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KOEBE SEQUENCES OF ARCS AND NORMAL MEROMORPHIC FUNCTIONS

BY

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ABSTRACT. Let f be a normal meromorphic function in the unit disk. An estimate for the growth of the modulus of f on a Koebe sequence of arcs is obtained; the estimate is in terms of the order of normality of f. An immediate consequence of the estimate is the following theorem due to F. Bagemihl and W. Seidel: A nonconstant normal meromorphic function has no Koebe values. Another consequence is that each level set of a nonconstant normal meromorphic function cannot contain a Koebe sequence of arcs provided the order of normality of f is less than a certain positive constant C^*.

1. **Introduction.** A meromorphic function f is normal in the unit disk D: $|z| < 1$ if and only if the family $\{f(S(z))\}$ is normal in D in the sense of Montel, where S is any conformal map of D onto itself. K. Noshiro [10, Theorem 1], and subsequently O. Lehto and K. I. Virtanen [6, Theorem 3], employed the spherical derivative

$$f^*(z) = \frac{|f'(z)|}{1+|f(z)|^2}$$

to obtain the following characterization of normal meromorphic functions: f is normal in D if and only if

$$C = C_f = \sup_{z\in D} (1 - |z|^2)f^*(z) < +\infty; \tag{1.1}$$

C is called the order of normality of f. Using integration, we deduce from (1.1) that an arc γ in D with non-Euclidean length ρ is mapped by f onto an arc with spherical length not exceeding $C\rho$. Thus, f is "more normal" than g when $C_f < C_g$.

Throughout this paper we make use of Pick's (differential) form of Schwarz's lemma. If ϕ is an analytic mapping of D into itself, then

$$(1 - |z|^2)|\phi'(z)| \leq 1 - |\phi(z)|^2; \tag{1.2}$$

thus $C_\phi \leq 1$, and $C_{f(\phi)} \leq C_f$ for any function f meromorphic in D. Equality holds in all cases if ϕ is a conformal map of D onto itself.

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If f is defined in D and ζ is a point of the unit circle Γ: $|z| = 1$, we set

$$|f(\zeta)| = \limsup_{z \to \zeta} |f(z)|.$$

In the terminology of cluster sets, $|f(\zeta)|$ is the radius of the smallest closed disk such that the disk has center zero and contains all the cluster values (boundary values) of f at ζ. If $|f(\zeta)| \leq M < +\infty$ for all points ζ in a subarc γ of Γ, then f is said to be bounded (by M) on γ. By combining the classical two-constants estimate for $|f(z)|$ and the estimate (1.1) for $|f'(z)|$, Lehto and Virtanen obtained a "two-constants theorem" for normal meromorphic functions bounded on a subarc γ. Several related results and examples are discussed in §2.

In §3 we prove the central result of this paper: An estimate for the growth of the modulus of a normal meromorphic function on a Koebe sequence of arcs. Following F. Bagemihl and W. Seidel [1], a sequence of disjoint Jordan arcs $\{\gamma_n\}$ in D is called a Koebe sequence of arcs relative to an open subarc γ of Γ provided (i) each neighborhood of γ contains all but finitely many of the arcs γ_n, and (ii) every open sector Δ of D subtending a subarc whose closure lies in γ has the property that all but finitely many of the arcs γ_n contain at least one Jordan subarc lying entirely in Δ except for its two endpoints which lie on distinct sides of Δ. If f is defined in D and $\{\gamma_n\}$ is a Koebe sequence of arcs, we set

$$m_n = m_n(\gamma_n; f) = \inf_{z \in \gamma_n} |f(z)| \quad \text{and} \quad M_n = M_n(\gamma_n; f) = \sup_{z \in \gamma_n} |f(z)|.$$

If $\lim M_n = 0$, zero is called a Koebe value of f. More generally, the complex number c is a Koebe value of f provided zero is a Koebe value of $f(z) - c$ or $1/f$, according as c is finite or $c = \infty$. Koebe's lemma states that a nonconstant bounded analytic function has no Koebe values.

Some estimates that we obtain for the growth of $|f|$ are the following (Theorem 1): If f is a meromorphic function in D with order of normality $C > 0$ and $\{\gamma_n\}$ is a Koebe sequence of arcs relative to an open subarc γ of Γ, then

$$\liminf M_n \geq \frac{1 + (1 + C^2)^{1/2}}{C} \exp[-(1 + C^2)^{1/2}] \tag{1.3}$$

or

$$\liminf M_n \geq \sup_{\zeta \in \gamma} |f(\zeta)|, \tag{1.4}$$

according as f is unbounded or bounded on γ. By combining (1.3) and (1.4) (or (1.3) and Koebe's lemma), we reestablish a result of Bagemihl and Seidel [1, Theorem 1]: A nonconstant normal meromorphic function cannot have zero (and hence any extended complex number) as a Koebe value.

