructed inductively. We begin with $M_0 = K_0$, $j(0) = 0$, $A_0 = K$. Suppose M_i, $j(i)$, A_i, a_{i-1}, C_i have already been obtained satisfying (1)–(6). Note that A_i/M_i is not bounded since otherwise $K/K_{j(i)}$ would be bounded. Let $n = \exp(K_{j(i)}/M_i)$, so $p^n K_{j(i)} \subset M_i$. Since $p^\omega(A_i/M_i) = 0$ and A_i/M_i is not bounded, there is $a_i \in A_i$ such that

(a) $A_i/M_i = P(a_i + M_i) \oplus B/M_i$ and $\exp(a_i + M_i) \geq k := n + i + 1$.

(For the existence of a_i note that every cyclic summand of a p-basic submodule of A_i/M_i is a direct summand of A_i/M_i by Kaplansky [6, Theorem 23].) Since $\bigcup K_r = K$ there is $j(i+1) > j(i)$ such that $K_{j(i+1)} \supset Pa_i + M_i$. Then it follows from (3) (Fuchs [3, p. 38, (b)]) that

(b) $K_{j(i+1)}/M_i = K_{j(i)}/M_i \oplus (A_i \cap K_{j(i+1)})/M_i$ and from (a) we obtain

(c) $(A_i \cap K_{j(i+1)})/M_i = P(a_i + M_i) \oplus (B \cap K_{j(i+1)})/M_i$.

Define $A_{i+1} = B + p^k K$ and $M_{i+1} = K_{j(i+1)} \cap A_{i+1}$. Then $M_i \subset K_{j(i)} \cap B \subset K_{j(i+1)} \cap A_{i+1} = M_{i+1}$. We have to verify in addition statements (1′)–(6′) which are obtained from (1)–(6) by replacing i by $i+1$. By construction (2′) is satisfied. Since $p^\omega(K/K_{j(i+1)}) = 0$ and $p^\omega(K/A_{i+1}) = 0$, we have $p^\omega(K/M_{i+1}) = 0$. So (1′) holds. Since $K_{j(i+1)} + A_{i+1} \supset K_{j(i)} + Pa_i + B \supset K_{j(i)} + A_i = K$ and $K_{j(i+1)} \cap A_{i+1} = M_{i+1}$ we have $K/M_{i+1} = K_{j(i+1)}/M_{i+1} \oplus A_{i+1}/M_{i+1}$, and (3′) holds. Note that $p^n K_{j(i)} \subset M_i$ and $K = K_{j(i)} + A_i$ imply $p^k K \subset p^n K = p^n K_{j(i)} + p^n A_i \subset M_i + A_i = A_i$, and so $A_{i+1} = B + p^k K \subset A_i$. Since $M_i \subset K_{j(i)} \cap M_{i+1} \subset K_{j(i)} \cap A_{i+1} \subset K_{j(i)} \cap A_i = M_i$, we have $M_i = K_{j(i)} \cap M_{i+1}$ and so (5′) holds. If we show (4′) then (6′) is clear from (5′) and (6). To show (4′), firstly note that $K_{j(i+1)} \supset K_{j(i)} + Pa_i + M_{i+1} = K_{j(i)} + Pa_i + A_{i+1} \cap K_{j(i+1)} = K_{j(i)} + Pa_i + (B + p^k K) \cap K_{j(i+1)} \supset K_{j(i)} + Pa_i + B \cap K_{j(i+1)} \supset K_{j(i)} + (A_i \cap K_{j(i+1)})$ (by (c)) $\supset K_{j(i+1)}$ (by (b)). So $K_{j(i+1)} = K_{j(i)} + M_{i+1} + Pa_i$. Secondly, $M_{i+1} \subset (K_{j(i)} + M_{i+1}) \cap (Pa_i + M_{i+1}) \subset (K_{j(i)} \cap (Pa_i + M_{i+1})) + M_{i+1} \subset (K_{j(i)} \cap A_i) + M_{i+1} = M_i + M_{i+1} = M_{i+1}$. Thus $K_{j(i+1)}/M_{i+1} = (K_{j(i)} + M_{i+1})/M_{i+1} \oplus C_{i+1}$ where $C_{i+1} := P(a_i + M_{i+1})$ is cyclic and $\exp C_{i+1} \leq k$ since $p^k a_i \in K_{j(i+1)} \cap p^k K \subset M_{i+1}$. To show $\exp C_{i+1} \geq i+1$ suppose $p^m a_i \in M_{i+1} \subset B + p^k K$. Then $p^m a_i = b + p^k x$ with $b \in B$, $x \in K$. Write $x = y + z$ with $y \in K_{j(i)}$, $z \in A_i$. Then $p^k x = p^k y + p^k z \equiv p^k z \bmod M_i$. Thus $p^m a_i \equiv b + p^k z \bmod M_i$, or $p^k z \equiv p^m a_i - b \bmod M_i$. From (a) it follows that $m \geq k$. Hence $\exp C_{i+1} = k = n + i + 1 \geq i + 1$. This proves (4′) and the construction of the M_i is finished.