Since f and $1/f$ have equal spherical derivatives, $C_f = C_{1/f}$. We infer from (1.3) that if $1/f$ is not bounded on γ, then

$$\limsup m_n \leq \frac{C}{1+(1+C^2)^{1/2}} \exp[(1+C^2)^{1/2}]. \tag{1.5}$$

In our investigations, a fundamental role is played by the number $C = C^*$ for which the right-hand sides of (1.3) and (1.5) are equal (and hence equal to one); $C^* \approx 2/3$. In particular, if both f and $1/f$ are unbounded on γ and $C_f < C^*$, then

$$\limsup m_n < \liminf M_n.$$

In §4 we consider the family $\mathfrak{L}$. A nonconstant meromorphic function $f \in \mathfrak{L}$ if and only if every level set of f does not contain a Koebe sequence of arcs. A lemma of G. R. MacLane [7, p. 11] states that each nonconstant bounded analytic function belongs to $\mathfrak{L}$; a brief proof of MacLane's lemma has been given by K. F. Barth and W. J. Schneider [2]. More generally, MacLane [7, Theorems 1 and 17] proved that each nonconstant normal analytic function belongs to $\mathfrak{L}$. We prove the following results: (Lemma 2) If the nonconstant normal meromorphic function f has a level set L_R,

$$L_R = \{z : |f(z)| = R\},$$

such that L_R contains a Koebe sequence of arcs relative to an open subarc γ of Γ, then

$$\frac{1+(1+C^2)^{1/2}}{C} \exp[-(1+C^2)^{1/2}] \leq R \leq \frac{C}{1+(1+C^2)^{1/2}} \exp[(1+C^2)^{1/2}], \quad C^* \leq C = C_f,$$

and each point of γ is a limit point of zeros and poles of f; thus (Theorem 2) if either f is a nonconstant normal analytic function in D or $C = C_f$ satisfies $0 < C < C^*$, then $f \in \mathfrak{L}$. Moreover, (Theorem 3)

$$C^* = \inf\{C_f : f \notin \mathfrak{L}, C_f > 0\}.$$

Whether or not there exists a normal meromorphic function $f \notin \mathfrak{L}$ such that $C^* = C_f$ is an open question.

Finally, in §5 we discuss the boundary behavior of normal meromorphic functions f for which $|f|$ has a Koebe value.

2. **Boundary estimates for normal functions.** Let G_α be a lens (crescent) bounded by the unit circle Γ: $|z| = 1$ and a circular arc in D that intersects Γ at an angle α, $0 < \alpha < \pi$. Excluding the vertices of G_α, the boundary of G_α consists of an open subarc Γ_β of Γ such that $\Gamma - \Gamma_\beta$ is viewed under the angle β $(0 < \beta < \pi)$ from Γ_β and an open circular arc Γ_δ in D such that $\Gamma - \Gamma_\beta$ is viewed under the angle δ $(\beta < \delta < \pi + \beta)$ from Γ_δ. Note that $\alpha = \delta - \beta$. The harmonic measure ω of Γ_β with respect to G_α is given by

$$\omega(z, \Gamma_\beta, G_\alpha) = \alpha^{-1}(\delta - \theta) \qquad (\beta < \theta < \delta),$$

where θ is the angle under which $\Gamma - \Gamma_\beta$ is viewed from z. In the theorem to follow, we make use of the expression

$$\lambda(z) = \frac{1}{1-|z|^2} \bigg/ \frac{\partial\omega}{\partial n} \qquad (z \in \Gamma_\delta),$$

where $\partial\omega/\partial n$ is the normal derivative of ω at $z \in \Gamma_\delta$. Using (1.2), we find that $\lambda(z)$ $(z \in G_\alpha)$ is invariant under a conformal map $z = \phi(z')$ of D onto itself. Therefore, we can assume that Γ_δ passes through $z = 0$ and that the positive real axis is the inner normal to Γ_δ at $z = 0$. Some elementary calculations yield

$$\lambda_\alpha = \lambda(z)|_{z\in\Gamma_\delta} = \frac{\alpha}{2\sin\alpha}.$$

In this setting, Lehto and Virtanen's "two-constants theorem" assumes the following form.

Theorem A. *Let f be meromorphic in D with order of normality C $(0 < C < +\infty)$, and suppose*

$$\sup_{z\in G_\alpha} |f(z)| \geq \mu.$$

If f is bounded by M on the open arc Γ_β, then

$$M \geq \mu \exp[-C\lambda_\alpha(\mu + 1/\mu)]. \tag{2.1}$$

If

$$\mu_\alpha = \frac{1 + (1 + 4C^2\lambda_\alpha^2)^{1/2}}{2C\lambda_\alpha},$$

then the best estimates in (2.1) *are obtained by setting*

$$\mu = \sup_{z\in G_\alpha} |f(z)|, \quad \text{if} \sup_{z\in G_\alpha} |f(z)| < \mu_\alpha,$$

and

$$\mu = \mu_\alpha, \quad \text{if} \sup_{z\in G_\alpha} |f(z)| \geq \mu_\alpha.$$

In their version of Theorem A, Lehto and Virtanen assumed that f is bounded by M on the closure of the arc Γ_β; our weaker form is easily verified by an obvious limiting argument using closed subarcs of Γ_β.

We briefly indicate once again how the best estimates are obtained. As a function of $\mu \geq 0$,

$$\mu \exp[-C\lambda_\alpha(\mu + 1/\mu)]$$

has its maximum at $\mu = \mu_\alpha$. Therefore, if

$$\sup_{z\in G_\alpha} |f(z)| \geq \mu_\alpha,$$

we can choose $\mu = \mu_\alpha$ in (2.1) and hence maximize its right-hand side. If

$$\sup_{z\in G_\alpha} |f(z)| < \mu_\alpha,$$

then

$$\mu = \sup_{z\in G_\alpha} |f(z)|$$

maximizes the right-hand side of (2.1).

If f is bounded by M on Γ with the (possible) exception of one point, then Theorem A, as stated, does not apply. However, by using a limiting process (as $\alpha \to 0$). Lehto and Virtanen deduced the following estimate of the Pharagmén-Lindelőf type.

Theorem B. *Let f be meromorphic in D with order of normality C $(0 < C < +\infty)$, and let f be bounded by M on $\Gamma - \{\zeta\}$, $\zeta \in \Gamma$. Then*

$$M \geq \frac{1 + (1 + C^2)^{1/2}}{C} \exp[-(1 + C^2)^{1/2}], \tag{2.2}$$

unless f is bounded in D, in which case $M \geq C$.

As noted in [6, p. 62], the estimate (2.2) is sharp. The function

$$f(z) = M \exp(b(1 + z)/(1 - z)) \qquad (M > 0,\ b > 0) \tag{2.3}$$

is bounded by M on $\Gamma - \{1\}$ and equality holds in (2.2) for $C = C_f$.

We now investigate the sharpness of the estimate (2.2) in case f is bounded by M on all of Γ and unbounded in D. To simplify notation, we henceforth let $C = C_M^*$ be the solution of

$$M = \frac{1 + (1 + C^2)^{1/2}}{C} \exp[-(1 + C^2)^{1/2}]. \tag{2.4}$$

As in the introduction, we set $C^* = C_1^*$. Using this notation, (2.2) is equivalent to the inequality $C_M^* \leq C$.

Example 1. *Let $M > 0$, and set $C_n = C_{f_n}$, where $f_n(z) = Mz^{-n}$ $(n = 2, 3, \cdots)$. Then $C_M^* \leq C_n = C_M^* + o(1)$.*

Proof. The inequality $C_M^* \leq C_n$ follows from Theorem B. Let

$$\Phi_n(z) \doteq (1 - |z|^2)f_n^*(z) = (1 - |z|^2)\frac{Mn|z|^{n-1}}{M^2 + |z|^{2n}}.$$

Then

$$C_n = \max_{0\leq r\leq 1} \Phi_n(r).$$

We have

$$\Phi_n'(r) = \frac{Mnr^{n-2}}{(M^2 + r^{2n})^2}[(n - 1)r^{2n+2} - (n + 1)r^{2n} - M^2(n + 1)r^2 + M^2(n - 1)].$$

Now, for all values of n,

$$\Phi_n'(n/(n+1)) < 0; \tag{2.5}$$

and, for all sufficiently large values of n,

$$\Phi_n'((n - M^{-2})/(n+1)) > 0. \tag{2.6}$$

To verify (2.6), we note that as $n \to +\infty$ in (2.6), we obtain the valid inequality

$$2 > (4 + 2M^{-2}) \exp[-2(1 + M^{-2})].$$

From (2.5) and (2.6), it follows that

$$\Phi_n'((n - t_n)/(n+1)) = 0 \tag{2.7}$$

for some t_n satisfying $0 < t_n < M^{-2}$.

Because Φ_n' has exactly one zero for $0 < r < 1$,

$$C_n = \Phi_n((n - t_n)/(n+1)). \tag{2.8}$$

If t' is a limit point of the sequence $\{t_n\}$ and we formally compute the limit as $n \to +\infty$ in (2.7), we find that t' satisfies

$$t + 2 = M^2 t e^{2(1+t)}. \tag{2.9}$$

However, (2.9) has exactly one solution for $t > 0$ and thus $t' = \lim t_n$. Moreover, since t' satisfies (2.9), t' maximizes the function

$$\psi(t) = \frac{2M(t+1)e^{1+t}}{1 + M^2 e^{2(1+t)}}$$

for which $\psi'(t) = 0$ reduces to (2.9). If we set $t' = (1 + C^2)^{1/2} - 1$, then some elementary computations show that

$$\psi(t') = \psi((1 + C^2)^{1/2} - 1) = C$$

and that C satisfies (2.4). Thus $C = C_M^*$ and we find

$$\lim C_n = \lim \Phi_n((n - t_n)/(n+1)) = \psi(t') = C = C_M^*.$$

Example 1 can be verified by another method that points out a connection between the functions Mz^{-n} and the function in (2.3); this method is based on the following lemma.

Lemma 1. *Let $\{g_n\}$ be a sequence of normal meromorphic functions that converges uniformly to g on compact subsets of D. Let $C_n = C_{g_n}$. Then*

$$C_g \le \liminf C_n. \tag{2.10}$$

If there exists an r $(0 \le r < 1)$ such that

$$C_n = (1 - |z_n|^2)g_n^*(z_n), \qquad |z_n| \le r, \tag{2.11}$$

then $C_g = \lim C_n$.