Now let $M = \bigcup M_i$. We have to show that K/M is reduced, unbounded and p-primary. We shall show that in fact $K/M \cong \bigoplus C_i$. By (4), we have $K_{j(i)} \subset K_{j(i-1)} + Pa_{i-1} + M_i \subset K_{j(i-2)} + Pa_{i-2} + Pa_{i-1} + M_i \subset \cdots \subset Pa_0 + Pa_1 + \cdots + Pa_{i-1} + M_i$. Since $K = \bigcup K_r$ we have $K = \Sigma Pa_r + M$ or $K/M = \Sigma P(a_r + M)$. Suppose $\Sigma_r \lambda_r a_r \equiv 0 \bmod M$. Since this sum is finite and $M = \bigcup M_r$ there is i such that $a_r \in K_{j(i)}$ for all r and $\Sigma \lambda_r a_r \equiv 0 \bmod M_i$. We rewrite this as $\Sigma_{r \leq i-1} \lambda_r a_r + \lambda_i a_i \equiv 0 \bmod M_i$. Now it follows from (4) that $\lambda_i a_i \equiv 0 \bmod M_i$, so $\lambda_i a_i \equiv 0 \bmod M$. Now we have $\Sigma_{r \leq i-1} \lambda_r a_r \in K_{j(i-1)} \cap M_i = M_{i-1}$. Arguing as before we get $\lambda_{i-1} a_{i-1} \equiv 0 \bmod M$ and $\Sigma_{r \leq i-2} \lambda_r a_r \in K_{j(i-2)} \cap M_{i-1} = M_{i-2}$. By induction $\lambda_r a_r \equiv 0 \bmod M$ for all r. Thus we have $K/M = \bigoplus P(a_r + M)$ as claimed.

4.5 **Remark.** In 4.4, P may be any complete discrete valuation ring with prime ideal (p). The proof uses no other property of P.

4.6 **Remark.** Considering K as a topological group with the p-adic topology, Theorem 4.3 can be expressed as follows: K *has a reduced unbounded p-primary epimorphic image if and only if* K *is the union of an ascending sequence of nowhere dense subgroups.* Hence *if* K *is of second category in the p-adic topology then every reduced p-primary epimorphic image of* K *is bounded.*

The converse to the last statement is not true since torsion-free groups of finite rank which are p-reduced are of first category (being countable) but have no unbounded reduced p-primary homomorphic image.

5. **An alternative P-hull.** A different embedding of a group in a P-module is the one described in Cartan-Eilenberg [2].

5.1 **Definition.** For any abelian group K let $K_P = P \otimes K$. The group K_P is a P-module with scalar multiplication given by $\lambda(\mu \otimes x) = \lambda\mu \otimes x$. For each homomorphism $f\colon K \to K'$ let $f_P = 1 \otimes f$.

The P-hull K_P has the following basic properties.

5.2 **Proposition.** (a) $-_P$ *is an exact functor on the category of abelian groups to the category of P-modules.*

(b) K *is embedded in* K_P *if and only if* $K[q] = 0$ *for all primes* $q \neq p$. *If* $K \subset K_P$, *then* $(K_P/K)[p] = 0$, $K_P = PK$ *and* K_P/K *is p-divisible.*

Proof. (a) It is well known that $-_P$ is a functor. Since $\mathrm{Tor}(P, X) = 0$ for any X, the functor $-_P$ is exact.