Proof. Let $\{g_{n_k}\}$ be a subsequence such that $\lim C_{n_k} = \liminf C_n$. The sequence $\{g_n'\}$ converges uniformly to g' on compact subsets of D; thus, for each $z \in D$,

$$(1 - |z|^2)g^*(z) = \lim(1 - |z|^2)g_{n_k}^*(z) \le \lim C_{n_k}.$$

Thus

$$C_g \le \liminf C_n.$$

If (2.11) holds for some r $(0 \le r < 1)$, then

$$\begin{aligned} \limsup C_n &= \limsup \left(\max_{|z| \le r} (1 - |z|^2)g_n^*(z) \right) \\ &= \max_{|z| \le r} (1 - |z|^2)g^*(z) \le C_g. \end{aligned} \tag{2.12}$$

By (2.10) and (2.12), we have $C_g = \lim C_n$.

Returning to the sequence $f_n(z) = Mz^{-n}$ $(n = 2, 3, \cdots)$, $C_n = C_{f_n}$, set

$$g_n(w) = f_n\left(\frac{w - x_n}{1 - x_n w}\right),$$

where x_n is determined from an arbitrary fixed number $b > 0$ by $x_n = e^{-b/n}$. Then $C_n = C_{g_n}$ by (1.2). From (2.8), if $|z| = (n - t_n)/(n + 1)$, then $C_n = (1 - |z|^2)f_n^*(z)$. We choose $z_n = -(n - t_n)/(n + 1)$. Then $C_n = (1 - |w_n|^2)g_n^*(w_n)$, where w_n is determined by $(w_n - x_n)/(1 - x_n w_n) = z_n$. We find

$$w_n = \frac{1 + t_n - b + o(1)}{1 + t_n + b + o(1)}.$$

Since $0 < t_n < M^{-2}$, the condition (2.11) in Lemma 1 holds for all sufficiently large values of n. Therefore,

$$\lim C_n = C_g,$$

where g is the limit of the sequence $\{g_n\}$. A direct calculation yields

$$g(w) = \lim M\left(\frac{1 - e^{-b/n}w}{w - e^{-b/n}}\right) = M \exp\left(b\,\frac{1 + w}{1 - w}\right).$$

As noted in (2.3), $C_g = C_M^*$. This reestablishes Example 1.

3. Estimates on Koebe sequences. We now obtain some estimates for the growth of the modulus of a normal function on a Koebe sequence of arcs.

Theorem 1. *Let f be meromorphic in D with order of normality C $(0 < C \le +\infty)$. Let γ be an open subarc of Γ and let $\{\gamma_n\}$ be a Koebe sequence of arcs relative to γ. Set $M = \sup_{\zeta\in\gamma} |f(\zeta)|$. If f is not bounded on γ, then*

$$\text{(3.1)} \qquad \liminf M_n \ge \frac{1 + (1 + C^2)^{1/2}}{C} \exp[-(1 + C^2)^{1/2}];$$

and if f is bounded on γ, then

$$\text{(3.2)} \qquad \liminf M_n \ge M.$$

Also, if $0 < C < +\infty$ and f is analytic in a domain $G \subset D$ for which γ is a free boundary arc, then

$$\text{(3.3)} \qquad \liminf M_n \ge M.$$

Proof. If $C = +\infty$, we let the right-hand side of (3.1) equal zero. Thus we assume that $0 < C < +\infty$. If f is not bounded on γ, we can choose a point $\zeta^* \in \gamma$ such that

$$|f(\zeta^*)| > \frac{1 + (1 + C^2)^{1/2}}{C}.$$

Let γ^* be an open arc such that $\zeta^* \in \gamma^*$ and the closure of γ^* lies in γ. Let ζ and ζ' denote the left and right endpoints, respectively, of γ^* as viewed from $z = 0$, and let r and r' be the radii at ζ and ζ', respectively. For all sufficiently large values of n, the arc γ_n has a Jordan subarc γ_n^* with an endpoint on each of r and r' and otherwise lying in the sector subtending γ^*. Let r_n and r_n' denote the segments on r and r', respectively, such that the arc

$$\Gamma_n = r_n \cup \gamma_n^* \cup r_n'$$

is a Jordan crosscut of D. The region bounded by $\Gamma_n \cup \Gamma - \gamma^*$ is denoted by D_n. Then $\lim D_n = D$ in the obvious sense. For all sufficiently large values of n, we can choose a point $z_n \in D_n$ such that

$$|f(z_n)| > \frac{1 + (1 + C^2)^{1/2}}{C}$$

and $\lim z_n = \zeta^*$. By the principle of monotoneity for harmonic measures,

$$\text{(3.4)} \qquad \omega(z_n, \gamma^*, D) \le \omega(z_n, \Gamma_n, D_n).$$

Let $z = \phi_n(w)$, $z_n = \phi_n(0)$, be a conformal map of $|w| < 1$ onto D_n. Then $\Gamma_n = r_n \cup \gamma_n^* \cup r_n'$ corresponds to a subarc $\Lambda_n \cup \Lambda_n^* \cup \Lambda_n'$ of $|w| = 1$ subtended by a central angle of measure $(\delta_n + \delta_n^* + \delta_n')/2\pi$, where r_n corresponds to the subarc Λ_n subtended by a central angle of measure $\delta_n/2\pi$, etc. Since $\{z_n\}$ converges to ζ^*, $\lim \omega(z_n, \gamma^*, D) = 1$; thus, by (3.4) and the invariance of harmonic measure under ϕ_n, (3.5) $$\lim (\delta_n + \delta_n^* + \delta_n')/2\pi = 1.$$

By performing rotations if necessary, we can assume that the arcs Λ_n have a common right endpoint as viewed from $w = 0$. The segments r_n converge uniformly to ζ; thus, the sequence $\{\phi_n\}$ converges uniformly to ζ on $\bigcap \Lambda_n$. If $\bigcap \Lambda_n$ is a proper subarc of $|w| = 1$, it follows that $\{\phi_n\}$ converges uniformly to ζ on compact subsets of $|w| < 1$ (see [4, Corollary 1] or [6, Theorem 10]). However, $\phi_n(0) = z_n$ and $\lim z_n = \zeta^* \neq \zeta$. Thus $\bigcap \Lambda_n$ is a singleton and $\lim \delta_n = 0$. In a similar fashion it is seen that $\lim \delta_n' = 0$. From (3.5),

(3.6) $$\lim \delta_n^* = 2\pi.$$

Let G_n be the lens bounded by Λ_n^* and the circle through $w = 0$ joining the endpoints of Λ_n^*. Since Λ_n^* is subtended by the central angle δ_n^*, it follows from some simple geometric considerations that G_n has vertex angle α_n given by

(3.7) $$\alpha_n = \pi - \tfrac{1}{2}\delta_n^*.$$

We now apply the estimate of Theorem A to the function $f(\phi_n(w))$. First, set

$$M_n^* = \sup_{w \in \Lambda_n^*} |f(\phi_n(w))| \leq M_n,$$

$$\lambda_n = \lambda_{\alpha_n} = \frac{\alpha_n}{2 \sin \alpha_n},$$

$$\mu_n = \mu_{\alpha_n} = \frac{1 + (1 + 4C^2\lambda_n^2)^{1/2}}{2C\lambda_n}.$$

Since

$$|f(\phi_n(0))| = |f(z_n)| > \frac{1 + (1 + C^2)^{1/2}}{C} > \mu_n,$$

we use $\mu = \mu_n$ in (2.1) to obtain the estimate

$$M_n \geq M_n^* \geq \mu_n \exp[-C\lambda_n(\mu_n + 1/\mu_n)].$$

By (3.6) and (3.7), $\lim \lambda_n = \tfrac{1}{2}$; thus,

$$\lim \mu_n = \frac{1 + (1 + C^2)^{1/2}}{C}.$$

The estimate (3.1) now follows.

Before we continue with the proof of Theorem 1, we cite Bagemihl and Seidel's result [1, Theorem 1] which shall be used later in our proof.

Corollary 1. *Let f be meromorphic in D with order of normality C $(0 < C < +\infty)$. Then f has no Koebe values.*

Proof. Let γ be any open subarc of Γ. If f is bounded on γ, then Koebe's lemma implies that $w = 0$ cannot be a Koebe value of f on a Koebe sequence of arcs relative to γ. If f is not bounded on γ, then the estimate (3.1) implies the same conclusion. If $w = c$ is a Koebe value of f, then we consider $F(z) = f(z) - c$ or $F(z) = 1/f(z)$, depending on whether c is finite or $c = \infty$. In either case, F is normal and therefore cannot have zero as a Koebe value; thus c cannot be a Koebe value of f and the corollary is proved.

We now return to the proof of Theorem 1. If f is bounded on γ, then, by Fatou's theorem, f has radial limits almost everywhere on γ. The estimate (3.2) now follows from the maximum principle as noted by MacLane [7, Theorem 9]. (Let γ^* be any closed subarc of γ such that f has a radial limit at each endpoint of γ^*. The maximum principle is applied to f in the region bounded by $\gamma_n^*, r', \gamma_{n+1}^*, r$; our notation here is that used in the proof of (3.1). It is worth noting that if M is the maximum of $|f(\zeta)|$ for ζ in the closure of γ, then equality holds in (3.2).)

Now suppose f is analytic in a domain $G \subset D$ for which γ is a free boundary arc. If M is finite, then we just proved the desired estimate (3.3) (even if $C = +\infty$). Therefore assume $M = +\infty$ and $0 < C < +\infty$. Then either (i) there is a point $\zeta^* \in \gamma$ at which f is unbounded or (ii) for each open subarc γ^* of γ,

$$M^* = \sup_{\zeta \in \gamma^*} |f(\zeta)| < +\infty,$$

and $\sup M^* = +\infty$. If (ii) occurs, then (3.2) implies that for each open subarc $\gamma^* \subset \gamma$,

$$\liminf M_n \geq M^*.$$

Thus $\lim M_n = +\infty = M$.