(b) Suppose $K[q] \neq 0$ for some prime $q \neq p$. Since every torsion element in a P-module has p-power order, K cannot be embedded in K_P. Now suppose $K[q] = 0$ for all primes $q \neq p$. Then it is a direct consequence of the definition of Tor [3, p. 264] that $\mathrm{Tor}(P/\mathbf{Z}, K) = 0$ since $(P/\mathbf{Z})[p] = 0$. Thus it follows from (2.1) that $0 \to \mathbf{Z} \otimes K \cong K \to K_P \to P/\mathbf{Z} \otimes K \to 0$ is exact, and K is embedded in K_P. Since $P/\mathbf{Z}$ is divisible and $(P/\mathbf{Z})[p] = 0$, $P/\mathbf{Z}$ is a direct sum of groups Q and $Z(q^\infty)$, $q \neq p$. Hence $P/\mathbf{Z} \otimes K$ is a direct sum of torsion-free groups $Q \otimes K \cong Q \otimes K/T(K)$ [3, 61.5] and q-groups $Z(q^\infty) \otimes K$, and therefore $(P/\mathbf{Z} \otimes K)[p] = 0$. Since $K_P/K \cong P/\mathbf{Z} \otimes K$, we have $(K_P/K)[p] = 0$, and also K_P/K divisible. Since $\{\lambda \otimes x \mid \lambda \in P,\ x \in K\}$ generates K_P as a group, and $\lambda \otimes x = \lambda(1 \otimes x)$, it is clear that $\{1 \otimes x \mid x \in K\} = K$ generates K_P.

Next we determine K_P in one case, and clarify the connection between K^P and K_P.

5.3 **Proposition.** (a) *If* K *is a P-module, then* $K_P = K \oplus [\bigoplus_{2^{\aleph_0}} (Q \otimes K/T(K))]$. *Thus* $K = K_P$ *if and only if* K *is torsion.*

(b) *If K is p-reduced, then $K^P = K_p/D$ where D is the maximal divisible submodule of K_p.*

Proof. (a) We have two homomorphisms $f\colon K \to P \otimes K\colon xf = 1 \otimes x$ and $g\colon P \otimes K \to K\colon (\lambda \otimes x)g = \lambda x$. Clearly $fg = 1$, hence $K_p = \operatorname{Im} f \oplus \operatorname{Ker} g$. Now $\operatorname{Im} f \cong K$ since f is injective, while $\operatorname{Ker} g \cong (P \otimes K)/\operatorname{Im} f = (P \otimes K)/\{1 \otimes x \mid x \in K\} \cong P/\mathbf{Z} \otimes K \cong P/\mathbf{Z} \otimes (K/T(K)) \cong \bigoplus_{2^{\aleph_0}}(Q \otimes K/T(K))$.

(b) We shall show that $K' = K_p/D$ satisfies (a)–(d) of 2.3. Since K is p-reduced and p-pure in K_p, $K \cap D = 0$, so K is embedded in K'. By definition K' is a reduced P-module. Since D, being divisible, is an absolute direct summand we have $K_p = L \oplus D$ with $L \supset K$. Hence $K'/K = K'/[(K \oplus D)/D] \cong K_p/(K \oplus D) \cong L/K \leq K_p/K$. Since $(K_p/K)[p] = 0$, we have $(K'/K)[p] = 0$. Since K generates K_p as a P-module it also generates K'.

From 5.3(c) it is clear Lemma 2.9 holds with lower Ps instead of upper Ps. Hence the application in §4 goes through with either hull. The same is true for the applications in §3, since we have the following crucial fact.

5.4 **Lemma.** *If T is a reduced p-group, then the groups* $\operatorname{Hom}(K_p, T)$ *and* $\operatorname{Hom}(K, T)$ *are naturally isomorphic.*

Proof. We use [3, p. 256 (J)]. $\operatorname{Hom}(K_p, T) = \operatorname{Hom}(K \otimes P, T) \cong \operatorname{Hom}(K, \operatorname{Hom}(P, T)) \cong \operatorname{Hom}(K, T)$ since $\operatorname{Hom}(P, T) \cong T$ by 2.2(a).

It is hard to say which hull is preferable. The hull K_p applies to a larger class of groups and is actually a functor. The disadvantage is that one has to consider nonreduced modules, and that the scalar multiplication functions in homological obscurity. We preferred the hull K^P because of its connection with the topological completion process for torsion-free K which motivated the whole construction and made it transparent.

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