Finally, suppose (i) occurs. (It can be deduced from a theorem of McMillan [8, Theorem 1] that if

$$\limsup M_{n_k} < +\infty$$

for any subsequence $\{\gamma_{n_k}\}$, then ∞ is a Koebe value of f on a Koebe sequence of arcs relative to a subarc of γ having ζ^* as one endpoint. Since $0 < C < +\infty$, f has no Koebe values; it follows that $\lim M_n = +\infty$ as required. We sketch a proof that in spirt is somewhat similar to that of McMillan. It is also similar to the type of argument used by Barth and Schneider [2] to prove MacLane's lemma.) Let us first assume that M' can be chosen so that

$$M_n < M' \qquad (n = 1, 2, \cdots). \tag{3.8}$$

Select an open subarc γ^* for which the closure lies in γ and such that $\zeta^* \in \gamma^*$. Let Δ be the open sector subtending γ^*. Since γ is a free boundary arc of G, there exists an $\epsilon > 0$ such that G contains the set

$$\Delta_\epsilon = \Delta \cap \{z : 1 - \epsilon < |z| < 1\}.$$

Choose a sequence $\{z_k\}$ in Δ_ϵ such that $\lim z_k = \zeta^*$ and such that

$$|f(z_k)| > M' + k \qquad (k = 1, 2, \cdots). \tag{3.9}$$

We can also assume that the Riemann surface of f over the extended plane (sphere) has no branch points over any of the radial segments

$$R_k = \{w : |w_k| \le |w| < +\infty, \arg w = \arg w_k, w_k = f(z_k)\}.$$

If the regular element $e_{z_k}(w, w_k)$ of $z = f^{-1}(w)$ is continued along R_k, there are two possibilities: either the continuation defines a transcendental singularity in a finite distance or at $w = \infty$, or the continuation defines a regular or algebraic element at $w = \infty$. In each case, the branch of f^{-1} defined by the continuation maps the segment of R_k in question onto a Jordan arc γ_k' with initial point z_k. By (3.8) and (3.9), $\gamma_k' \cap \gamma_n = \emptyset$ $(n = 1, 2, \cdots)$. In the first case, γ_k' terminates at a point of $\Gamma - \gamma$; in the second case, γ_k' terminates at a pole of f outside Δ_ϵ. It is apparent that from the sequence $\{\gamma_k'\}$ we can find a Koebe sequence of arcs on which f has Koebe value ∞. This contradiction shows that the condition (3.8) cannot hold; that is, $\lim\sup M_n = +\infty$. By the same argument, $\lim\sup M_{n_k} = +\infty$ for any subsequence $\{\gamma_{n_k}\}$. Thus, $\lim M_n = +\infty$, and this completes the proof of (i).

We shall make use of the estimate (3.3) in the following form.

Corollary 2. *Let f be meromorphic and normal in D. Let γ be a subarc of Γ such that f is analytic in a domain $G \subset D$ and γ is a free boundary arc of G. If f is bounded on a Koebe sequence of arcs relative to γ, then f is bounded on γ; moreover,*

$$M = \sup_{\zeta \in \gamma} |f(\zeta)| \le \lim\inf M_n < +\infty.$$

Since f and $1/f$ have the same order of normality, we can apply Theorem 1 to $1/f$ to obtain the following result.

Corollary 3. *Let f be meromorphic in D with order of normality C $(0 < C \le +\infty)$. Let γ be an open subarc of Γ and let $\{\gamma_n\}$ be a Koebe sequence of arcs relative to γ. Set*

$$m = \inf_{\zeta \in \gamma} \left(\liminf_{z \to \zeta} |f(z)| \right).$$

If $w = 0$ is a boundary value of f on γ, then

$$\lim\sup m_n \le \frac{C}{1 + (1 + C^2)^{1/2}} \exp[(1 + C^2)^{1/2}]; \tag{3.10}$$

and if $w = 0$ is not a boundary value of f on γ, then

$$\lim\sup m_n \le m.$$

Also, if $0 < C < +\infty$ *and* f *omits* $w = 0$ *is a domain* $G \subset D$ *for which* γ *is a free boundary arc, then*

$$\limsup m_n \le m.$$

The following example is concerned with the sharpness of the estimate (3.1) of Theorem 1.

Example 2. Let $w = \phi(z)$ be an analytic map of D into $|w| < 1$ such that $w = 0$ is a boundary value of ϕ at $\zeta = 1$ and

$$\lim_{|z|\to 1;\; z\in\sigma} |\phi(z)| = 1,$$

where σ is a spiral in D (see [11, p. 14]). Let $g_n(w) = Mw^{-n}$ and set $f_n(z) = g_n(\phi(z))$. By (1.2) and Example 1,

$$C_{f_n} \le C_{g_n} = C_M^* + o(1).$$

Also, f_n is unbounded at $\zeta = 1$ so that the estimate (3.1) applies. Since

$$\lim_{|z|\to 1;\; z\in\sigma} |f_n(z)| = M,$$

(3.1) can be written as $C_M^* \le C_{f_n}$. Thus

$$C_M^* \le C_{f_n} \le C_M^* + o(1).$$

4. Level sets of normal functions. We now consider the family $\mathfrak{L}$ defined in the introduction.

Lemma 2. *Suppose the nonconstant normal meromorphic function* $f \notin \mathfrak{L}$. *Then*

(i) $$C^* \le C = C_f.$$

If the level set L_R *of* f *contains a Koebe sequence of arcs* $\{\gamma_n\}$ *relative to the open subarc* γ *of* Γ, *then the following hold*:

(ii) $$\frac{1 + (1 + C^2)^{1/2}}{C} \exp[-(1 + C^2)^{1/2}] \le R \le \frac{C}{1 + (1 + C^2)^{1/2}} \exp[(1 + C^2)^{1/2}],$$

(iii) *if* ζ *is any point of* γ *and*

$$R' < \frac{1 + (1 + C^2)^{1/2}}{C} \exp[-(1 + C^2)^{1/2}],$$

then, in each neighborhood of ζ, *there is a component* U *of the set* $f^{-1}(|w| < R')$ *and a component* V *of the set* $f^{-1}(|w| > 1/R')$ *such that the closures of* U *and* V *lie in* D.

Proof. We first verify (ii). Both f and $1/f$ must be unbounded on γ (and in fact on each subarc of γ), otherwise the assumption concerning L_R is impossible by MacLane's lemma. Since $R = \lim M_n = \lim m_n$, the inequality (ii) follows from (3.1) and (3.10).

If $f \notin \mathfrak{L}$, then there exists an R such that (ii) holds for $C = C_f$; thus $C^* \leq C$.

Finally, we verify (iii). Let $\zeta \in \gamma$. Since f and $1/f$ are unbounded on each subarc of γ, Corollary 2 implies that ζ is a limit point of poles and zeros of f. For each positive integer k, let U_k be a component of $f^{-1}(|w| < R')$ containing a zero z_k of f, and let V_k be a component of $f^{-1}(|w| > 1/R')$ containing a pole p_k of f, where $z_k \to \zeta$ and $p_k \to \zeta$. By (ii), the level sets $L_{R'}$ and $L_{1/R'}$ cannot contain a Koebe sequence of arcs. Thus the diameters of U_k and V_k tend to zero as $k \to +\infty$. Because L_R contains the Koebe sequence $\{\gamma_n\}$, the closures of U_k and V_k lie in D for all sufficiently large values of k.

The next result follows from (i) and (iii) of Lemma 2.

Theorem 2. *Let f be meromorphic in D with order of normality C $(0 < C < +\infty)$. If either f is analytic in D or $C < C^*$, then $f \in \mathfrak{L}$.*

We now show that C^* cannot be replaced by any larger number in Lemma 2 and Theorem 2.

Theorem 3. $C^* = \inf \{C_f : f \notin \mathfrak{L},\ C_f > 0\}$.

Proof. Lemma 2 states that C^* is a lower bound for the set in question. For each value $k = 3, 4, \cdots$, let T_k be a regular non-Euclidean polygon in D with k sides and center $z = 0$ and such that each vertex angle of T_k has magnitude $2\pi/3$. Let n be a fixed positive integer, and let $w = f_k(z)$ be an analytic n-to-1 map of T_k onto $|w| < 1$ such that $f_k(z) = 0$ only for $z = 0$. We extend f_k from T_k to all of the unit disk D by the usual reflection technique. It is apparent that the level set L_1 of f_k consists of the boundary of T_k and all its reflections. Thus L_1 contains a Koebe sequence of arcs relative to Γ and hence $f_k \notin \mathfrak{L}$.

We intend to show that f_k can be constructed so that

$$\lim_{k \to +\infty} f_k(z) = z^n$$

uniformly on compact subsets of D and

$$C_{f_k} = (1 - |z_k|^2) f_k^*(z_k),$$

where $|z_k| \leq r < 1$. Then, by Lemma 1,

$$\lim C_{f_k} = C_n,$$

where C_n is the order of normality of z^n. By Example 1,

$$\lim C_n = C^*.$$

Since $f_k \notin \mathfrak{L}$, this implies the assertion of our theorem.

Because the expression $(1-|z|^2)f_k^*(z)$ is invariant under reflections through non-Euclidean lines of the unit disk D,

$$C_{f_k} = \sup_{z\in T_k} (1-|z|^2)f_k^*(z).$$

Also D is the kernel of the domains T_k. Thus, we need only concern ourselves with the construction of f_k on T_k. Let T_k be positioned so that there is a vertex of T_k located on the positive real axis at r_k. The conformal map $z = h_k(w)$ of $|w| < 1$ onto T_k such that $h_k(0) = 0$ and $h_k(1) = r_k$ can be given in terms of hypergeometric functions as follows (see [5, p. 83]):

$$z = h_k(w) = r_k s_k \psi_2(w)/\psi_1(w),$$

where

$$\psi_1(w) = F\left(\frac{1}{6} - \frac{1}{k}, \frac{1}{6}, 1 - \frac{1}{k}; w^k\right), \qquad \psi_2(w) = wF\left(\frac{1}{6} + \frac{1}{k}, \frac{1}{6}, 1 + \frac{1}{k}; w^k\right),$$

and

$$s_k = \frac{\psi_1(1)}{\psi_2(1)} = \frac{\Gamma(1-1/k)\Gamma(5/6+1/k)}{\Gamma(1+1/k)\Gamma(5/6-1/k)}.$$

Let $w = g_k(z)$ be the inverse of h_k and set $w = f_k(z) = [g_k(z)]^n$. We have $\lim h_k(w) = w$ uniformly on $|w| \le 1$. Thus,

$$\lim f_k(z) = z^n$$

uniformly on compact subsets of D. By Lemma 1, C_n, the order of normality of z^n, satisfies

$$0 < C_n \le \liminf C_{f_k}. \tag{4.1}$$

We now show that f_k', and hence f_k^*, is bounded independent of k on the closure of T_k. First,

$$h_k'(w) = r_k s_k \frac{\psi_1\psi_2' - \psi_2\psi_1'}{\psi_1^2} kw^{k-1}$$

$$= r_k s_k \frac{1/k}{(w^k)^{1-1/k}(1-w^k)^{1/3}\psi_1^2} kw^{k-1} = r_k s_k \frac{1}{(1-w^k)^{1/3}\psi_1^2}.$$

The quantity $\psi_1\psi_2' - \psi_2\psi_1'$ has the above simplification using the technique of Carathéodory [3, p. 164]. Thus,

$$|h_k'(w)| \ge \frac{r_3 s_3}{2^{1/3}} \frac{1}{|\psi_1|^2} \ge \frac{r_3 s_3}{2^{1/3}} \frac{1}{F^2(1/6, 1/6, 2/3; 1)},$$

and

$$|f_k'(z)| \leq n|g_k'(z)| \leq n\frac{2^{1/3}}{r_3 s_3} F^2\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}; 1\right).$$

It now follows that there exists a compact set $|z| \leq r < 1$ such that $C_{f_k} = (1 - |z_k|^2) f_k^*(z_k)$, where $|z_k| \leq r$; for if $|z_k| \to 1$, then (since f_k^* is bounded independent of k on the closure of T_k) $\lim C_{f_k} = 0$. This contradicts (4.1).

5. **Functions bounded on Koebe sequences.** Although a nonconstant normal meromorphic function f cannot have a Koebe value, $|f|$ can have a Koebe value R as evidenced by the functions of Examples 1 and 2 and Theorem 3. It must be the case that $0 < R < +\infty$. Also, a function f can be "very normal" and yet $|f|$ can have a Koebe value. For example, if $M > 0$ and $f(z) = z/M$ then $|f|$ has a Koebe value and $C_f = 1/M$.

Let f be meromorphic and normal in D. Suppose $|f|$ has R as a Koebe value on a Koebe sequence of arcs $\{\gamma_n\}$ relative to an open subarc γ of Γ. If no point of γ is a limit of poles of f, then Corollary 2 yields $|f(\zeta)| \leq R$ $(\zeta \in \gamma)$. By Fatou's theorem, f has radial limits at almost every point of γ; these radial limits all have modulus R. If no point of γ is a limit point of zeros of f, then Corollary 2 applied to $1/f$ yields

$$\limsup_{z\to\zeta} |f(z)|^{-1} \leq R^{-1} \qquad (\zeta \in \gamma);$$

that is, f has no boundary values w on γ such that $|w| < R$. On the other hand suppose $\zeta \in \gamma$ is a limit point of zeros of f; let $z_k \to \zeta$ be such a sequence of zeros. Choose R' $(0 < R' < R)$ such that the Riemann surface of f has no branch points over the circle $|w| = R'$. Let U_k be the component of $f^{-1}(|w| < R')$ containing z_k. For all sufficiently large values of k, U_k cannot intersect any of the arcs γ_n since

$$\lim m_n(\gamma_n; f) = R > R'.$$

Since f has radial limits of modulus $R > R'$ at almost every point of γ, we conclude that the diameter of U_k tends to zero as $k \to +\infty$. Thus $|w| \leq R'$ lies in the set of boundary values of f at ζ. Since R' can be chosen arbitrarily close to R, we conclude that $|w| \leq R$ is the set of boundary values of f at ζ.

In the foregoing discussion, we have assumed that no point of γ is a limit point of poles of f. The following theorem shows that this is the case if C_f is sufficiently small.

Theorem 4. *Let f be meromorphic in D with order of normality $C > 0$. Let $|f|$ have the Koebe value R on a Koebe sequence of arcs $\{\gamma_n\}$ relative to an open subarc γ of Γ. If $C < C^*$, then either f or $1/f$ is bounded on γ.*

Proof. Suppose both f and $1/f$ are unbounded on γ. Then, by (3.1),

$$R = \lim M_n \geq \frac{1 + (1 + C^2)^{1/2}}{C} \exp\left[-(1 + C^2)^{1/2}\right]$$

and by (3.10),

$$R = \lim m_n \leq \frac{C}{1 + (1 + C^2)^{1/2}} \exp\left[(1 + C^2)^{1/2}\right].$$

Because $C < C^*$, these inequalities are inconsistent. Thus, either f or $1/f$ is bounded on γ.

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